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EXACT SOLUTIONS IN BRANS-DICKE THEORY: A DYNAMICAL SYSTEM APPROACH

by

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ABSTRACT

A method is furnished for constructing isotropic and homogeneous solutions in Brans-Dicke theory based on the analysis of the dynamical system formed by the field equations. Large classes of solutions found in the literature are recovered and shown to be special cases of those generated by this method.

Key-words: Brans-Dicke theory; Dynamical system; Cosmological solutions.

Introduction

Dynamical system theory has been applied with great success in cosmology and astrophysics within the context of general relativity[1].

Recently, its range of applicability has been enlarged by considering alternative theories of gravity, such as those which do not obey the principle of minimal coupling[2], Kaluza--Klein[3] and scalar-tensor theories[4].

In this paper we are concerned with Brans-Dicke theory[5], and look for cosmological solutions presenting homogeneous, isotropic and spatially flat geometry having a perfect fluid as source.

The starting point of our work is to reduce the field equations to a planar autonomous dynamical system (the phase space of this system is given in ref.[6]). From the knowledge of some mathematical features of the system (without having to solve any differential equation) we have developed a simple method for generating exact solutions of Brans-Dicke equations. Some of these solutions are valid for fluids satisfying a rather general equation of state and for arbitrary values of w. The restrictions which should be imposed on the solutions if Brans-Dicke theory is to be reduced to Einstein's general relativity in the limit $w \to \infty$ and $\phi = G^{-1}$ are discussed at the end of the article.

1. Brans-Dicke field equations

The field equations in Brans-Dicke theory are given[7] by

$$R_{\mu\nu} = -\frac{8\pi}{\phi}\left[T_{\mu\nu} - \left(\frac{\omega+1}{2\omega+3}\right)T g_{\mu\nu}\right] - \frac{\omega}{\phi^2}\phi_{;\mu}\phi_{;\nu} - \frac{1}{\phi}\phi_{;\mu;\nu} \qquad (1.a)$$

$$\Box\phi = \frac{8\pi}{2\omega+3}\,T\,, \qquad (1.b)$$

where ϕ is the scalar field, $T_{\mu\nu}$ is the energy-momentum tensor and T its trace.

For perfect fluids $T_{\mu\nu} = (\rho+p)V_\mu V_\nu - p\,g_{\mu\nu}$, ρ denoting the energy density, p the pressure and V^μ is the 4-velocity of the fluid. Isotropic, homogeneous and spatially flat geometries may be described by the Friedmann-Robertson-Walker line element written in the standard form

$$ds^2 = dt^2 - R^2(t)[d\chi^2 + \chi^2(d\theta^2 + \mathrm{sen}^2\theta\, d\phi^2)]. \qquad (2)$$

Naturally, homogeneity and isotropy of spacetime also implies $\phi = \phi(t)$. Choosing a co-moving coordinate system and adopting as the equation of state $p = \lambda\rho\,(0 \le \lambda \le 1)$, the equations (1.a) and (1.b) become:

$$\dot\theta + \frac{\theta^2}{3} = -\frac{8\pi\rho}{\phi}\left[1 - \frac{\omega+1}{2\omega+3}(1-3\lambda)\right] - \omega\frac{\dot\phi^2}{\phi^2} - \frac{\ddot\phi}{\phi}\,, \qquad (3.a)$$

$$\frac{\dot\theta}{3} + \frac{\theta^2}{3} = -\frac{8\pi\rho}{\phi}\left[\lambda + \frac{\omega+1}{2\omega+3}(1-3\lambda)\right] - \frac{\theta}{3}\frac{\dot\phi}{\phi}\,, \qquad (3.b)$$

$$\ddot\phi + \theta\dot\phi = \frac{8\pi\rho}{2\omega+3}[1-3\lambda]\,. \qquad (3.c)$$

where $\theta = 3\dot R/R$ is the expansion factor.

2. The dynamical system

Eliminating ρ from equation (3.c), we arrive at a pair of equations which give rise to an autonomous dynamical system in two dimensions:

$$\dot{\theta} = F_{\lambda\omega}(\theta,\psi) \quad , \qquad (4.a)$$

$$\dot{\psi} = H_{\lambda\omega}(\theta,\psi) \quad , \qquad (4.b)$$

where we have defined the new variable $\psi = \frac{\dot{\Phi}}{\phi}$, and

$$F_{\lambda\omega}(\theta,\psi) = \frac{1}{2\omega+3}\left[-(2+\omega+\lambda\omega)\theta^2 + \frac{3\omega}{2}(\lambda\omega-\omega-1)\psi^2 + \omega(1-3\lambda)\theta\psi\right]$$

$$H_{\lambda\omega}(\theta,\psi) = \frac{1}{2\omega+3}\left[\left(\frac{1-3\lambda}{3}\right)\theta^2 - \left(\frac{6+5\omega-3\lambda\omega}{2}\right)\psi^2 - (2\omega+2+3\lambda)\theta\psi\right].$$

As it has been pointed out previously, the phase portraits of the system (4) were analysed in ref.[6]. To carry out a general qualitative analysis of dynamical systems of this type, i.e., homogeneous in the variables (θ,ψ) it is useful to work with the polar coordinates r and α ($\theta = r\cos\alpha$, $\psi = r\,\mathrm{sen}\,\alpha$), defined in the phase plane. In these new variables (4) is transformed into

$$\dot{r} = r^2 Z_{\lambda\omega}(\alpha) \quad , \qquad (5.a)$$

$$\dot{\alpha} = r N_{\lambda\omega}(\alpha) \quad , \qquad (5.b)$$

where $N_{\lambda\omega}(\alpha) = H_{\lambda\omega}(\cos\alpha,\mathrm{sen}\,\alpha)\cos\alpha - F_{\lambda\omega}(\cos\alpha,\mathrm{sen}\,\alpha)\,\mathrm{sen}\,\alpha$ and $Z_{\lambda\omega}(\alpha) = H_{\lambda\omega}(\cos\alpha,\mathrm{sen}\,\alpha)\,\mathrm{sen}\,\alpha + F_{\lambda\omega}(\cos\alpha,\mathrm{sen}\,\alpha)\cos\alpha$.

The roots of the equation

$$N_{\lambda\omega}(\alpha) = 0 \qquad (6)$$

are referred to as the invariant rays of the phase plane and consist of straight lines $\alpha = \text{const.}$, passing through the origin of the plane $(\theta = 0,\ \psi = 0)$. It turns out that the invariant rays are solutions of the dynamical system (see, for example, ref. [8]). Therefore, the knowledge of the invariant rays automatically lead us to find out classes of solutions of the system without needing to solve analytically any differential equation. Thus, let us apply these ideas to our case. Here, the function $N_{\lambda\omega}(\alpha)$ will be given by

$$N_{\lambda\omega}(\alpha) = \frac{1}{2\omega+3}\Big[\Big(\frac{1-3\lambda}{3}\Big)\cos^3\alpha + (\lambda\omega - 3\lambda - \omega)\cos^2\alpha\ \text{sen}\,\alpha + \Big(\frac{9\lambda\omega - 7\omega - 6}{2}\Big)\cos\alpha\ \text{sen}^2\alpha + \frac{3\omega}{2}(1+\omega-\lambda\omega)\,\text{sen}^2\alpha\Big] \quad . \tag{7}$$

If $\lambda = \frac{1}{3}$ (radiation case), then we see that $\text{sen}\,\alpha = 0$ is a root of eq.[6]. Conversely, if $\text{sen}\,\alpha = 0$ is a root of [6], then $\lambda = \frac{1}{3}$.

Initially we take $\lambda \neq \frac{1}{3}$ and assume also $w \neq -\frac{3}{2}$. Thus, to solve equation [6] is equivalent to solving

$$\Big(\frac{1-3\lambda}{3}\Big)\xi^3 + (\lambda\omega - 3\lambda - \omega)\xi^2 + \Big(\frac{9\lambda\omega - 7\omega - 6}{2}\Big)\xi + \frac{3\omega}{2}(1+\omega-\lambda\omega) = 0 \tag{8}$$

where we have put $\xi = \theta/\psi = \text{cotg}\,\alpha$.

It is possible to factorize eq. [8] in the following way:

$$\Big[\Big(\frac{1-3\lambda}{3}\Big)\xi + \lambda\omega - \omega - 1\Big]\Big[\xi^2 + 3\xi - \frac{3\omega}{2}\Big] = 0. \tag{9}$$

Thus, if $w < -3/2$, only one real root exists, namely,

$$\xi_3 = \frac{3(1+\omega-\lambda\omega)}{1-3\lambda} \quad . \tag{10}$$

On the other hand, when w > - 3/2 we must include the two real roots:

$$\xi_1 = -\frac{3}{2}\left(1 + \sqrt{1 + \frac{2\omega}{3}}\right) \tag{11}$$

and

$$\xi_2 = -\frac{3}{2}\left(1 - \sqrt{1 + \frac{2\omega}{3}}\right) \tag{12}$$

3. Generating solutions from the invariant rays

The generate solutions from the invariant rays we must recall that, as it has been pointed out before, the invariant rays themselves are integral curves of the phase plane, i.e., solutions of the dynamical system. This clearly provides a means for obtaining solutions of Brans-Dicke equations, as we shall illustrate with the following examples.

Consider, for instance, the well known Brans-Dicke solution[9] corresponding to a pressureless fluid and to zero spatial curvature:

$$\phi(t) = \phi_o t^{\frac{2}{3\omega+4}} , \tag{13.a}$$

$$R(t) = R_o t^{\frac{2\omega+2}{3\omega+4}} , \tag{13.b}$$

where ϕ_o and R_o are constants. After evaluating θ and ψ in this case, we get

$$\theta = 3\frac{\dot{R}}{R} = \frac{3(2\omega+2)}{3\omega+4}\,\frac{1}{t} , \tag{14.a}$$

$$\psi = \frac{\dot{\phi}}{\phi} = \frac{2}{3\omega+4}\,\frac{1}{t} . \tag{14.b}$$

But, noticing that $\frac{\theta}{\psi} = \text{const.} = 3(1+w)$, we conclude that (14.a) and (14.b) refer necessarily to a solution of the dynamical system (4) lying just on one of the invariant rays. In fact, settint $\lambda = 0$ lead us to identify this invariant ray with $\alpha_3 = \text{arc cotg}\ \xi_3$ (see eq. (10)).

Another example is provided by Nariai's solutions (10, 11,12), which constitute a class of solutions valid for spatially flat Friedmann-Robertson-Walker metrics and perfect fluids with equation of state $p = \lambda\rho$:

$$R(t) = R_o t^{p_1} \quad , \tag{15.a}$$

$$\phi(t) = \phi_o t^{p_2}$$

with $p_1 = \frac{2+2\omega(1-\lambda)}{4+3\omega(1-\lambda^2)}$ and $p_2 = \frac{2(1-3\lambda)}{4+3\omega(1-\lambda^2)}$.

Here, again, a simple calculation of θ and ψ immediately shows that (15.a) e (15.b) define curves located on the invariant ray α_3.

A brief analysis of these two examples suggests us to do an almost obvious generalization. Let us consider the following class of solutions lying on the invariant ray α_3:

$$R(t) = R_o t^{\frac{1+\omega-\lambda\omega}{f(\lambda,\omega)}} \tag{16.a}$$

$$\phi(t) = \phi_o t^{\frac{1-3\lambda}{f(\lambda,\omega)}}$$

where $f(\lambda,\omega)$ is an arbitrary function of λ and ω. Each solution of the above class is a solution of the dynamical system (4) and satisfies Brans-Dicke equations. In the preceeding example, Nariai's solutions correpond to the particular choice $f(\lambda,\omega) =$

$= 2 + \frac{3\omega}{2}(1 - \lambda^2)$.

Analogously, equations (11) and (12) suggest further generalizations:

$$R(t) = R_o t^{\frac{1 \pm \sqrt{1 + \frac{2\omega}{3}}}{g(\omega)}} , \qquad (17.a)$$

$$\phi(t) = \phi_o t^{-\frac{2}{g(\omega)}} , \qquad (17.b)$$

with $g(\omega)$ arbitrary.

3. Vacuum solutions and the energy density equation

From equations (3.a) e (3.b) we can deduce the following expression for $\rho(t)$:

$$\frac{8\pi\rho}{\phi} = \frac{\theta^2}{3} - \frac{\omega}{2}\psi^2 + \theta\psi \quad . \qquad (18)$$

Thus, for $\phi(t)$ and $R(t)$ as given by the class of solutions (16.a) and (16.b) we have

$$\frac{8\pi\rho}{\phi} = \frac{1}{f^2(\lambda,\omega)t^2}\left[3(1+\omega-\lambda\omega)^2 - \frac{\omega}{2}(1-3\lambda)^2 + (1-3\lambda)\,.\right.$$
$$\left.\cdot\,(1+\omega-\lambda\omega)\right] \qquad (19)$$

For the classes of solutions given by equations (17.a) and (17.b), lying respectively on the invariant rays $\alpha_1 = \text{arc cotg}\ \xi_1$ and $\alpha_2 = \text{arc cotg}\ \xi_2$, we conclude from eq. (18) that they represent vacuum solutions ($\rho = 0$). In reality, these classes contain as particular cases the vacuum solutions first obtained by O'Hanlon and Tupper[13] given by

$$R(t) = R_o t^q \quad , \tag{20.a}$$

$$\phi(t) = \phi_o t^r \quad , \tag{20.b}$$

where $\frac{1}{r} = -\frac{1}{2}[1 \pm \sqrt{3(2\omega+3)}$ and $q = \frac{1}{3}(1-r)$.
Also, it is interesting to notice that the special solution $\omega = -4/3$ found by these authors, which corresponds to de Sitter model in Brans-Dicke theory without cosmological constant

$$R(t) = R_o \exp[t] \quad , \tag{21.a}$$

$$\phi(t) = \phi_o \exp[-3t] \quad , \tag{21.b}$$

lies exactly on the invariant ray α_2. Indeed, if $\omega = -4/3$ then $\xi_2 = -1$, which means that $\theta = -\psi$. Now, if we look at equations (4.a) and (4.b) we see that for $\omega = -4/3$ and $\theta = -\psi$, we have

$$\dot{\theta} = \dot{\psi} = 0.$$

The phase plane (θ,ψ) has a line $(\theta = -\psi)$ of multiple equilibrium points (see ref.[14]), each one representing the de Sitter-type solutions of eq. (21.a) and (21.b).

5. The general relativity limit

Brans-Dicke's theory of gravity is formulated in such a way as to reproduce Einstein equations in the limit when ω goes to infinity and $\phi = G^{-1} = \text{const.}$. This arises the question: do Brans--Dicke solutions tend to Einstein solutions when the same limit is required? The answer to this question is no, as it will be shown in this section.

Friedmann models with spatially flat metric and perfect fluid with equation of state $p = \lambda\rho$ are given by

$$R(t) = R_o t^{\frac{2}{3(1+\lambda)}} \tag{22.a}$$

$$\rho(t) = \frac{G}{8\pi} \cdot \frac{4}{3(1+\lambda)^2} \cdot \frac{1}{t^2} \tag{22.b}$$

Hence, requiring that Brans-Dicke solutions reduce to Einstein solutions when $\omega \to \infty$ and $\phi = G^{-1}$ amounts to impose restrictions on the arbitrariness of the functions $f(\lambda,\omega)$. Comparison of eq. (16.a) and (22.a) prescribes the following form for $f(\lambda,\omega)$:

$$f(\lambda,\omega) = h(\lambda) + \frac{3\omega}{2}(1-\lambda^2) + O(\lambda,\omega) \quad , \tag{23}$$

where $\lim_{\omega \to \infty} (\frac{O(\lambda,\omega)}{\omega} = 0$, and $h(\lambda)$ is an arbitrary function of λ. If $f(\lambda,\omega)$ cannot be put in this form, surely the general relativistic limit will not be attained by the solutions of equations (16). Clearly, Nariai's solutions are a particular case of (23). Furthemore, if $f(\lambda,\omega)$ has the form of (23), then $\phi \to \phi_o$ = const. automatically when $\omega \to \infty$.

In the same way, we have to impose restrictions on the functions $g(\omega)$ in equations (17.a) and (17.b). Nevertheless, since we are dealing now with vacuum solutions, we must obtain a static geometry (Minkowski spacetime) when the limit $\omega \to \infty$ is taken. This condition implies

$$\lim_{\omega \to \infty} \frac{\sqrt{\omega}}{g(\omega)} = 0. \tag{24}$$

It is worth mentioning that O'Hanlon and Tupper vacuum solutions referred previously do not satisty the above condition.

6. The radiation case

When $\lambda = 1/3$ we have a special case since the invariant ray $\alpha_3 = \text{arc cotg}\, \xi_3$ does not depend on ω. In effect, as it can be inferred from eq. (7), $\lambda = 1/3$ implies that $\alpha_3 = 0$ or π, which is to say $\psi = 0$ (or, $\phi = \text{const.}$). Equation (4.a), then, reduce to

$$\dot{\theta} = -\frac{2}{3}\theta^2 \quad , \tag{25}$$

which, after immediate integration, yields

$$R(t) = R_o(t+m)^{1/2} \quad . \tag{26}$$

Thus, we conclude that in this case the class of solutions represented by the invariant ray α_3 is nothing but Friedmann's radiation solution with flat spatial sections. Furthemore, it is interesting to notice that this result is independent of taking the limit $\omega \to \infty$. The reason for this lies on the fact that the scalar field is source-free ($T = 0$) for radiation, and, then, all solutions of general relativity satisfy Brans-Dicke equations.

Finally, let us make a brief comment on the radiation solutions found in the literature and which are related to the invariant rays of the dynamical system (4). It can be shown that when $\omega = -3/2$ the lines BB' and CC' coincide in the phase plane and satisfy the equation $\frac{\theta}{\psi} = -3/2$. On the other hand, the radiation solution for $\omega = -3/2$ found out by Singh and Deo[15], given by the equations

$$R(t) = R_o t \quad , \tag{27.a}$$

$$\phi(t) = \phi_o t^{-2} \quad , \tag{27.b}$$

$$\rho(t) = \frac{F}{R^4(t)} \quad , \tag{27.c}$$

where F is supposed to be constant, constitutes a case in which $\frac{\theta}{\psi} = -3/2$. Thus, Singh and Deo's solution (incidentally it does not satisfy Dirac's hypothesis since the gravitational constant $G = \frac{1}{\phi}$ increases with the age of the universe) lies just on lines BB' and CC' containing the invariant rays BM and CM. Furthermore, the constant F (left undetermined in Singh and Deo's paper) must be necessarily null as a consequence of equation (18).

7. Final comments

We should like to add some final comments on the classes of solutions represented by equations (16) and (17):

i) As far as singularities are concerned, we should say that all solutions constructed from the invariant rays present singularities in the geometry (i.e., collapse of spacetime). This may be proved by recalling that, by its own nature, the invariant rays extend over regions of the phase plane (θ,ψ) where θ and (or) ψ are infinite. The only exception is when we have $\omega = -4/3$ for, in this case, we get an entire line of multiple equilibrium points implying that θ and ψ remain constant for all the time.

ii) An interesting class of solutions in which the geometry is static, even though the gravitational constant changes with time, is obtained if we set $\omega = \frac{1}{\lambda - 1}$ in equations (16). Then, we have

$$\phi = \phi_0 t^{\frac{-2-3/\omega}{f(\lambda,\omega)}} \tag{28}$$

The static solution reported by Raychaudhuri (see ref. (15)) corresponds to the particular choice $f(\lambda,\omega) = -1 - \frac{3}{2\omega}$, which, in turn, belongs to the class of Nariai's solutions discussed in sec. 3.

Conclusion

Obtaining classes of exact solutions for a general equation of state and arbitrary ω in Brans-Dicke theory is not usually an easy task even if one works with homogeneous and isotropic spacetimes. In this article we have developed a method of finding solutions whose simplicity is rather surprising. This fact alone stresses the importance of dynamical system theory as a powerfull mathematical tool to be used in the theoretical investigation of cosmology.

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