Gravitational Wave Emission and Mass Extraction from a Perturbed Schwarzschild Black Hole

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ABSTRACT

A relativistic model for the emission of gravitational waves from an initially unperturbed Schwarzschild black hole, or spherical collapsing configuration, is completely integrated. The model consists basically of gravitational perturbations of the Robinson-Trautman type on the Schwarzschild spacetime. In our scheme of perturbation, gravitational waves may extract mass from the collapsing configuration, and the amount of mass extracted depends on the particular gravitational wave l-poles emitted ($l \geq 2$). The formulation allows us to define a mass emissity function $e_l = \dfrac{3b_0}{2} \Big/ \left[\dfrac{l(l+1)}{2}\left(\dfrac{l(l+1)}{2} - 1\right)\right]$, which measures the ratio of mass taken out from the source by a l-pole (b_0 is a small positive parameter, of the order of the perturbations, characteristic of the mechanism of mass emission in a Schwarzschild background spacetime). The quadrupole emission mode is the dominant mechanism in extracting mass from the configuration. Robinson-Trautmann perturbations also include another mode of emission of mass, which we denote shell emission mode: in the equatorial plane of the configuration, a timelike $(1 + 2)$ shell of matter may be present, whose stress-energy tensor is modelled by neutrinos and strings emitted radially on the shell; no gravitational waves are present in this model. The invariant characterization of gravitational wave perturbations and of the gravitational wave zone is made through the analysis of the structure of the curvature tensor and the use of the Peeling Theorem. The time behaviour of a l-pole gravitational wave has the form $\exp[-(u/3M)\, l(l+1)/2\, (l(l+1)/2 - 1)]$, where M is the initial mass of the collapsing configuration and u may be interpreted asymptotically as the retarded time. It follows that, for a Galilean observer at infinity, the horizon of the final black hole configuration takes an infinite time to form.

Key-words: Gravitational waves; Black holes; Schwarzschild geometry perturbations; Robinson-Trautman perturbations.

1   Introduction

In the study of the dynamics of formation of black holes, the final state of the collapsing configuration is fixed by the so-called Wheeler's lemma "A black hole has no hair" [1]. For a Schwarzschild black hole this lemma is sustained by the complete analysis of scalar, electromagnetic and gravitational perturbations on the background geometry realized in Refs. [2–7] and extended for non-classical fields in Refs. [8–10]. The analysis carried out in these references treat the perturbations as test fields. However, in the particular case of gravitational perturbations, the approach does not consider two important issues: (i) how do the gravitational perturbations of radiative character extract mass, and what is the amount of mass carried out by a particular gravitational radiation pole emitted; (ii) what is the fate of the compact event horizon due to the presence of gravitational perturbations. In this paper, our aim is to examine these issues by considering a simple class of gravitational perturbations of the radiative type. Our approach here will be based on perturbations of the Robinson-Trautman type made on a Schwarzschild background. This class of perturbations may be understood as belonging to the family of Robinson-Trautman metrics [11], having the Schwarzschild geometry as a particular limit, the advantage of which is that the definition of a mass function appears naturally. A further advantage in the use Robinson-Trautman perturbations is that a gravitational wave zone is well characterized in the perturbed spacetime, such that it constitutes a simple and suitable model for the exam of the issues discussed above. Also, the Robinson-Trautman perturbations, if by one hand they contain just a particular set of even-parity perturbations (in the terminology of Regge-Wheeler), by the other hand they are more general in the sense that they contain a coordinate gauge dependent piece which allows us to obtain information on how gravitational waves extract mass from the collapsing configuration (indeed, invariant information).

The paper is organized as follows. In Section 2, the Robinson-Trautman metrics are introduced together with the corresponding vacuum Einstein equations, and after particularized as perturbations of the Schwarzschild metric. In Section 3 we examine the structure of the curvature tensor for the Robinson-Trautman perturbations, in order to obtain under what conditions they are true gravitational wave perturbations and have a well defined gravitational wave zone. The use of the Peeling Theorem is made in this characterization. In Section 4, the integration of field equations is completely realized, based on pertinent initial conditions, and physical results discussed. In Section 5, the temporal gauge group for Robinson-Trautman metrics is introduced, and an invariant mass function defined, allowing a characterization of mass loss from the configuration by emission of gravitational waves. Finally in Section 6, a summing up of the resulting scenario is made.

Throughout the paper, the units are such that $c = 8\pi G = 1$.

2   The Geometry and the Field Equations

We start by considering the family of Robinson-Trautman spacetimes [11], with metric given by

$$ds^2 = A^2(u,r,\theta)du^2 + 2du\ dr - r^2 K^2(u,\theta)(d\theta^2 + \sin^2\theta d\phi^2)\ . \qquad (1)$$

This geometry is non-stationary and axially symmetric, admitting the obvious Killing vector $\partial/\partial\phi$. The components $G_{22} = 0$ and $G_{33} = 0$ of Einstein's equations in vacuum, together with $G_{02} = 0 = G_{12}$, gives that

$$A^2(u,r,\theta) = L(u,\theta) + B(u)/r + 2rK'(u,\theta)/K(u,\theta)\ , \qquad (2)$$

with L and K arbitrary functions of u and θ, and B an arbitrary function of u. Here a prime denotes $\partial/\partial u$. The remaining vacuum equations $G_{00} = G_{01} = G_{11} = 0$ lead us to choose the function $L(u,\theta)$ as

$$L(u,\theta) = 1/K^2 - K_{\theta\theta}/K^3 + K_\theta^2/K^4 - K_\theta \mathrm{cotg}\ \theta/K^3\ , \qquad (3)$$

where a subscript θ denotes now $\partial/\partial\theta$, resulting

$$3B(u)K'/K + B'(u) + \frac{1}{2K^2\sin\theta}(L_\theta\sin\theta)_\theta = 0\ . \qquad (4)$$

Equations (3) and (4) are the basis of our analysis in this paper.

It is easy to see that

$$L = 1 = K\quad,\quad B = -2M = \mathrm{const.} \qquad (5)$$

is a solution of (3) and (4), corresponding to the Schwarzschild metric in outgoing Eddington-Finkelstein coordinates [1]. These coordinates are most convenient for our analysis of a non-spherical, axially symmetric collapse with emission of gravitational waves. We remark here that the Eddington-Finkelstein retarded coordinate u may be interpreted as the Newtonian time of an observer a rest at infinity $(r \to \infty)$ [12, 13], and r is the parameter distance along the *outgoing* null geodesics determined by the vector field $\partial/\partial r$. Of course, our description is valid only outside the apparent horizon defined by $A^2(u,r,\theta) = 0$.

Let us now introduce Robinson-Trautman perturbations on the Schwarzschild solution (5), namely, we take a geometry of the form (1), (2), (3) given by

$$ds^2 = \left[1 + \varepsilon W(u,\theta) + \frac{-2M + \varepsilon Z(u)}{r} + 2\varepsilon r\partial/\partial u\ Y(u,\theta)\right]du^2$$
$$+\ 2dudr - r^2[1 + \varepsilon Y(u,\theta)]^2(d\theta^2 + \sin^2\theta d\phi^2)\ , \qquad (6)$$

where ε is a small parameter. As we will show later, the curvature tensor calculated from (6) guarantees that, in general, these perturbations do not result from mere coordinate transformations but they are physical in the sense that the invariants constructed with the curvature tensor calculated from (6) are distinct from the ones of the curvature of the Schwarzschild solution (5).

The components of the metric perturbations $h_{ab}(a, b = 0, 1, 2, 3)$ may be expressed

$$h_{ab} = \begin{pmatrix} W(u,\theta) + Z(u)/r + 2r & \partial_u Y(u,\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2r^2 Y(u,\theta) & 0 \\ 0 & 0 & 0 & 2r^2 Y(u,\theta) \sin^2\theta \end{pmatrix} \quad (6.a)$$

After the integration of the field equations (3) and (4) for the perturbed metric, when the functions $W(u,\theta)$ and $Y(u,\theta)$ are determined, it will result that perturbations (6.a) are actually split into two parcels, only one of which will have angular dependence. This latter parcel is a particular type of even-parity perturbations, in the terminology of Regge and Wheeler (Ref. [2]), or of polar-type perturbations in the terminology of Chandrasekhar (Ref. [14]), and are already in the canonical form for even-parity perturbations, as given in Ref. [2]. The other parcel will contain, as its entries, the function $Z(u)/r$ and separation constants appearing in the integration. We might be tempted to interpret $B(u) = -2M + \varepsilon Z(u)$ as a time-dependent mass term, or $Z(u)$ as a perturbation in the mass M of the Schwarzschild background. However, as will be discussed in Section 5, we can always choose a temporal gauge in which $B(u)$ is a constant. Therefore it is not possible to attach a meaning to the time dependence of B, unless the time coordinate u may be fixed in an independent way, for instance, as the retarded time coordinate of the Schwarzschild background. This is a crucial point in our paper, which shall be dealt with in Section 5. There we will show that we can define an invariant mass function which coincides with $B(u)$ for large values of u.

Inserting (6) in the field equations (3) and (4), we obtain in first order in ε,

$$dZ/du - 6M \ \partial Y/\partial u + \frac{1}{2\sin\theta}[(\sin\theta \ W_\theta)_\theta] = 0 \ , \qquad (7)$$

$$Y_{\theta\theta} + Y_\theta \cotg\theta + 2Y = -W \ . \qquad (8)$$

The form of the above equations suggests the following separation Ansatz

$$\begin{aligned} Y(u,\theta) &= y(\theta)N(u) \ , \\ W(u,\theta) &= w(\theta)N(u) \ , \end{aligned} \qquad (9)$$

yielding from (7) and (8),

$$-\frac{6M}{N} \ [dN/du] = a_0 = \text{const.} \ , \qquad (10.a)$$

$$\frac{1}{N} \ [dZ/du] = b_0 = \text{const.} \ , \qquad (10.b)$$

$$w_{\theta\theta} \ + \ w_\theta \cotg\theta + 2a_0 y + 2b_0 = 0 \ , \qquad (11.a)$$

$$y_{\theta\theta} \ + \ y_\theta \cotg\theta + 2y = -w \ . \qquad (11.b)$$

Equations (10) and (11) determine uniquely (up to a temporal gauge freedom to be discussed later) the Robinson-Trautman gravitational perturbations of Schwarzschild black hole of mass M.

Before proceeding into the task of integrating equations (10) and (11), taking into account the initial conditions connected to the emission of gravitational waves, we must discuss the physical nature of the perturbations considered and what are the new features present in the spacetime described by (6). In this way, we examine, in the next section, the structure of the Weyl tensor of (6).

3 The Structure of the Weyl Tensor and the Gravitational Wave Zone

Let us introduce the semi-null tetrad basis determined by the 1-forms

$$
\begin{aligned}
O^0 &= du , \\
O^1 &= A^2/2 \ du + dr, \\
O^2 &= r \ K \ d\theta , \\
O^3 &= r \ K \ \sin\theta \ d\phi ,
\end{aligned}
\tag{12}
$$

where the metric (1) assumes the from $ds^2 = g_{AB}O^A O^B$, with

$$
g_{AB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .
$$

The tetrad basis has the physical property that it is parallely propagated along the null geodesics determined by $\partial/\partial r$. As we shall see, these *outgoing* null geodesics are the propagation direction of the gravitational waves. In this basis, the non-zero Weyl tensor components are given by

$$
\begin{aligned}
C_{2323} &= -C_{0101} = 2 \ C_{0212} = B(u)/r^3 = [-2M + Z(u)]2/r^3 , \tag{13.a} \\
C_{0303} &= -C_{0202} = A(u,\theta)/r^2 \ , \quad C_{0303} = L_\theta/2Kr^2 , \tag{13.b} \\
C_{0202} &= -C_{0303} = -D(u,\theta)/r , \tag{13.c}
\end{aligned}
$$

where the functions A and D are

$$
A(u,\theta) = \frac{1}{4K^2} \left(-L_{\theta\theta} + 2L_\theta K_\theta/K + L_\theta\mathrm{cotg}\ \theta\right) , \tag{14}
$$

$$
D(u,\theta) = \frac{1}{2K^2} \partial_u[(K_\theta/K)_\theta - K_\theta/K\mathrm{cotg}\ \theta - (K_\theta/K)^2] .
$$

We note that the r-dependence of the components (13.a,b,c) is , respectively, $1/r^3$, $1/r^2$ and $1/r$. Indeed, from (13), we may express

$$
C_{ABCD} = D_{ABCD}/r^3 + III_{ABCD}/r^2 + N_{ABCD}/r , \tag{15}
$$

where D_{ABCD}, III_{ABCD} and N_{ABCD} are of algebraic type D, type III and type N in the Petrov classification [15, 16, 17], respectively, and have the vector field $k = \partial/\partial r$ as a principal null direction (cf. (17) below). In the coordinate basis, they have the property of being covariantly constant along the null direction k [11]. For the Schwarzschild geometry, only type D terms are present, and the non-zero components of the Weyl tensor are

$$C_{2323} = -C_{0101} = 2\ C_{0212} = -4M/r^3\ . \tag{16}$$

Comparing (13) with (16) we can see that, indeed, the metric (6) is a true perturbation of the Schwarzschild geometry.

From (15) we can now establish an invariant characterization of the presence of gravitational waves in the perturbed spacetime (6), and of a corresponding gravitational radiation wave zone. This is based on two pillars:

(i) the Peeling Theorem (for the linearized Riemann tensor of retarded multipole fields, see Refs. [21, 22]; for the general case, see Ref. [23]; for a review, including peeling properties of the Maxwell tensor, see Ref. [20]);

(ii) the analysis of the spacetime of gravitational wave solutions of Einstein's equations, and their relation to electromagnetic waves in Maxwell's theory [1, 11, 18, 19, 20].

The Peeling Theorem states that the Weyl tensor (or vacuum Riemann tensor) of a radiative gravitational bounded source, expanded in powers of $(1/r)$, has the general form

$$C_{ABCD} = N_{ABCD}/r + III_{ABCD}/r^2 + II_{ABDC}/r^3 + I_{ABCD}/r^2 + \cdots \tag{17}$$

where r is the parameter distance defined along the null geodesics determined by the null vector field $k = \partial/\partial r$. The quantitites N_{ABCD}, III_{ABCD}, II_{ABCD} and I_{ABCD}, when expressed in the coordinate basis, have vanishing covariant derivatives along the null vector field k. They are of the algebraic type N, III, II and I, respectively, in the Petrov classification. The direction of propagation k is a repeated principal null direction [16, 17], of the Weyl tensor to order r^{-4}, and satisfies

$$\begin{aligned} N_{ABCD}k^D &= 0\ , \\ III_{ABC[D}k^C k_{E]} &= 0\ , \\ II_{ABC[D}k_{E]}k^B k^C &= 0\ , \\ k_{[E}I_{A]BC[D}k_{F]}k^B k^C &= 0\ . \end{aligned} \tag{18}$$

If the spacetime is such that N_{ABCD} is non-zero, then, for large values of the distance parameter, the curvature tensor has the approximate asymptotic expression $C_{ABCD} = N_{ABCD}/r$, that is, it is of Petrov type N, with the degenerate principal null direction given by k. This is the curvature tensor of a gravitational wave spacetime [1, 11, 18, 19, 20], with propagation vector k (cf. (18)). In other words, the field looks like a plane wave at large distances. The non-vanishing of the scalars N_{ABCD} is then taken as a invariant criterion for the presence of gravitational waves, and the asymptotic region (where the $0(1/r)$ term in (17) is dominant) defined as the wave zone.

Now if we compare (15) with (17), we see that the invariant condition for gravitational wave perturbations in (6) is that (cf. (14))

$$D(u,\theta) = -\varepsilon/2[y_{\theta\theta} - y_\theta \cot g \ \theta] dN/du = -\varepsilon(a_0/12M)[y_{\theta\theta} - y_\theta \cot g \ \theta]N \neq 0 \ . \quad (19)$$

From the comparison of (13) and (15) with (17), we can see that the perturbations (6) are not general, in the algebraic sense. In fact, (6) is algebraically special [24]. However, the credo in the literature is that exact radiation fields must be algebraically general. This comes from the analysis of the linearized vacuum Riemann tensor of a retarded multipole field [20, 22], the form of which is the same as (17), and the consideration that exact radiation fields are at least as complicated as their linearized approximation. For instance, in the linear approximation, a static quadrupole gives rise to terms proportional to r^{-5}, while terms going as r^{-1}, r^{-2}, r^{-3} and r^{-4} arise if the quadrupole moment varies in time [21]. Nevertheless the problem of the structure of the source of a radiation field in the full non-linear theory is still open (we should mention a tentative definition of multipole structure of the gravitational source, based upon a detailed analysis of the linearized theory [25]). From the point of view of Einstein's equations, the expression (15) may well be sustained while it is inconsistent from the point of view of the linearized theory. We assume that (15) can well represent a particular structure of the bounded source of the field, which indeed radiates even-parity multipoles, as we will show. For our purposes, these even-parity perturbations of the Schwarzschild geometry are sufficient to give us an answer about the question of how a collapsing star (or black hole), when perturbed, may loose mass by emitting a particular gravitational radiation l-pole.

4 The Integration of Field Equations and Initial Conditions

Equation (10) can be immediately integrated to

$$
\begin{aligned}
N &= N_0 \exp[(-a_0/6M)u] \ , \\
Z &= Z_0 - (6b_0 M/a_0)N_0 \exp(-a_0 u/6M) \ ,
\end{aligned}
\tag{20}
$$

where N_0 and Z_0 are integration constants. The model envisaged for the initial conditions which fix the integration constants Z_0 and N_0 is that of an initially collapsing spherical star, or unperturbed Schwarzschild black hole of mass M, which, in a given time, say $u = 0$, starts to emit gravitational radiation in the Robinson-Trautman regime, that is, in accordance to (6). In view of this, gravitational waves are emitted with *initial amplitude* (this is a provisional denomination, justified in Refs. [18, 19, 20]) N_0 and we must have $Z(u = 0) = 0$ (cf. Ref. [26]). The mass perturbation (21) is then fixed to

$$Z = (6b_0 M/a_0)[1 - N_0 \exp(-a_0 u/6M)] \ . \tag{21}$$

The gravitational waves emitted carry the information that the system was switched on in $u = 0$, through the finite discontinuity of the $O(1/r)$ components of the Riemann tensor in the first gravitational wave front $u = 0$ emitted. This discontinuity is determined by

$$[N_{ABCD}](u = 0) = -\varepsilon a_0 N_0/6M \ , \tag{22}$$

up to an angular factor (cf. (13c)). We remark that the discontinuity appearing in (6), due to the time derivative $\frac{dN}{du}(u=0)$, can be properly eliminated by a coordinate transformation, but its presence in the scalars (13) of the Weyl curvature tensor is unavoidable.

In the integration of the angular equations (11) we must distinguish two cases:
(i) $a_0 \neq 0$; the requirement of solutions regular at $p = 0$ and $p = \pi$ demands that

$$a_0 \quad = \quad : A_l = 2\,\frac{l(l+1)}{2}\left[\frac{l(l+1)}{2} - 1\right] \tag{23}$$

$$w_l \quad = \quad 2\,b_0/A_l + w_{0l}P_l(\cos\theta) \tag{24.a}$$

$$y_l \quad = \quad -b_0/A_l + \left[\frac{l(l+1)}{2A_l}\right]w_{0l}P_l(\cos\theta)\;, \tag{24.b}$$

where l is a non-negative integer and w_{0l} is an arbitrary non-zero constant [27]. Here $P_l(\cos\theta)$ is the Legendre polynomial with angular momentum l. The condition $a_0 =: A_l \neq 0$ defining case (i) implies that

$$l \geq 2\;, \tag{25}$$

that is, only quadrupole or higher-order poles gravitational radiation fields are emitted, as should be expected. Obviously outgoing gravitational waves are present in the space-time: condition (19) for holds $a_0 \neq 0$, with wave zone defined by the $O(1/r)$ non-zero components of the Weyl tensor

$$C_{0202} = -C_{0303} = -\frac{\varepsilon}{r}\,\frac{l(l+1)w_{0l}}{24M}\left(-2\cos\theta dP_l/d\theta + l(l+1)P_l\right)N_0\exp(-A_l u/6M)\;. \tag{26}$$

In sum, the general Robinson-Trautmann perturbations h_{ab} for this case can be split into

$$h_{ab} = h^{(1)}_{ab} + h^{(2)}_{ab} \tag{27}$$

where

$$h^{(1)}_{ab} = \begin{pmatrix} A_l/(l(l+1)) - (A_l/6M)r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}\cdot \tag{28}$$

$$\cdot\,\frac{l(l+1)}{A_l}w_{0l}P_l(\cos\theta)N_{0l}\exp(-A_l u/6M)\;,$$

and

$$h^{(2)}_{ab} = \frac{Z(u)}{r} + \left(\frac{2}{A_l} + \frac{r}{3M}\right)b_0 N_{0l}\exp[(-A_l/6M)u])\delta_a^0\delta_b^0 +$$

$$- \frac{2r^2}{A_l}b_0 N_{0l}\exp[(-A_l/6M)u](\delta_a^2\delta_a^2 + \sin^2\theta\delta_a^3\delta_b^3)\;, \tag{29}$$

with $Z(u)$ given by (21). The perturbations (28) are already in the canonical form for even parity perturbations, in the terminology of Regge and Wheeler [2]. Under the inversion

operation $\theta \rightarrow \pi - \theta$, they transform as $h_{ab}^{(1)} \rightarrow (-1)^l h_{ab}^{(1)}$. Perturbations (29) are denoted "mass perturbations". Although dependent on the temporal gauge, they will cause an effective decrease of the mass of the system. This analysis will be done in Section 5, where an *invariant* mass perturbation function (due to outgoing gravitational waves) is defined. The final configuration will be a Schwarzschild geometry with mass smaller than the original mass, the effective decrease of mass being dependent on the particular pole emitted.

The non-radiatable modes correspond to the case

(ii) $a_0 = 0$: as expected, no gravitational radiation is present (N = const, $C_{0202} = -C_{0303} = 0$, cf. (19) and (20)). Solutions of equations (11) for $a_0 = 0$ and $b_0 \neq 0$ are, in general, singular at $p = 0$ and and $p = \pi$. We select the particular set

$$
\begin{aligned}
w &= w_0 + 2b_0 \ ln(1 + \cos \theta) \ , \\
y &= -(w_0 + b_0)/2 - b_0 \ ln(1 + \cos \theta) + y_0 P_1(\cos \theta) \ ,
\end{aligned}
\tag{30.a}
$$

which are regular at $\theta = 0$, and

$$
\begin{aligned}
w &= w_0 + 2b_0 \ ln(1 - \cos \theta) \ , \\
y &= (w_0 + b_0)/2 - b_0 \ ln(1 - \cos \theta) + y_0 P_1(\cos \theta),
\end{aligned}
\tag{30.b}
$$

which are regular at $\theta = \pi$. We remark that the parcel $y_0 P_1(\cos \theta)$ appearing in both Eqs. (30) can be eliminated by a convenient coordinate transformation, which asymptotically is interpreted as an infinitesimal Lorentz boost with velocity parameter εy_0. Solutions (30) coincide at $p = \pi/2$. We therefore define the continuous solution in $[0, \pi]$ as the union of (30.a) in $[0, \pi/2]$, and (30.b) in $[\pi/2, \ \pi]$. Although continuous in $p = \pi/2$, its first derivative has a finite discontinuity, defining physically a shell of matter at the equatorial plane. This shell can be modelled by neutrinos and strings being emitted radially in the $1 + 2$ spacetime of the shell. This configuration was examined in Ref. [28], and will be referred to here as the shell emission mode for $b_0 \neq 0$. The parameter b_0 is proportional to the neutrino flux emitted on the shell; this, in fact, gives us a direct mechanism for the measurement of b_0. The mass function pertubation of this mode is

$$
Z(u) = b_0 N_0 u + \text{const.}
\tag{31}
$$

We interpret b_0 as the mass emissivity parameter of the collapsing configuration, either in gravitational wave emission modes $l \geq 2$, or in a shell emission mode $a_0 = 0$. This parameter appears to be characteristic of mass variation in the Schwarzschild background, and we conjecture that its existence is independent of the particular perturbations considered. In the next Section, this parameter will allow us to define a mass emissivity function associated to a l-pole. Physical considerations implies obviously $b_0 \geq 0$.

We must now comment on the superposition of the modes $l \geq 2$, and also of the shell emission mode with a $l \geq 2$ mode. It is easy to verify that the solutions

$$
W = \sum_l C_l W_l \quad , \quad Y = \sum_l C_l Y_l \quad , \quad Z = \sum_l C_l Z_l \ ,
$$

where W_l, Y_l and Z_l are given by

$$\begin{pmatrix} W_l \\ Y_l \end{pmatrix} = \begin{pmatrix} 2b_0/A_l + w_{0l}P_l(\cos\theta) \\ -b_0/A_l + [l(l+1)/2A_l]w_{0l}P_l(\cos\theta) \end{pmatrix} \cdot N_0 \exp(-A_l u/6M) , \quad (32)$$

$$Z_l = (6b_0 M/A_l)N_{0l}[1 - \exp(-A_l u/6M)] ,$$

satisfy the field equations (7) and (8). Note that necessarily the same coeficients C_l appear in the three linear combinations. Also a linear combination of a set solutions (32) with a solution for the shell emission mode satisfy (7) and (8).

5   The Temporal Gauge Group, the Invariant Mass Function and The Mass Emissity

The Robinson-Trautmann metrics described by (1), (2), as well the field equations (3) and (4), are invariant under a subgroup of coordinate transformations $(u,r,\theta,\phi) \rightarrow (\bar{u},\bar{r},\theta,\phi)$ described by

$$\bar{r} = rF ,$$
$$d\bar{u} = du/F , \qquad (33.a)$$

where F is an arbitrary function of u. Under (33.a), the quantities appearing in the metric (1)–(2) transform as

$$\bar{L} = LF^2 ,$$
$$\bar{B} = BF^3 , \qquad (33.b)$$
$$\bar{K} = K/F ,$$

In other words, under (33) it is enough to replace, into Equations (1) to (4), unbarred coordinates and variables by the corresponding barred ones. We remark the generality of (33), because they are general transformations leaving the Weyl scalars (13) invariant.

From (33.b) we can see that the time dependence of the mass function $B(u)$ is not an invariant, but we define the invariant mass aspect or mass function [29]

$$m(u,\theta) = -BK^3/2 \qquad (34)$$

which, for our purposes, will provide an useful invariant characterization of the mass variation due to the emission of gravitational waves. It is interesting to note how the Weyl curvature scalars (13.a), associated to the Newtonian component of the field (cf. (16)), is expressed in terms of this invariant function. Indeed,

$$C_{2323} = -C_{0101} = 2C_{0212} = \frac{BK^3}{(rK)^3} ,$$

justifying our characterization of BK^3 as invariant mass function. We also note that these are the sole Weyl scalars where the mass variation function is present. For the case of the

Schwarzschild spacetime, it follows immediately that the invariant definition (34) yields exactly the constant mass parameter M. This can be checked either for the expression (1) with (5), or for the expression of the Schwarzschild geometry given by (6), with (20), (21) and (24), and $w_{0l} = 0$ (cf. also Ref. [27]).

From (6), (9), (20), (21) and (24.b) we obtain

$$m(u, \theta) = M - \frac{3\varepsilon M N_{0l}}{A_l} \left\{ b_0 - \frac{l(l+1)}{2} \; w_{0l} P_l(\cos \theta) \exp[(-A_l/6M)u] \right\} \; . \qquad (35)$$

This invariant mass function (35) will coincide with the gauge-dependent one given from (21), for $u \to \infty$, resulting in the constant value

$$m = M - 3\varepsilon b_0 M N_{0l}/A_l \; . \qquad (36)$$

Therefore (36) can be properly interpreted as the invariant mass of the final configuration. Hence, in the limit $u \to \infty$, the geometry will be the one of a Schwarzschild black hole with invariant mass given by (36), smaller than the mass M of the original configuration ($u < 0$). We note that (35) can be interpreted as mass only for some specific limits, for instance $u \to \infty$, when no gravitational waves are present. The total amount of mass extracted by each gravitational wave pole emitted is given by

$$\Delta m = 3\varepsilon b_0 M N_{0l}/A_l = \frac{3\varepsilon b_0 M N_{0l}}{2[l(l+1)/2(l(l+1)/2 - 1)]} \; . \qquad (37)$$

The $l = 2$ quadrupole emission appears then as being the most effective mode in the mechanism of extracting mass by emission of gravitational waves. From (37) we are led to define the mass emissivity factor e_l as the mass fraction extracted by a l-pole gravitational wave of unit amplitude, that is,

$$e_l =: \Delta m / M N_{0l} = \frac{3\varepsilon b_0}{2[l(l+1)/2(l(l+1)/2 - 1)]} \; . \qquad (38)$$

We conjecture that the l-pole dependence given in (38) is a characteristic of the Schwarzschild black hole, and independent of the perturbations which gave rise to the gravitational waves emitted.

The following remarks are pertinent here. As we have seen, in our perturbative scheme for the Schwarzschild geometry, a temporal gauge infinitesimal transformation can always be performed such that $\bar{Z} = 0$. From this it could be incorrectly inferred that the results (36)-(37) on mass decrease may be eliminated, because the parameter b_0 would then not appear in the solutions in the new gauge. This is not the case, since the perturbations in the new gauge are

$$\bar{W} = W + \frac{1}{3M} Z$$
$$\bar{Y} = Y - \frac{1}{6M} Z$$
$$\bar{Z} = 0$$

where Z is given in (21). Therefore b_0 is still present in the solutions and in the limit $\bar{u} \to \infty$ when no gravitational waves are present, the usual Schwarzschild mass definition yields (36). Analogously we could have started the integration directly in a temporal gauge where $Z = 0$. Eqs. (7) and (8), and the separation Ansatz $Y = y(\theta)N(u) + L(u)$, $W = w(\theta)N(u) - 2L(u)$ yields the same result (36) in the limit $u \to \infty$.

We must finally comment the existence of solutions with $b_0 = 0$ and $a_0 \neq 0$, corresponding to outgoing gravitational waves which do not extract mass from the configuration. No gravitational wave experiments will distinguish the cases $b_0 = 0$ and $b_0 > 0$. The only direct experiment to measure b_0 is the detection of the shell emission mode; in this case, b_0 is proportional to the flux of neutrinos emitted radially on the shell. If we adhere to the general formulation of Bondi, Van der Burg and Metzner [29] for the emission of gravitational waves by an axially symmetric bounded source, we should then discard the $b_0 = 0$ solutions as being physically not meaningful. Their presence is possibly due to the simple form we have assumed to the Robinson-Trautmann perturbations. We intend to return to this point in the future.

6   Conclusions and Final Remarks

In this paper we discussed gravitational wave perturbations of a static black hole or a spherical collapsing star, the exterior spacetime of which is described by the Schwarzschild metric, by using the so-called Robinson-Trautman perturbations. This class of radiative type perturbations belongs to the family of Robinson-Trautman metrics, which have the Schwarzschild geometry as a particular limit. Einstein's field equations are integrated completely. We obtain that Robinson-Trautman perturbations are constitued of two distinct pieces, one of them being a particular set of even-parity perturbations (in the terminology of the formalism of Regge-Wheeler). Although the second piece is coordinate gauge dependent, it allows us to obtain information on how mass is extracted from the configuration by the emission of gravitational waves. An invariant mass function is defined, and it results that gravitational waves may extract mass from the collapsing configuration, and the amount of mass extracted depends on the particular gravitational wave l-poles emitted. We are able to define a mass emissivity function

$$e_l = \frac{3b_0}{\{l(l+1)[l(l+1)/2 - 1]\}} \; ,$$

which measures the ratio of mass taken out from the source by a l-pole. Here b_0 is a small positive parameter, of the order of the perturbations and characteristic of the mechanism of emission. The quadrupole emission mode ($l = 2$) is the dominant mechanism in extracting mass from the configuration. Robinson-Trautman perturbations also includes a mechanism of extracting mass from the collapsing configuration, denoted shell emission mode, where no gravitational waves are present; this mode corresponds to neutrinos and strings emitted radially on a shell of mass at the equatorial plane of the configuration. Although this mode might be superposed to gravitational wave models ($l \geq 2$), we could have used it as a mechanism to perturb the configuration, previous to the emission of gravitational waves. In the Robinson-Trautman regime, the presence of gravitational waves,

and of a gravitational wave zone are well characterized by the analysis of the asymptotic structure of the Weyl curvature tensor. From the result of the integration of Einstein's equations, namely Eqs. (11), and from the condition $a_0 \neq 0$ necessary for the presence of gravitational waves and a gravitational wave zone (cf. Eq. (19)), only, it follows that the lowest gravitational wave pole emitted is the quadrupole ($l = 2$). This result was expected from classical radiation theory for a spin-2 field and from Einstein's linearized theory for gravitational waves.

Extraction of mass appears to be an essential characteristic of gravitational wave emission and we conjecture that the l-dependence expressed in Eq. (38) is a characteristic of the Schwarzschild exterior geometry only, and does not dependent on the class of perturbations which gave rise to the gravitational waves emitted.

The time behaviour of a l-pole gravitational wave has the form

$$\exp[-(u/3M)l(l + 1)/2(l(l + 1)/2 - 1] \ ,$$

where M is the initial mass of the collapsing configuration. It follows that, for a Galilean observer at infinity, the horizon of the final black hole configuration takes an infinite time to form.

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[26] In this scenario we do note take into account that the perturbation of the collapsing star or of a static black hole does actually occur in a finite period of time: indeed, for $u < u_0 < 0$, we have the Schwarzschild geometry; in the interval $u_0 < u < 0$, the black hole or star is perturbed so that in $u = 0$ the configuration starts to emit gravitational waves. Whatever the details of the geometry and of the mechanism of perturbation in the interval $u_0 < u < 0$, the fact that the configuration begins to emit gravitational waves in $u = 0$, and goes on emitting according to (6), necessarily implies that the curvature tensor will have a finite discontinuity in the first emitted gravitational wave front $u = 0$. This is basically independent of the perturbating mechanisms in the interval $u_0 < u < 0$, and legitimates the limit $u_0 \to 0$, provided that the total mass M in $u = 0$ eventually includes the contribution of the perturbing mechanism in that interval.

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