New Solutions for Evolution of the Scale Factor with a Variable Cosmological Term

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The cosmological term is assumed to be a function of time such as $\Lambda = Ba^{-2}$ where $a(t)$ means the scale factor of standard cosmology. Analytical solutions for radiation dominated epoch and open universe are found. For closed universe, $k = +1$, and for flat universe, $k = 0$, we show a numerical solution. In general the scenario of Big Bang is preserved in our case for the Friedmann-Robertson-Walker cosmology.

Key-words: Gravitation; Cosmological term.

Introduction

In the last few years several paper [1, 5] have appeared discussing the cosmological constant problem when that term is considered as part of the content of the universe. In the original Einstein's proposal the cosmological term was exactly a constant introduced by hand in Einstein Hilbert lagrangean as a kind of force opposite to that of gravitation. In the Einstein's equation the cosmological term appears on the left side and its meaning is only geometrical, as must be in the general relativity. However changing the cosmological constant to the right side in Einstein's equations the situation is really quite different. Now there is a possibility that the cosmological term may be time dependent.

In the new context the cosmological term could be a function of time with large possibilities for evolution in time to cosmological term [1].

The central equation which describes that evolution for scale factor with cosmological term whether or not it is a function of time is shown here (7) and elsewhere [1, 2].

It is possible to find solutions for special cases. In general one can solve the equation (7) after successive variable transformations on scale factor variable until the original equation (non linear differential equation) is written as an ordinary linear differential equation. However the solutions that are found for almost flat space time, $k = 0$ [1] or under particular hypothesis and numerically are approximate solutions.

Here we present some new analytical solutions for the general case, $k = 0$, and $k = \pm 1$. We obtain a direct solution of the equation without variable transformations for radiation epoch and a numerical solution for a perfect fluid.

Differently from [1] we find solutions for flat, open and closed universes and the big bang scenario is preserved [2]. We consider the usual assumption for a Friedman-Robertson-Walker universe like a homogeneous and isotropic universe.

Such a universe is seen as a perfect fluid with a pressure P and energy density ρ.

The Einstein's equations are given by

$$G_{\mu\nu} = 8\pi G \tilde{T}_{\mu\nu} \qquad (1)$$

where $\tilde{T}^{\mu\nu}$ is the energy-momentum tensor written as

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{\Lambda}{8\pi G}\, g_{\mu\nu} \qquad (2)$$

It is clear that Λ is a part of the matter content of the universe in our case. On the right side, $T_{\mu\nu}$ is the usual

energy momentum tensor for perfect fluid and G is the gravitational constant.

The effective energy-momentum tensor describes a perfect fluid and thus give an effective pressure

$$\tilde{p} = p - \frac{\Lambda}{8\pi G} \tag{3}$$

and an effective density as

$$\tilde{\rho} = \rho + \frac{\Lambda}{8\pi G} \tag{4}$$

Then $\tilde{T}_{\mu\nu}$ satisfies energy momentum conservartion [3, 4]

$$\nabla^{\nu}\tilde{T}_{\mu\nu} = 0 \tag{5}$$

The equation of state is written as

$$P = (\gamma - 1)\rho \tag{6}$$

where γ is a constant.

Now, considering the initial hypothesis, eq. (1) and (2), the Robertson-Walker line element and the fact that the cosmological term here is not a constant but a function of time, one can show [1] that the equation which governs the behavior of the scale factor in the presence of a cosmological term, Λ, a constant like in the original Einstein model is given by.

$$\frac{\ddot{a}}{a} = \left(1 - \frac{3\gamma}{2}\right)\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) + \frac{\gamma}{2}\Lambda \tag{7}$$

There are numerous possibilites to choose the evolution low for the cosmological term [1, 4]. In accordance with [2] we choose $\Lambda = \Lambda(a)$, where "a" means the scale factor for Friedmann-Robertson-Walker universe. Thus the cosmological term is written as

$$\Lambda(a) = Ba^{-2} \tag{8}$$

where B is a pure number of the order 1 and "a" is a function of time.

By combining eq. (7) and (8) we get

$$a\frac{d^2a}{dt^2} + \alpha\left(\frac{da}{dt}\right)^2 + \lambda = 0 \tag{9}$$

where

$$\alpha = -\left(1 - \frac{3\gamma}{2}\right) \quad \text{and}$$

$$\lambda = -\left(1 - \frac{3\gamma}{2}\right)k + \frac{\gamma B}{2} \tag{10}$$

There are three free parameters γ, k and B to be fixed. The B parameter may be fixed to be 1 based on dimensional analysis arguments [2].

The parameter k, the three spatial curvature assumes the values $+1$, 0 and -1 for closed, flat and open universe respectively.

Some solutions for equation (9)

The dynamic equation for the scale factor $a(t)$ have solutions for particular values of α.

On taking $\alpha = -1$ the following solutions is found.

$$a(t) = \frac{\sqrt{\lambda}\,sinh[c(t - t_0)]}{c}$$

$$\tag{11}$$

$$a(t) = \frac{\sqrt{\lambda}\,cosh[c(t - t_0)]}{c}$$

where c is a constant.

One fixing α it is easy to verify that $\gamma = 0$, and because of eq. (6) and equation for λ we obtain

$$p = -\rho \tag{12}$$

and

$$\lambda = -k \ . \tag{13}$$

On taking now $\alpha = -1/2$ the solution is written as

$$a(t) = -\frac{\lambda}{2c} + c(t - t_0)^2 \tag{14}$$

and c is a constant.

In this case we find $\gamma = 1/3$. The state equation is givne now by

$$p = -\frac{2}{3}\rho \tag{15}$$

and

$$\lambda = \frac{-3k + 1}{6} \tag{16}$$

Next for $\alpha = -2$ the solutions are given by Jacoby elliptic functions

$$a(t) = \sqrt{\frac{\lambda}{2}}\,\frac{S_n\left[c(t - t_0),\,\frac{1}{2}\right]d_n\left[c(t - t_0),\,\frac{1}{2}\right]}{c\,c_n\left[c(t - t_0),\,\frac{1}{2}\right]} \ ,$$

$$a(t) = \sqrt{-2\lambda}\,\frac{d_s\left[c_1(t - t_0),\frac{1}{2}\right]}{c_1} \tag{17}$$

and

$$a(t) = \frac{\sqrt{-\lambda}}{c_1\left[c_n(t - t_0),\,\frac{1}{2}\right]}$$

where c, c_1 are constants and S_n, d_n c_n and d_s are two argument elliptic functions.

In this case we find that $\gamma = -2/3$ and again the equation of state and λ are written as

$$P = -\frac{5}{3} \rho , \qquad (18)$$

$$\lambda = -\frac{1}{3} (6k + 1) . \qquad (19)$$

Consider now the case $\alpha = 1$. The solution is found to be

$$a(t) = \pm\sqrt{c - \lambda(t - t_0)^2} \qquad (20)$$

where c is a constant.

For this case we have $\gamma = 4/3$ and the equation for pressure p and λ parameter are shown respectively as

$$p = \frac{1}{3} \rho , \qquad (21)$$

$$\lambda = k + \frac{2}{3} . \qquad (22)$$

The following gives the case for $\alpha = -3/2$. The solution is a three argument Weierstrass function written as.

$$a(t) = \rho\left(c_1(t - t_0) , \quad 0 \quad , \quad -\frac{2}{3}\frac{\lambda}{c_1^2}\right) \qquad (23)$$

where c_1 is a constant. The same way using eq. (10) and eq. (6) we can find immediately

$$p = \left(-\frac{4}{3}\right) \rho , \qquad (24)$$

$$\lambda = -\left(2k + \frac{1}{6}\right) . \qquad (25)$$

Finally, we shall consider the case $\alpha = 1/2$. In this case it is possible to find an analytical solution for the open universe $k = -1$. Using again eq. (6) and eq. (10) we get

$$p = 0 , \qquad (26)$$

$$\lambda = \frac{1}{2} (k + 1) . \qquad (27)$$

The solution is shown as

$$a(t) = a_0 \left(1 + \frac{3}{2}\frac{v_0}{a_0} (t - t_0)\right)^{2/3} \qquad (28)$$

with a_0 being a constant when we take $a(t)$ at $t = t_0$ and v_0 being another constant when $\dot{a}(t)$ is considered at $t = t_0$. The *dot* means time derivative.

It's easy to see that for an open universe there is an origen in time and the big bang scenario is preserved. We find the singularity $a(t) = 0$ as time is expressed as

$$t = t_0 - \frac{2}{3}\frac{a_0}{V_0} \qquad (29)$$

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[1] see the Plot for $\alpha = 1/2$ and $k = 0, +1$ at the end of the paper

For the cases $k = 0$ and $k = +1$ we do not find analytical solutions but we show a numerical solution[1]. It is clear that all possibilities have an origin in time exactly the same way as the standard cosmology where the cosmological term is a constant.

The Hubble constant may be found as usual i.e.

$$H = \frac{\dot{a}}{a} = H_0 \left(\frac{da}{ad\tau}\right) \qquad (30)$$

where H_0 is the present value of Hubble parameter and $\tau = H_0 t$ is the measure of time in units of Hubble time [1].

The analysis of the universe's age follows as usual [2].

Conclusions and Comments

We have found direct solutions of a non linear differential equation which describes the dynamics for scale factor when the cosmological term is a function of time. In particular the solution with $\alpha = 1$ describing the radiation dominated epoch and $\alpha = 1/2$ describing a perfect fluid for open universe both preserve the scenario of the standard cosmology. The possibility that cosmological term is a time varying function does not change the usual picture of the Friedmann cosmology when we choose the evolution law eq. (8), except for a different choice for evolution of Λ the final evolution of the universe can be different from the usual model.

The solutions for $\alpha = -1, -1/2, -2, -3/2$ all have an appropriete equation of state for vacuum energy for any value of k but these may not be physical.

Finally the case $\alpha = 1/2$ for $k = 0, +1$ is plotted in figure I. Clearly we have the big bang scenario for both cases. Only with our initial hypothesis, the equation of state, eq. (6), and the dynamical equation for Λ, the complete behaviour of our model is similar to the Friedmann cosmology. The analysis of the age of the universe follows the same steps as in [2]. The analyse of this problem with $\Lambda = Ba^m$ for $m \neq 2$ is presently being considered for the same initial hypothesis and the equation of state will be obtained.

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