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Supergravity and the Gauging of $N = 1$ Supersymmetric Non-Linear σ-Models in the Atiyah-Ward Space-Time

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Abstract

We study the coupling of N=1 supergravity to a class of supersymmetric nonlinear σ-models in an N=1 superspace which is based on the Atiyah-Ward space-time of (2+2)-signature metric. We also discuss the conditions for the implementation of the gauging of isometries of the associated Kählerian target spaces and present the resulting gauge-covariant supersymmetric action functional in the curved superspace. In an Appendix, we aim at clarifying some aspects of the ordinary (1+3)-signature version of the very same problem.

Key-words: Supersymmetry; Supergravity.

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1 Introduction

In the recent past, a good deal of attention has been directed towards the construction of field theoretical models in the Atiyah-Ward space-time of (2+2)-signature metric [1]. One of the reasons for such an interest stems from the striking result obtained in refs.[2, 3], where a consistent N=2 superstring theory was shown to require a complex (Kahlërian) two-dimensional manifold as its relevant space-time background.

From a pure mathematical standpoint, the Atiyah-Ward space-time turns out to be one of the most suitable environments for the introduction of self-dual Yang-Mills connections [4, 5]. As it is well-known by now, these field configurations are of remarkable usefulness in the Donaldson's programme on algebraic geometry [6], as well as in the Ward's classification scheme of lower-dimensional integrable models [7].

As shown by Gates et al. in refs.[8], it may be also very interesting to construct and investigate both self-dual super-Yang-Mills and self-dual supergravity theories in the Atiyah-Ward space-time. To achieve this goal in an elegant manner, the authors of [8] found it desirable to develop a superspace formalism adapted to the (2+2)-signature: the so-called Atiyah-Ward superspace. Additional aspects concerning this new class of models were further discussed in an unified approach in ref.[9]. Furthermore, in refs.[10, 11] one addressed the issue of the construction of gauged supersymmetric non-linear σ-models in the flat Atiyah-Ward superspace. More specifically, the coupling of these theories to the super-Yang-Mills gauge sector was performed through the gauging of the isometries of their target manifolds [12, 13, 14, 15, 16, 17, 18, 19]. Now, concerning the curved superspace situation, it seems indeed quite appealing to analyze the possible couplings of matter and gauge fields in the presence of Atiyah-Ward supergravity. Let us observe, however, that the class of models focused here must be necessarily understood in the sense of the dimensional reduction framework employed by Ward in [7] (see also ref.[20]).

This is the purpose of the present work: to undertake the curved N=1 superspace extension of the supersymmetric non-linear σ-models of refs.[10, 11] which were built up in the context of the (2+2)-signature for the Atiyah-Ward space-time and, subsequently, to derive their gauged versions by means of the procedure carried out in [15].

The paper is organized as follows: in Section 2 we describe in detail the necessary steps in the formulation of N=1 supergravity in the Atiyah-Ward superspace by using the prepotential method of ref.[21], i.e. by determining the appropriate covariant derivatives from the transformations of the matter superfields; in Section 3 we recall the basic notions on Kähler's geometry needed for the gauging of isometries and use the elements introduced in the previous section to obtain the curved superspace version of the fully covariant supersymmetric σ-model action; Section 4 is devoted to an interpretation of our results and to the presentation of our conclusions. We also include an Appendix in which some aspects of the gauging of isometries in (1+3)-signature are clarified. In particular, besides the consistency conditions obeyed by the matter superpotentials (see ref.[22]), we determine further constraints for the chiral and antichiral matrices of the non-minimal gauge kinetic terms.

2 N=1 Supergravity in the Atiyah-Ward Superspace

We begin our study by describing the construction of the relevant N=1 supergravity supermultiplets in the Atiyah-Ward superspace. The notation and conventions of a superspace with a base space-time possessing a (2+2)-signature metric we are the same as in ref.[23]. To start with, we introduce supersymmetric matter in terms of a pair of chiral-antichiral complex scalar superfields, Φ and Ξ, with their component field expansions writing in accordance with[†]:

$$\Phi = A + i\theta\psi + i\theta^2 F + i\tilde{\theta}\tilde{\sigma}^m\theta\partial_m A + \tfrac{1}{2}\,\theta^2\tilde{\theta}\tilde{\sigma}^m\partial_m\psi - \tfrac{1}{4}\,\theta^2\tilde{\theta}^2\Box A, \tag{2.1}$$

$$\Xi = B + i\tilde{\theta}\tilde{\chi} + i\tilde{\theta}^2 G + i\theta\sigma^m\tilde{\theta}\partial_m B + \tfrac{1}{2}\,\tilde{\theta}^2\theta\sigma^m\partial_m\tilde{\chi} - \tfrac{1}{4}\,\theta^2\tilde{\theta}^2\Box B, \tag{2.2}$$

where A and B are complex scalar fields, ψ and $\tilde{\chi}$ are Weyl spinors and F and G are complex auxiliary fields. One has to notice here an essential feature of the Atiyah-Ward superspace: the scalar superfiels do not change their chirality properties under the complex conjugation operation:

$$\begin{aligned} \widetilde{D}_{\dot{\mu}}\Phi &= \widetilde{D}_{\dot{\mu}}\Phi^* = 0, \\ D_\mu\Xi &= D_\mu\Xi^* = 0, \end{aligned} \tag{2.3}$$

with the supersymmetric covariant derivatives[‡]

$$\begin{aligned} D_\mu &= \partial_\mu - i\tilde{\theta}^{\dot{\mu}}\partial_{\mu\dot{\mu}}, \\ \widetilde{D}_{\dot{\mu}} &= \widetilde{\partial}_{\dot{\mu}} - i\theta^\mu\widetilde{\partial}_{\mu\dot{\mu}}, \end{aligned} \tag{2.4}$$

obeying the graded differential algebra

$$\begin{aligned} \{D_\mu,\ \widetilde{D}_{\dot{\mu}}\} &= -2i\ \sigma^m_{\mu\dot{\mu}}\ \partial_m, \\ \{D_\mu,\ D_\nu\} &= \{\widetilde{D}_{\dot{\mu}},\ \widetilde{D}_{\dot{\nu}}\} = 0, \\ [D_\mu,\ \partial_m] &= [\widetilde{D}_{\dot{\mu}},\ \partial_m] = 0. \end{aligned} \tag{2.5}$$

In order to parametrize supertranslations in the Atiyah-Ward superspace, we choose a couple of chiral-antichiral real scalar superfields[§] , Λ and Γ, possessing the following structure:

$$\Lambda = \Lambda^M D_M, \qquad \Gamma = \Gamma^M D_M, \tag{2.6}$$

[†]The Grassmann coordinates, θ and $\tilde{\theta}$, are Majorana-Weyl spinors in (2+2) dimensions.

[‡]We use $\partial_{\mu\dot{\mu}} = \sigma^m_{\mu\dot{\mu}}\ \partial_m$.

[§]A crucial observation about the Atiyah-Ward superspace is the fact that chiral or antichiral superfields may be taken as real non-constant quantities, which is in contrast with the more familiar Minkowskian situation.

with $D_M = \{\partial_m,\ D_\mu,\ \widetilde{D}_{\dot\mu}\}$ standing for the supertranslation generators. The desired transformation laws for both chiral-antichiral matter superfields and their complex conjugated counterparts write then as:

$$
\begin{aligned}
\Phi' &= e^{i\Lambda}\ \Phi\ e^{-i\Lambda}, &\qquad \Xi' &= e^{i\Gamma}\ \Xi\ e^{-i\Gamma},\\
\Phi^{*\prime} &= e^{i\Lambda}\ \Phi^*\ e^{-i\Lambda}, &\qquad \Xi^{*\prime} &= e^{i\Gamma}\ \Xi^*\ e^{-i\Gamma},
\end{aligned}
\tag{2.7}
$$

It should be observed that the supertranslational parameters in (2.6) cannot be arbitrary, being indeed demanded to obey chirality-antichirality type constraints, namely,

$$
[\widetilde{D}_{\dot\mu},\ \Lambda] = 0, \qquad\qquad [D_\mu,\ \Gamma] = 0, \tag{2.8}
$$

which, in turn, may be rephrased as follows:

$$
\begin{aligned}
\widetilde{D}_{\dot\mu}\Lambda^\nu &= 0, &\qquad D_\mu\Gamma^{\dot\nu} &= 0,\\
\widetilde{D}_{\dot\mu}\Lambda^{\nu\dot\nu} + 2i\ \Lambda^\nu\delta_{\dot\mu}{}^{\dot\nu} &= 0, &\qquad D_\mu\Gamma^{\nu\dot\nu} + 2i\ \Gamma^{\dot\nu}\delta_\mu{}^\nu &= 0.
\end{aligned}
\tag{2.9}
$$

From eqs.(2.9) above one may determine the solutions

$$
\begin{aligned}
\Lambda^\mu &= \widetilde{D}^2 L^\mu, &\qquad \Gamma^{\dot\mu} &= D^2 M^{\dot\mu},\\
\Lambda^{\mu\dot\mu} &= 4i\ \widetilde{D}^{\dot\mu} L^\mu, &\qquad \Gamma^{\mu\dot\mu} &= 4i\ D^\mu M^{\dot\mu},
\end{aligned}
\tag{2.10}
$$

where the spinor superfields L^μ and $M^{\dot\mu}$ are totally arbitrary and mutually independent; and where the superfields $\Lambda^{\dot\mu}$ and Γ^μ are left undetermined for the moment.

To covariantize matter superfield monomials under supertranslations, one makes use of a real superfield H, the supergravity prepotential, which behaves under supercoordinate transformations as indicated below:

$$
e^{iH'} = e^{i\Gamma}\ e^{iH}\ e^{-i\Lambda}, \tag{2.11}
$$

and introduces the redefined quantities:

$$
\widetilde{\Xi} = e^{-iH}\ \Xi\ e^{iH}, \qquad\qquad \widetilde{\Xi}^* = e^{-iH}\ \Xi^*\ e^{iH}. \tag{2.12}
$$

As an illustration, one can easily verify that matter monomials such as $\Phi\ \widetilde{\Xi}^*$ and $\widetilde{\Xi}\ \Phi^*$ are truly covariant (Λ-transforming) objects.

The next step consists of defining a set of supergravity covariant derivatives. To accomplish this task we start by defining two non-covariant spinorial differential operators[¶‖]:

$$
\hat{E}_{\dot\mu} \equiv \widetilde{D}_{\dot\mu}, \qquad\qquad \hat{E}_\mu \equiv e^{-iH}\ D_\mu\ e^{iH}. \tag{2.13}
$$

[¶]We adopt a curved superspace notation in which letters from the early greek alphabet are used as spinorial frame indexes, while letters from the middle and beyond stand for the curved supermanifold ones.

[‖]For the sake of simplicity, we work throughout the paper in a chiral type supergravity representation.

Moreover, a vector derivative can also be obtained from the anticommutator of the spinorial ones:

$$\hat{E}_{\mu\dot{\mu}} = \tfrac{i}{2}\left\{\hat{E}_\mu,\ \hat{E}_{\dot{\mu}}\right\}. \tag{2.14}$$

However, we stress once more that these derivatives are not supergravity covariant. To see this, we simply let $\hat{E}_{\dot{\mu}}$ vary infinitesimally to get the following:

$$\begin{aligned} \delta\hat{E}_{\dot{\mu}} &= e^{i\Lambda}\ \hat{E}_{\dot{\mu}}\ e^{-i\Lambda} - \left(\hat{E}_{\dot{\mu}}\Lambda^{\dot{\nu}}\right)\hat{E}_{\dot{\nu}} \\ &= \left[i\Lambda,\ \hat{E}_{\dot{\mu}}\right] + \omega_{\dot{\mu}}{}^{\dot{\nu}}\hat{E}_{\dot{\nu}} + \Sigma\ \hat{E}_{\dot{\mu}}, \end{aligned} \tag{2.15}$$

where

$$\omega_{\dot{\mu}}{}^{\dot{\nu}} = -\tfrac{1}{2}\ \widetilde{D}_{(\dot{\mu}}\Lambda^{\dot{\nu})}, \qquad \Sigma = -\tfrac{1}{2}\ \widetilde{D}_{\dot{\mu}}\Lambda^{\dot{\mu}}. \tag{2.16}$$

To overcome this difficulty, we define a chiral density compensator Ψ transforming such as

$$\delta\Psi = [i\Lambda,\ \Psi] - \Sigma\ \Psi, \tag{2.17}$$

and construct a modified spinorial derivative

$$\check{E}_{\dot{\mu}} = \Psi\ \widetilde{D}_{\dot{\mu}}, \tag{2.18}$$

which transforms as

$$\delta\check{E}_{\dot{\mu}} = \left[i\Lambda,\ \check{E}_{\dot{\mu}}\right] + \omega_{\dot{\mu}}{}^{\dot{\nu}}\check{E}_{\dot{\nu}}. \tag{2.19}$$

Furthermore, one may also introduce an antichiral density compensator Υ transforming as

$$\delta\Upsilon = [i\Gamma,\ \Upsilon] - \Theta\ \Upsilon, \tag{2.20}$$

with

$$\Theta = -\tfrac{1}{2}\ D_\mu\Gamma^\mu, \tag{2.21}$$

and then write

$$\check{E}_\mu = e^{-iH}\ \Upsilon\ D_\mu\ e^{iH} = \widetilde{\Upsilon}\hat{E}_\mu, \tag{2.22}$$

where

$$\widetilde{\Upsilon} = e^{-iH}\ \Upsilon\ e^{iH}. \tag{2.23}$$

The transformation is now given by:

$$\delta\check{E}_\mu = \left[i\Gamma,\ \check{E}_\mu\right] + \tilde{\lambda}_\mu{}^\nu\check{E}_\mu, \tag{2.24}$$

with

$$\lambda_\mu{}^\nu = -\tfrac{1}{2}\ D_{(\mu}\Gamma^{\nu)}, \tag{2.25}$$

and

$$\tilde{\lambda}_\mu{}^\nu = e^{-iH}\ \lambda_\mu{}^\nu\ e^{iH}. \tag{2.26}$$

From the frame viewpoint, the spinorial derivatives E_α and $E_{\dot{\alpha}}$ read as:

$$E_\alpha = N_\alpha{}^\mu\ \check{E}_\mu, \qquad E_{\dot{\alpha}} = N_{\dot{\alpha}}{}^{\dot{\mu}}\ \check{E}_{\dot{\mu}}, \tag{2.27}$$

where one will later choose the gauge-fixings $N_\alpha{}^\mu = \delta_\alpha{}^\mu$ and $N_{\dot\alpha}{}^{\dot\mu} = \delta_{\dot\alpha}{}^{\dot\mu}$ upon these compensating fields. In the frame language, the vector derivative writes then as:

$$E_{\alpha\dot\alpha} = N_\alpha{}^\mu N_{\dot\alpha}{}^{\dot\mu} \left[\check{E}_{\mu\dot\mu} + \tfrac{i}{2}\, \Phi_{\mu\dot\mu}{}^{\dot\nu} \check{E}_{\dot\nu} + \tfrac{i}{2}\, \Phi_{\dot\mu\mu}{}^{\nu} \check{E}_\nu \right], \tag{2.28}$$

with the fermionic spin-connections given in terms of certain holonomy coeficients:

$$\Phi_{\mu\dot\mu}{}^{\dot\nu} = -\tfrac{1}{2}\, C_{\mu,\ \nu(\dot\mu}{}^{\nu\dot\nu)}\,, \qquad \Phi_{\dot\mu\mu}{}^{\nu} = \tfrac{1}{2}\, C_{(\mu\dot\nu,\ \dot\mu}{}^{\nu)\dot\nu}\,. \tag{2.29}$$

Under supertranslations one gets then the following:

$$\delta E_{\alpha\dot\alpha} = [i\Lambda,\ E_{\alpha\dot\alpha}] + \tilde\lambda_\alpha{}^\beta E_{\beta\dot\alpha} + \omega_{\dot\alpha}{}^{\dot\beta} E_{\alpha\dot\beta}. \tag{2.30}$$

To build up the fully covariant derivatives, one needs two further fermionic spin-connections:

$$\Phi_{\alpha\beta}{}^{\gamma} = -\tfrac{1}{4}\, C_{\alpha,\ (\beta\dot\alpha}{}^{\gamma)\dot\alpha}\,, \qquad \Phi_{\dot\alpha\dot\beta}{}^{\dot\gamma} = -\tfrac{1}{4}\, C_{\dot\alpha,\ \beta(\dot\beta}{}^{\alpha\dot\gamma)}\,. \tag{2.31}$$

With (2.29) and (2.31) one can finally obtain the fermionic derivatives:

$$\nabla_\alpha \equiv E_\alpha + \Phi_{\alpha\beta}{}^{\gamma} M_\gamma{}^\beta + \Phi_{\alpha\dot\beta}{}^{\dot\gamma} \widetilde{M}_{\dot\gamma}{}^{\dot\beta}, \tag{2.32}$$

and

$$\widetilde\nabla_{\dot\alpha} \equiv E_{\dot\alpha} + \Phi_{\dot\alpha\beta}{}^{\gamma} M_\gamma{}^\beta + \Phi_{\dot\alpha\dot\beta}{}^{\dot\gamma} \widetilde{M}_{\dot\gamma}{}^{\dot\beta}, \tag{2.33}$$

with the Lorentz group generators $M_\gamma{}^\beta$ and $\widetilde{M}_{\dot\gamma}{}^{\dot\beta}$ obeying respectively to the two algebraic relations:

$$\left[M_{\gamma\beta},\ M^{\delta\varepsilon}\right] = \delta_{(\gamma}{}^{(\delta} M_{\beta)}{}^{\varepsilon)}, \qquad \left[\widetilde{M}_{\dot\gamma\dot\beta},\ \widetilde{M}^{\dot\delta\dot\varepsilon}\right] = \delta_{(\dot\gamma}{}^{(\dot\delta} \widetilde{M}_{\dot\beta)}{}^{\dot\varepsilon)}. \tag{2.34}$$

At this stage, the covariant vector derivative is determined from

$$\left\{\nabla_\alpha,\ \widetilde\nabla_{\dot\alpha}\right\} = -2i\ \nabla_{\alpha\dot\alpha}, \tag{2.35}$$

its expression reading then as:

$$\nabla_{\alpha\dot\alpha} = E_{\alpha\dot\alpha} + \Phi_{\alpha\dot\alpha,\beta}{}^{\gamma} M_\gamma{}^\beta + \Phi_{\alpha\dot\alpha,\dot\beta}{}^{\dot\gamma} \widetilde{M}_{\dot\gamma}{}^{\dot\beta}, \tag{2.36}$$

with the vector spin-connections possessing the following structure

$$\Phi_{\alpha\dot\alpha,\beta}{}^{\gamma} = \tfrac{i}{2}\left[E_\alpha \Phi_{\dot\alpha\beta}{}^{\gamma} + \Phi_{\alpha\dot\alpha}{}^{\dot\delta} \Phi_{\dot\delta\beta}{}^{\gamma} + E_{\dot\alpha} \Phi_{\alpha\beta}{}^{\gamma} + \Phi_{\dot\alpha\alpha}{}^{\delta} \Phi_{\delta\beta}{}^{\gamma} + \Phi_{\alpha\beta}{}^{\delta} \Phi_{\dot\alpha\delta}{}^{\gamma} + \Phi_{\alpha}{}^{\gamma\delta} \Phi_{\dot\alpha\delta\beta}\right], \tag{2.37}$$

and

$$\Phi_{\alpha\dot\alpha,\dot\beta}{}^{\dot\gamma} = \tfrac{i}{2}\left[E_\alpha \Phi_{\dot\alpha\dot\beta}{}^{\dot\gamma} + \Phi_{\alpha\dot\alpha}{}^{\dot\delta} \Phi_{\dot\delta\dot\beta}{}^{\dot\gamma} + E_{\dot\alpha} \Phi_{\alpha\dot\beta}{}^{\dot\gamma} + \Phi_{\dot\alpha\alpha}{}^{\delta} \Phi_{\delta\dot\beta}{}^{\dot\gamma} + \Phi_{\alpha\dot\beta}{}^{\dot\delta} \Phi_{\dot\alpha\dot\delta}{}^{\dot\gamma} + \Phi_{\alpha}{}^{\dot\gamma\dot\delta} \Phi_{\dot\alpha\dot\delta\dot\beta}\right]. \tag{2.38}$$

Now, in view of the expressions (2.18), (2.22) and (2.28), the vielbein superdeterminant may be obtained. One finds:

$$E \ = \ \mathrm{sdet}\left[E_A{}^M\right] \ = \ \Psi^2\ \widetilde\Upsilon^2\ \mathrm{sdet}\left[\hat{E}_A{}^M(H)\right]. \tag{2.39}$$

The tensor $\widetilde{R}$ may be obtained through the anticommutator relation:

$$\{\nabla_\alpha,\ \nabla_\beta\} \ = \ \widetilde{R}_{\alpha\beta\gamma}{}^{\delta} M_\delta{}^{\gamma}, \tag{2.40}$$

where

$$\begin{aligned}\widetilde{R}_{\alpha\beta\gamma}{}^{\delta} \ &= \ \check{E}_\alpha \Phi_{\beta\gamma}{}^{\delta} \ + \ \Phi_{\alpha\beta}{}^{\varepsilon}\Phi_{\varepsilon\gamma}{}^{\delta} \ + \ \Phi_{\alpha\gamma}{}^{\varepsilon}\Phi_{\beta\varepsilon}{}^{\delta} \ + \ \Phi_{\alpha}{}^{\delta\varepsilon}\Phi_{\beta\gamma\varepsilon} \ + \\ &+ \ \check{E}_\beta \Phi_{\alpha\gamma}{}^{\delta} \ + \ \Phi_{\beta\alpha}{}^{\varepsilon}\Phi_{\varepsilon\gamma}{}^{\delta} \ + \ \Phi_{\beta\gamma}{}^{\varepsilon}\Phi_{\alpha\varepsilon}{}^{\delta} \ + \ \Phi_{\beta}{}^{\delta\varepsilon}\Phi_{\alpha\gamma\varepsilon}.\end{aligned} \tag{2.41}$$

One has to observe that (2.40) above is assuming implicitly the conventional constraint:

$$T_{\alpha\beta}{}^{\gamma} \ = \ C_{\alpha\beta}{}^{\gamma} \ + \ \Phi_{(\alpha\beta)}{}^{\gamma} \ = \ 0, \tag{2.42}$$

where the anholonomy coeficients are given by:

$$\{E_\alpha,\ E_\beta\} \ = \ C_{\alpha\beta}{}^{\gamma}\ E_\gamma. \tag{2.43}$$

Eq.(2.42) entails, in turn, the relation:

$$\Phi_{\alpha\beta\gamma} \ = \ \tfrac{1}{2}\left(C_{\beta\gamma\alpha} - C_{\alpha(\beta\gamma)}\right). \tag{2.44}$$

In the gauge $N_\alpha{}^{\mu} = \delta_\alpha{}^{\mu}$, the anholonomy coeficients may be determined from:

$$\left\{\check{E}_\alpha,\ \check{E}_\beta\right\} \ = \ \check{C}_{\alpha\beta}{}^{\gamma}\ \check{E}_\gamma, \tag{2.45}$$

with the following result:

$$C_{\alpha\beta}{}^{\gamma} \ = \ \check{C}_{\alpha\beta}{}^{\gamma} \ = \ \delta_{(\alpha}{}^{\gamma}\ \check{E}_{\beta)}\ ln\ \Upsilon. \tag{2.46}$$

By substituting this into (2.44) one gets:

$$\Phi_{\alpha\beta}{}^{\gamma} \ = \ -\tfrac{1}{2}\delta_{(\alpha}{}^{\gamma}\ \check{E}_{\beta)}\ ln\ \Upsilon^2. \tag{2.47}$$

We get then the following:

$$\widetilde{R} \ = \ \tfrac{1}{3}\ \widetilde{R}_{\alpha\beta}{}^{\alpha\beta} \ = \ \tfrac{1}{3}\ \left(\ -3\ \hat{E}_\alpha \hat{E}^\alpha\ \Upsilon^2\ \right) \ = \ \left(\hat{E}_\alpha\right)^2\ \Upsilon^2. \tag{2.48}$$

The tensor R writes similarly as:

$$R \ = \ \left(\hat{E}_{\dot\alpha}\right)^2\ \Psi^2. \tag{2.49}$$

To parametrize a restricted class of the super-Weyl transformations in the Atiyah-Ward superspace one introduces two independent complex superfields: the covariantly chiral and antichiral superfields, Ω and Π, respectively. The inverse of the vielbein superdeterminant and the curvature field-strengths will vary infinitesimally as:

$$\delta E^{-1} \ = \ -\left(\Omega + \Omega^* + \Pi + \Pi^*\right)\ E^{-1}, \tag{2.50}$$

and

$$\delta R^{-1} \ = \ -[\ 2(\Omega+\Omega^*) \ - \ (\Pi+\Pi^*)\ ]\ R^{-1} \ + \ [\ (\widetilde{\nabla}^2\Pi) \ + \ (\widetilde{\nabla}^2\Pi^*)\ ]\ R^{-2}, \tag{2.51}$$

$$\delta\widetilde{R}^{-1} \ = \ -[\ 2(\Pi+\Pi^*) \ - \ (\Omega+\Omega^*)\ ]\ \widetilde{R}^{-1} \ + \ [\ (\nabla^2\Omega) \ + \ (\nabla^2\Omega^*)\ ]\ \widetilde{R}^{-2}. \tag{2.52}$$

In the next section, we shall make use of the fundamentals of the Atiyah-Ward N=1 supergravity introduced here to perform the coupling of a class of supersymmetric non-linear σ-models. As it was mentioned in the Introduction, the combined notions of Kähler geometry of the target space, local isometry transformations of matter superfields and the super-Weyl dilatations will allow us to attain important information on the structure of the corresponding gauge invariant action functional.

3 N=1 Supergravity and Gauged Isometries

We now concentrate our attention on the specific issue of constructing gauged σ-models in the Atiyah-Ward superspace. Following [10, 11], we introduce the N=1 supersymmetric action governing the dynamics of two sets of complex chiral and antichiral superfields of the type described in the beginning of the previous section. We take**:

$$S = 2\int d^4x\ d^2\theta\ d^2\tilde{\theta}\ K(\Phi^i, \Xi^i; \Phi^{*i}, \Xi^{*i}), \tag{3.1}$$

with the Kähler potential K decomposing into two conjugated pieces:

$$K(\Phi^i, \Xi^i; \Phi^{*i}, \Xi^{*i}) = H(\Phi^i, \Xi^{*i}) + H^*(\Phi^{*i}, \Xi^i). \tag{3.2}$$

The pure scalar sector steming from the projection of (3.1) into component fields is given by:

$$S_{scalar} = 2\int d^4x \left(\frac{\partial^2 K}{\partial A^i \partial B^{*j}} \partial_\mu A^i \partial^\mu B^{*j} \ + \ \frac{\partial^2 K}{\partial A^{*i} \partial B^j} \partial_\mu A^{*i} \partial^\mu B^j \right). \tag{3.3}$$

Dimensional reduction and proper field truncations upon S_{scalar} above will give rise to a sensible (ghost free) scalar kinetic term in D=1+2 space-time dimensions (see ref.[20]).

The target spaces corresponding to the action S_{scalar} above are $4n$-dimensional Kähler manifolds, their Hermitean metric tensor possessing a four-block structure of the following type:

$$g_{\mathcal{I}\mathcal{J}} = \begin{pmatrix} 0 & 0 & 0 & g_{i\bar{\jmath}} \\ 0 & 0 & g_{\tilde{\imath}\bar{\jmath}} & 0 \\ 0 & g_{\bar{\imath}\tilde{\jmath}} & 0 & 0 \\ g_{\bar{\tilde{\imath}}j} & 0 & 0 & 0 \end{pmatrix}, \tag{3.4}$$

where

$$g_{i\bar{\jmath}} = \frac{\partial^2 H}{\partial \Phi^i \partial \Xi^{*j}}, \qquad g_{\tilde{\imath}\bar{\jmath}} = \frac{\partial^2 H^*}{\partial \Xi^i \partial \Phi^{*j}}, \qquad g_{\bar{\imath}\tilde{\jmath}} = \frac{\partial^2 H^*}{\partial \Phi^{*i} \partial \Xi^j}, \qquad g_{\bar{\tilde{\imath}}j} = \frac{\partial^2 H}{\partial \Xi^{*i} \partial \Phi^j}, \tag{3.5}$$

and

$$\mathcal{I}, \mathcal{J} = 1, ..., 4n \quad \text{with} \quad i, j = 1, ..., n. \tag{3.6}$$

The particular form of $g_{\mathcal{I}\mathcal{J}}$ in (3.4) above will entail a number of consequences for the geometry of our Kählerian target manifold. In fact, the most general type of Kähler transformation one can perform upon the potential K while keeping the action (3.1) invariant and the metric (3.4) unchanged is:

$$K \ \longrightarrow \ K' \ = \ K + \eta(\Phi) + \eta^*(\Phi^*) + \rho(\Xi) + \rho^*(\Xi^*), \tag{3.7}$$

** $\int d^4x d^2\theta d^2\tilde{\theta} \equiv \frac{1}{16}\int d^4x D^\alpha \widetilde{D}^{\dot{\alpha}} \widetilde{D}_{\dot{\alpha}} D_\alpha$

with $(\eta,\ \eta^*)$ and $(\rho,\ \rho^*)$ standing for arbitrary chiral and antichiral functions respectively. Moreover, the complex structures can be parametrized as follows:

$$J_{\mathcal{I}}{}^{\mathcal{J}} = \begin{pmatrix} i\delta_i{}^j & 0 & 0 & 0 \\ 0 & i\delta_{\hat{i}}{}^{\hat{j}} & 0 & 0 \\ 0 & 0 & -i\delta_{\bar{i}}{}^{\bar{j}} & 0 \\ 0 & 0 & 0 & -i\delta_{\bar{\hat{i}}}{}^{\bar{\hat{j}}} \end{pmatrix}. \tag{3.8}$$

In regard to isometries, they will be symmetries of (3.1) provided their action upon K writes in a form compatible with the Kähler transformation in (3.7). The Killing vectors $(\kappa^i_a(\Phi), \tau^i_a(\Xi), \kappa^{*i}_a(\Phi^*), \tau^{*i}_a(\Xi))$ are the generators of the isometry group $\mathcal{G}$. A global isometry transforms the target coordinates as:

$$\Phi'^i = \exp\left(L_{\lambda\cdot\kappa}\right)\ \Phi^i, \qquad \Phi'^{*i} = \exp\left(L_{\lambda\cdot\kappa^*}\right)\ \Phi^{*i},$$

$$\Xi'^i = \exp\left(L_{\lambda\cdot\tau}\right)\ \Xi^i, \qquad \Xi'^{*i} = \exp\left(L_{\lambda\cdot\tau^*}\right)\ \Xi^{*i}, \tag{3.9}$$

where λ is a real isometry parameter and $L_{\lambda\cdot\kappa}$ (resp. $L_{\lambda\cdot\tau}$) is for the Lie derivative along the vector field $\lambda\cdot\kappa \equiv \lambda^a\kappa^i_a\partial_i$ (resp. $\lambda\cdot\tau \equiv \lambda^a\tau^i_a\partial_{\hat{i}}$). As explained in detail in ref.[10], the gauging of the model demands the introduction of a complex Killing potential Y_a writing as:

$$Y_a \ = \ 2\ f_{ab}{}^c\kappa^i{}_d\tau^{*j}_c\ \frac{\partial^2 H}{\partial\Phi^i\partial\Xi^{*j}}\ g^{bd}, \tag{3.10}$$

where $f_{ab}{}^c$ and g^{bd} are the structure constants and the Killing metric of the isometry gauge group, respectively. Here, local isometries are parametrized in terms of a couple $(\Lambda_g,\ \Gamma_g)$ of real chiral and antichiral superfields, respectively[††]. The matter superfields transform under local isometries as defined below:

$$\Phi'^i = \exp\left(L_{\Lambda_g\cdot\kappa}\right)\ \Phi^i, \qquad \Xi'^i = \exp\left(L_{\Gamma_g\cdot\tau}\right)\ \Xi^i. \tag{3.11}$$

The gauge sector is constructed from a Lie algebra valued prepotential V which is a real superfield transforming as:

$$\exp\left(L_{V'\cdot\tau}\right) \ = \ \exp\left(L_{\Lambda_g\cdot\tau}\right)\ \exp\left(L_{V\cdot\tau}\right)\ \exp\left(-L_{\Gamma_g\cdot\tau}\right). \tag{3.12}$$

The action (3.1) is then modified by replacing the antichiral fields $(\Xi,\ \Xi^*)$ with the redefined quantities $(\tilde{\Xi},\ \tilde{\Xi}^*)$:

$$\tilde{\Xi}^i \ \equiv \ \exp\left(L_{V\cdot\tau}\right)\ \Xi^i, \qquad \tilde{\Xi}^{*i} \ \equiv \ \exp\left(L_{V\cdot\tau^*}\right)\ \Xi^{*i}. \tag{3.13}$$

Furthermore, in order to implement the gauging correctly one is enforced to introduce a pair of antichiral superfields $(v,\ v^*)$ playing the role of fictitious coordinates of the target. These superfields transform as:

$$\delta v \ = \ \lambda^a\rho_a(\Xi), \qquad \delta v^* \ = \ \lambda^a\rho^*_a(\Xi^*). \tag{3.14}$$

[††]These gauge parameters should not be confused with the supertranslational ones, i.e. Λ and Γ.

The isometry covariant action functional is [10]:

$$S_{cov} \;=\; 2 \int d^4x \; d^2\theta \; d^2\tilde{\theta} \Big[\; H(\Phi^i, \; \tilde{\Xi}^{*j}) \;+\; H^*(\Phi^{*i}, \; \tilde{\Xi}^j) \;-\; \tilde{\upsilon} \;-\; \tilde{\upsilon}^* \; \Big] . \qquad (3.15)$$

By substituting the original variables and introducing the chiral and antichiral integrals, one gets the complete gauged action functional:

$$\begin{aligned} S_{cov} \;&=\; 2 \int d^4x \; d^2\theta \; d^2\tilde{\theta} \Big\{ \; H(\Phi^i, \; \Xi^{*j}) \;+\; H^*(\Phi^{*i}, \; \Xi^j) \;+\; 2 \; Re\Big[\; \frac{e^L - 1}{L} \; V^a Y_a^* \; \Big] \; \Big\} + \\ &+ \int d^4x \; d^2\theta \; \big[\; F(\Phi^i, \; \Phi^{*i}) \;-\; \tfrac{1}{16} \, g_{ab} \; W^{a\alpha} W^b_\alpha \; \big] + \\ &+ \int d^4x \; d^2\tilde{\theta} \; \Big[\; G(\Xi^j, \; \Xi^{*j}) \;-\; \tfrac{1}{16} \, g_{ab} \; \widetilde{W}^{a\dot{\alpha}} \widetilde{W}^b_{\dot{\alpha}} \; \Big] , \end{aligned} \qquad (3.16)$$

where $L \equiv L_{V \cdot \tau}$, F and G are real chiral and antichiral superpotentials, respectively, and $(W_\alpha, \; \widetilde{W}_{\dot{\alpha}})$ are the gauge field-strengths with:

$$W_\alpha \;\equiv\; i\widetilde{D}^2 \left(e^{iV} D_\alpha \; e^{-iV} \right) , \qquad\qquad \widetilde{W}_{\dot{\alpha}} \;\equiv\; iD^2 \left(e^{-iV} \tilde{D}_{\dot{\alpha}} \; e^{iV} \right) . \qquad (3.17)$$

In the curved Atiyah-Ward superspace, the gauged action (3.16) will be coupled to supergravity and will write as (see the Appendix A):

$$\begin{aligned} S_{cov} \;&=\; -\tfrac{6}{k^2} \int d^4x \; d^2\theta \; d^2\tilde{\theta} \; E^{-1} \; e^{-\frac{k^2}{3} \left\{ \; H(\Phi^i, \; \Xi^{*j}) \;+\; H^*(\Phi^{*i}, \; \Xi^j) \;+\; 2 \; Re\left[\; \frac{e^L - 1}{L} \; V^a Y_a^* \; \right] \; \right\}} + \\ &+ \int d^4x \; d^2\theta \; d^2\tilde{\theta} \; E^{-1} R^{-1} \; \big[\; F(\Phi^i, \; \Phi^{*i}) \;-\; \tfrac{1}{16} \, g_{ab} \; W^{a\alpha} W^b_\alpha \; \big] \; + \\ &+ \int d^4x \; d^2\theta \; d^2\tilde{\theta} \; E^{-1} \widetilde{R}^{-1} \; \Big[\; G(\Xi^j, \; \Xi^{*j}) \;-\; \tfrac{1}{16} \, g_{ab} \; \widetilde{W}^{a\dot{\alpha}} \widetilde{W}^b_{\dot{\alpha}} \; \Big] . \end{aligned} \qquad (3.18)$$

Under an infinitesimal isometry transformation, the action (3.18) will vary such as:

$$\begin{aligned} \delta S_{cov} \;&=\; -\tfrac{6}{k^2} \int d^4x \; d^2\theta \; d^2\tilde{\theta} \; E^{-1} \; \Big[-\tfrac{k^2}{3} \; \big(-\Lambda_g{}^b \eta_b \;-\; \Lambda_g{}^b \eta_b^* \;-\; \Gamma_g{}^b \rho_b \;-\; \Gamma_g{}^b \rho_b^* \; \big) \; \Big] \; \times \\ &\times \; e^{-\frac{\kappa^2}{3} \left\{ H(\Phi^i, \; \Xi^{*j}) \;+\; H^*(\Phi^{*i}, \; \Xi^j) \;+\; 2 \; Re\left[\; \frac{e^L - 1}{L} \; V^a Y_a^* \; \right] \; \right\}} + \\ &+ \int d^4x \; d^2\theta \; d^2\tilde{\theta} \; E^{-1} R^{-1} \big[\; \Lambda_g{}^a \kappa_a^i F_i \;+\; \Lambda_g{}^a \kappa_a^{*i} F_{\bar{\imath}} \;-\; \tfrac{1}{16} \, g_{ab} \; W^{a\alpha} W^b_\alpha \; \big] \; + \\ &+ \int d^4x \; d^2\theta \; d^2\tilde{\theta} \; E^{-1} \bar{R}^{-1} \Big[\; \Gamma_g{}^a \tau_a^i G_{\hat{\imath}} \;+\; \Gamma_g{}^a \tau_a^{*i} G_{\bar{\hat{\imath}}} \;-\; \tfrac{1}{16} \, g_{ab} \; \widetilde{W}^{a\dot{\alpha}} \widetilde{W}^b_{\dot{\alpha}} \; \Big] . \end{aligned} \qquad (3.19)$$

On the other hand, a restricted super-Weyl transformation of the type discussed in the end of Section 2 will induce the following variation:

$$\delta_W S_{cov} = -\frac{6}{k^2}\int d^4x\, d^2\theta\, d^2\tilde{\theta}\, E^{-1}\,\left[-\,(\,\Omega\,+\,\Omega^*\,+\,\Pi\,+\,\Pi^*\,)\right]\times$$
$$\times\, e^{-\frac{k^2}{3}\left\{ H(\Phi^i,\,\Xi^{*j})\,+\,H^*(\Phi^{*i},\,\Xi^j)\,+\,2\,Re\left[\frac{e^L-1}{L}\,V^aY^*_a\right]\right\}}\,+$$
$$-\int d^4x\, d^2\theta\, d^2\tilde{\theta}\, E^{-1}R^{-1}\,(\,3\Omega\,+\,3\Omega^*)\left[\,F(\Phi^i,\,\Phi^{*i})\,-\,\frac{1}{16}\,g_{ab}\,W^{a\alpha}W^b_\alpha\,\right]+$$
$$-\int d^4x\, d^2\theta\, d^2\tilde{\theta}\, E^{-1}\widetilde{R}^{-1}\,(\,3\Pi\,+\,3\Pi^*)\left[\,G(\Xi^j,\,\Xi^{*j})\,-\,\frac{1}{16}\,g_{ab}\,\widetilde{W}^{a\dot{\alpha}}\widetilde{W}^b_{\dot{\alpha}}\,\right]. \tag{3.20}$$

The cancellation between (3.19) and (3.20) implies the differential constraints upon the superpotentials F and G:

$$\Lambda_g^{\ a}\,\kappa^i_a\,F_i\,-\,k^2\,\Lambda_g^{\ a}\,\eta_a\,F\,=\,0,\qquad \Lambda_g^{\ a}\,\kappa^{*i}_a\,F_{\bar{\imath}}\,-\,k^2\,\Lambda_g^{\ a}\,\eta^*_a\,F\,=\,0, \tag{3.21}$$

$$\Gamma_g^{\ a}\,\tau^i_a\,G_{\hat{\imath}}\,-\,k^2\,\Gamma_g^{\ a}\,\rho_a\,G\,=\,0,\qquad \Gamma_g^{\ a}\,\tau^{*i}_a\,G_{\hat{\bar{\imath}}}\,-\,k^2\,\Gamma_g^{\ a}\,\tau^*_a\,G\,=\,0. \tag{3.22}$$

The introduction of non-minimal kinetic terms for the gauge fields may also be taken into account along the lines envisioned in the Appendix A. In this case, chiral and antichiral coupling matrices would also obey to non-trivial conditions (see the equations (A.22) and (A.23)).

4 Concluding Remarks

We have explicitly constructed and discussed the essential elements of N=1 supergravity in the Atiyah-Ward superspace. The method employed here consisted of determining the correct covariant derivatives of supergravity by postulating the convenient supertranslational transformations for the matter superfields (see ref.[21]). In a second step, we presented the consistent coupling of the so-called Atiyah-Ward supergravity to a class of N=1 supersymmetric non-linear σ-models. Subsequently, one implemented the gauging of isometries of the corresponding target manifolds by making use of a general framework introduced by Hull et al. in [15]. One observes that, in order to attain local isometry invariance of the gauged action functional, the matter superpotentials must obey constraint equations for both chiralities, in a similar form as it was obtained in the Minkowskian case. We also indicate the conditions which have to be imposed onto the chiral and antichiral matrices of the non-minimal couplings for the gauge kinetic terms.

The gauge-invariant supersymmetric σ-models constructed here may be of interest in the study of the dynamics of lower-dimensional supersymmetric field theories. Indeed, the supression of one time coordinate of these action functionals allows one to uncover new examples of three-dimensional Minkowskian field models. Moreover, it seems quite reasonable to believe that the coupling to supergravity entails new consequences for these dimensionally reduced theories.

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A Appendix: N=1 Supergravity and Gauged Isometries in D=1+3 Dimensions

In the present appendix we discuss some aspects of the coupling of supergravity to a general gauged (isometry covariant) N=1 supersymmetric non-linear σ-model in the usual Wess-Zumino superspace with base space-time of (1+3)-signature. As mentioned in the introduction, after the component field formulation developed by Bagger in ref.[24], this issue has been focused in the literature by some authors. Indeed, the first curved superspace formulation for such a construction was developed by Samuel in ref.[25] and, later on, by Grimm in [26] in the so-called Kähler superspace framework (see refs.[27, 28] for a more complete account on that structure) and by Bagger and Wess in [22]. Here, we aim at clarifying some aspects concerning the use of the restricted superscale transformations of supergravity which, being employed in conjunction with the local isometry ones, characterize a combined invariance of the theory. In what follows, we switch to the notation and conventions of ref.[21] and exploit the strategy and results obtained by Hull et al. in ref.[15] for the flat N=1 superspace situation.

In the flat N=1 superspace case, the σ-model action gorverning the dynamics of the complex scalar superfields involves the Kähler potential of a complex manifold of coordinates $(\phi^i,\ \bar{\phi}_j)$ and is given by (see ref.[29]):

$$S = \int d^4x\ d^4\theta\ K(\phi^i,\ \bar{\phi}_j), \tag{A.1}$$

which is known to be invariant under global isometry transformations of that internal space which, at the infinitesimal level write as below

$$\delta\phi^i = \lambda^A k^i{}_A, \qquad \delta\bar{\phi}_j = \lambda^A \bar{k}_{Aj}, \tag{A.2}$$

with $k^i{}_A(\phi)$ and $\bar{k}_{Aj}(\bar{\phi})$ standing for holomorphic and antiholomorphic Killing vectors obeying the Lie algebraic relations

$$k^j_{[A} k^i{}_{B]\ ,j} = f_{AB}{}^C k^i{}_C, \qquad \bar{k}_{[Aj}\bar{k}_{B]i}{}^{,j} = f_{AB}{}^C \bar{k}_{Ci}. \tag{A.3}$$

However, the action (A.1) is not invariant under isometry transformations of the local type:

$$\delta\phi^i = \Lambda^A k^i{}_A, \qquad \delta\bar{\phi}_j = \Lambda^A \bar{k}_{Aj}, \tag{A.4}$$

where the parameters Λ and $\bar{\Lambda}$ are now promoted to chiral and antichiral superfields respectively. To covariantize (A.1) one introduces a real functional, the Killing potential $X(\phi^i,\ \bar{\phi}_j)$, which satisfies the equations

$$k^i{}_A K_i = iX_A + \eta_A, \qquad \bar{k}_{Ai} K^i = -iX_A + \bar{\eta}_A, \tag{A.5}$$

with $\eta_A(\phi)$ and $\bar{\eta}_A(\bar{\phi})$ being the chiral and antichiral functions of a general Kähler transformation. In the specific case of a semi-simple isometry group, the expression for the Killing potential can be explicitly determined:

$$X_A = 2i\ f_{AB}{}^C k^i{}_D \bar{k}_{Cj} K^i{}_j g^{BD}, \tag{A.6}$$

where g^{BD} is the inverse Killing metric. Furthemore, one is enforced to introduce a couple of "auxiliary" variables ζ and $\bar{\zeta}$ which vary under infinitesimal isometries as follows [15]:

$$\delta\zeta = \eta_A \ \Lambda^A, \qquad \delta\bar{\zeta} = \bar{\eta}_A \ \bar{\Lambda}^A. \tag{A.7}$$

Next, we redefine the antichiral matter fields $\bar{\phi}_j$ to transform in terms of the chiral gauge parameter Λ rather than $\bar{\Lambda}$. One sets then:

$$\widetilde{\phi}_j = e^{iL_{V.\bar{k}}}\bar{\phi}_j, \tag{A.8}$$

where $L = iL_{V.\bar{k}}$ is the Lie derivative along the vector field direction $iV.\bar{k}$, and assuming moreover the additional transformation law for the prepotential V

$$e^{iL_{V'.\bar{k}}} = e^{L_{\Lambda.\bar{k}}} \ e^{iL_{V.\bar{k}}} \ e^{-L_{\bar{\Lambda}.\bar{k}}}. \tag{A.9}$$

By collecting all these facts one arrives at the gauged action given below:

$$S = \int d^4x \ d^4\theta \ \left[\ K(\phi^i, \ \tilde{\phi}_j) - \zeta - \tilde{\zeta} \ \right], \tag{A.10}$$

which, with the aid of the relations (see ref.[15])

$$K(\phi^i, \ \tilde{\phi}_j) = K(\phi^i, \ \bar{\phi}_j) + \frac{e^L - 1}{L} \ K^i \bar{k}_{Ai}(iV^A), \tag{A.11}$$

and

$$\tilde{\zeta} = \bar{\zeta} + i \ \frac{e^L - 1}{L} \ \bar{\eta}_A V^A, \tag{A.12}$$

may be rewritten as

$$\begin{aligned} S \ &= \ \int d^4x \ d^4\theta \ \left[\ K(\phi^i, \ \bar{\phi}_j) \ + \ \frac{e^L - 1}{L} \ X_A V^A \ \right] + \\ &+ \ \int d^4x \ d^2\theta \ \left[\ P(\phi^i) \ + \ \tfrac{1}{4} \ Q_{AB}(\phi^i) \ W^{A\alpha} W^B_\alpha \ \right] + \\ &+ \ \int d^4x \ d^2\bar{\theta} \ \left[\ P(\bar{\phi}_j) \ + \ \tfrac{1}{4} \ \bar{Q}_{AB}(\bar{\phi}_j) \ \bar{W}^{A\dot{\alpha}} \bar{W}^B_{\dot{\alpha}} \ \right], \end{aligned} \tag{A.13}$$

where we have added the chiral and antichiral integrals. Here, one has to observe that the two auxiliary fields dropped out from the final gauged action due to their chiral and antichiral characters respectively. Equation (A.13) represents the most general version of the gauged N=1 supersymmetric non-linear sigma model in flat superspace. In the curved superspace, (A.13) will assume the form:

$$\begin{aligned} S \ &= \ -\tfrac{3}{\kappa^2} \int d^4x \ d^4\theta \ E^{-1} e^{-\frac{\kappa^2}{3} \left[\ K(\phi^i, \ \bar{\phi}_j) \ + \ \frac{e^L - 1}{L} X_A V^A \ \right]} + \\ &+ \ \int d^4x \ d^4\theta \ E^{-1} R^{-1} \left[\ P(\phi^i) \ + \ \tfrac{1}{4} \ Q_{AB}(\phi^i) \ W^{A\alpha} W^B_\alpha \ \right] + \\ &+ \ \int d^4x \ d^4\theta \ E^{-1} \bar{R}^{-1} \left[\ P(\bar{\phi}_j) \ + \ \tfrac{1}{4} \ \bar{Q}_{AB}(\bar{\phi}_j) \ \bar{W}^{A\dot{\alpha}} \bar{W}^B_{\dot{\alpha}} \ \right], \end{aligned} \tag{A.14}$$

where R^{-1} and $\bar{R}^{-1}$ are the inverses of the chiral and anti-chiral curvature field-strengths respectively. Here, we attribute zero super-Weyl weights to the matter and gauge superfields*. An infinitesimal isometry will induce the following variation upon S:

$$\begin{aligned}\delta S \;=\; & -\tfrac{3}{\kappa^2}\int d^4x\; d^4\theta\; E^{-1}\;\Big[\,-\tfrac{\kappa^2}{3}\;\left(-\Lambda^B\eta_B-\bar{\Lambda}^B\bar{\eta}_B\right)\Big]\; e^{-\frac{\kappa^2}{3}\left[\,K(\phi^i,\,\bar{\phi}_j)\,+\,\frac{e^L-1}{L}X_AV^A\,\right]}+\\ & +\;\int d^4x\; d^4\theta\; E^{-1}R^{-1}\,[\;\Lambda^A k^i_A P_i\;+\;\tfrac{1}{4}\;(\;\Lambda^C k^i_C Q_{ABi}\;-\;f_{[AC}{}^D\Lambda^C Q_{DB]}\;)\;W^{A\alpha}W^B_\alpha\;]\,+\\ & +\;\int d^4x\; d^4\theta\; E^{-1}\bar{R}^{-1}\,[\;\bar{\Lambda}^A\bar{k}_{Ai}\bar{P}^i\;+\;\tfrac{1}{4}\;(\;\bar{\Lambda}^C\bar{k}_{Ci}\bar{Q}_{AB}{}^i\;-\;f_{[AC}{}^D\bar{\Lambda}^C\bar{Q}_{DB]}\;)\;\bar{W}^{A\dot{\alpha}}\bar{W}^B_{\dot{\alpha}}\;]\,.\end{aligned} \tag{A.15}$$

To check that (A.14) above is invariant under the local isometries in (A.4), one is enforced to simultaneously implement a restricted type of superscale transformation [30, 31]. Under the action of these specific superscalings, we get the following infinitesimal transformations:

$$\delta E^{-1}=-\left(\Sigma+\bar{\Sigma}\right)E^{-1}, \tag{A.16}$$

for the inverse supervielbein superdeterminant, and

$$\delta R^{-1}=-\left(2\Sigma-\bar{\Sigma}\right)R^{-1}\;+\;\left(\bar{\nabla}^2\bar{\Sigma}\right)R^{-2}, \tag{A.17}$$

$$\delta\bar{R}^{-1}=-\left(2\bar{\Sigma}-\Sigma\right)\bar{R}^{-1}\;+\;\left(\nabla^2\Sigma\right)R^{-2}, \tag{A.18}$$

for the inverse curvature field-strengths. The parameters Σ and $\bar{\Sigma}$ are chiral and anti-chiral superfields respectively. The gauged action S in (A.14) will vary then as:

$$\begin{aligned}\delta_W S \;=\; & \tfrac{3}{\kappa^2}\int d^4x\; d^4\theta\; E^{-1}\left(\Sigma+\bar{\Sigma}\right)e^{-\frac{\kappa^2}{3}\left[\,K(\phi^i,\,\bar{\phi}_j)\,+\,\frac{e^L-1}{L}X_AV^A\,\right]}+\\ & -\;\int d^4x\; d^4\theta\; E^{-1}R^{-1}(\;3\Sigma\;)\,[\;P(\phi^i)\;+\;\tfrac{1}{4}\;Q_{AB}(\phi^i)\;W^{A\alpha}W^B_\alpha\;]\,+\\ & -\;\int d^4x\; d^4\theta\; E^{-1}\bar{R}^{-1}(\;3\bar{\Sigma}\;)\,[\;P(\bar{\phi}_j)\;+\;\tfrac{1}{4}\;\bar{Q}_{AB}(\bar{\phi}_j)\;\bar{W}^{A\dot{\alpha}}\bar{W}^B_{\dot{\alpha}}\;]\,,\end{aligned} \tag{A.19}$$

where we observe that the non-homogeneous parts of the transformations (A.17) and (A.18) give rise to surface terms which do not contribute to $\delta_W S$. Now, to obtain the cancellation between the two variations one is led to impose the following conditions upon the potentials $P(\phi^i)$, $Q_{AB}(\phi^i)$ and their complex conjugated counterparts:

$$\Lambda^A k^i_A P_i\;-\;\kappa^2\Lambda^A\eta_A P\;=\;0, \tag{A.20}$$

$$\bar{\Lambda}^A\bar{k}_{Ai}\bar{P}^i\;-\;\kappa^2\bar{\Lambda}^A\bar{\eta}_A\bar{P}\;=\;0, \tag{A.21}$$

and

$$\Lambda^C k^i_C Q_{ABi}\;-\;f_{[AC}{}^D\Lambda^C Q_{DB]}\;-\;\kappa^2\Lambda^C\eta_C Q_{AB}\;=\;0, \tag{A.22}$$

*In general, super-Weyl weights for matter are arbitrary.

$$\bar{\Lambda}^C \bar{k}_{Ci} \bar{Q}_{AB}{}^i \;-\; f_{[AC}{}^D \bar{\Lambda}^C \bar{Q}_{DB]} \;-\; \kappa^2 \bar{\Lambda}^C \bar{\eta}_C \bar{Q}_{AB} \;=\; 0. \tag{A.23}$$

The two conditions (A.20) and (A.21) above have been derived in ref.[12], while (A.22) and (A.23) were lacking in the literature. They represent highly non-trivial constraints upon the potentials and turn out to be essential for the gauge invariance of the N=1 supersymmetric σ-model in curved superspace.

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