# Black Hole entropy in $D = 2 + 1$ dimensions from extended Chern-Simons term in a gravitational background

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We study the contribution to entropy of Black Holes in $D = 2 + 1$ dimensions from an extension of the Chern Simons theory including higher derivative in a curved space-time [2].

**Key-words:** Black holes; Chern-Simons; Gravitation; Entropy.

## Introduction

The topologicaly theory like Chern-Simons in $D = 2 + 1$ dimensions has been studied in various different approaches in quantum Field theory, in particularly in perturbative quantum gravity [6].

In general, topological action such as $\varepsilon^{\mu\nu\alpha} A_\mu \partial_\nu A_\alpha$ where $A_\mu$ means the abelian potential vector, or the action $\varepsilon^{\mu\nu\alpha}\left(\Gamma^\lambda_{\mu\beta}\partial_\nu\Gamma^\beta_{\alpha\lambda} + \frac{2}{3}\,\Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\gamma}\Gamma^\gamma_{\alpha\sigma}\right)$ with $\Gamma^\mu_{\alpha\beta}$ being the connection, that do not contribute to the entropy of black holes in $D = 2 + 1$. Furthermore these terms do not contribute dynamically in a quantum field theory [6]. Rich physics can be explored in quantum field theory [6, 2] when Chern-Simons terms is combined with Maxwell or Einstein Hilbert Lagrangian. The extension of Chern-Simons theory including highest derivative in flat space time or in curved space time is carried out by Jackiw and Deser [2]. The higher derivative Chern-Simons extensions has a strong dependence on the local field strengh, $F_{\mu\nu}$, and not on the vector potential, thus the gauge information can be lost [2].

On the other hand "extensions" such as the usual Chern-Simons term do not contribute to any change in the original value of entropy for black holes in $D = 2+1$ dimensions [3, 4]. In contrast to that, with the extension of Chern-Simons term in a gravitational background [2] some interesting things happen. We intend to find contributions to entropy of black holes in $D = 3$ dimensions.

Introducing the $I_{ECS}$ (extension for Chern-Simons with Higher Derivative in a gravitational background) and applying the same procedure as in [3] we compute contributions to entropy of black holes in $D = 3$ and we find the inverse e Hawking evaporation temperature, partition function and stress energy-momentum tensor. Although the $I_{ECS}$ is not globally topological [2] due to its energy-momentum tensor $T^{\mu\nu}_{ECS}$ being different from zero the contribution from $I_{ECS}$ to entropy of black holes can be computed.

The non Abelian case will be treated in the next letter again without any linkage with topology associated, with metric in accordance with [5].

Let us begin by writing the funcional integral

$$Z = \int \mathcal{D}g\; e^{-(I+I_{ECS})} \tag{1}$$

where $I$ and $I_{ECS}$ are respectively the action for the three dimensional gravity with a negative cosmological constant $\Lambda = -\frac{2}{\ell^2}$ and action for higher derivative Chern-Simons extension in a curved space time given by

$$I = \frac{1}{16\pi G}\int\left(R + \frac{2}{\ell^2}\right)dx^3 \quad \text{and} \tag{2}$$

$$I_{ECS} = -(2m)^{-1}\int \varepsilon^{\alpha\beta\gamma} f_\alpha \partial_\beta f_\gamma dx^3 \tag{3}$$

with $f_\alpha$ written as

$$f_\alpha = \frac{1}{\sqrt{g}}\, g_{\alpha\beta}\varepsilon^{\beta\mu\nu}\partial_\mu A_\nu \ . \tag{4}$$

In according with [2] $f_\alpha$ is a covariant vector and $f^\alpha$ being contravariant vector. The metric dependence in $I_{ECS}$ is completely contained in $f_\alpha$.

The equations of motion derived from this action (2) are solved [1, 3] for the three-dimensional black hole whose metric is

$$ds^2 = -\left(-8MG + \frac{r^2}{\ell^2}\right)dt^2 + \left(-8MG + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\varphi^2 \tag{5}$$

where the quantities $R$, $\varepsilon^{\alpha\beta\gamma}$, $A_\mu(x)$ are the scalar curvature, the Levi-Civita tensor $\varepsilon^{012} = +1$ and the vector potential respectively.

The three components of $f_\alpha$ are

$$\begin{aligned} f_0 &= \frac{g_{00}}{\sqrt{-g}}\,\varepsilon^{012}\left(\partial_1 A_2 - \partial_2 A_1\right) \\ f_1 &= \frac{g_{11}}{\sqrt{-g}}\,\varepsilon^{102}\left(\partial_0 A_2 - \partial_2 A_0\right) \\ f_2 &= \frac{g_{22}}{\sqrt{-g}}\,\varepsilon^{201}\left(\partial_0 A_2 - \partial_1 A_0\right) \end{aligned} \tag{6}$$

On considering the antisymmetry of the Levi-Civita tensor, the action $I_{ECS}$, can be written as

$$I_{ECS} \sim \int d^3x \left[f_0\left(\partial_1 f_2 - \partial_2 f_1\right) - f_1\left(\partial_0 f_2 - \partial_2 f_0\right) + f_2\left(\partial_0 f_1 - \partial_1 f_0\right)\right] \ . \tag{7}$$

We recall that in $D = 3$ we have

$$A_\mu = A_\mu(x^\alpha) = (A_0, A_i) = (\varphi, A_i) \quad i = 1, 2$$

and

$$x^\alpha = (x^0, x^1, x^2) = (t, r, \varphi)$$

and that the electric and magnetic field are pseudo vector and scalar respectively.

Thus, we introduce definitions for magnetic and electric fields as

$$\begin{aligned} \vec{E} &= -\vec{\nabla}\varphi + \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla} \times \vec{A} \ . \end{aligned} \tag{8}$$

Then $f_\alpha$ are given as

$$\begin{aligned} f_0 &= \frac{g_{00}}{\sqrt{-g}}\, B \ , \\ f_1 &= -\frac{g_{11}}{\sqrt{-g}}\, E_\varphi \quad \text{and} \\ f_2 &= \frac{g_{22}}{\sqrt{-g}}\, E_r \end{aligned} \tag{9}$$

where $E_\varphi$ and $E_r$ are components of the electric field and $B$ is the magnetic field in a gravitational background.

Then the "Chern-Simons action" as a function of the metric, electric and magnetic fields is

$$\frac{g_{00}}{\sqrt{-g}}\, B\left[E_r + r\left(\frac{\partial E_r}{\partial r}\right) + \frac{g_{11}}{\sqrt{-g}}\left(\frac{\partial E_\varphi}{\partial \varphi}\right)\right] \ , \tag{10.a}$$

$$\left(-\frac{g_{11}}{\sqrt{-g}}\, E_\varphi\right)\left[r\left(\frac{\partial E_r}{\partial t}\right) - \frac{g_{00}}{\sqrt{-g}}\left(\frac{\partial B}{\partial \varphi}\right)\right] \ , \tag{10.b}$$

$$-\frac{g_{11}}{\sqrt{-g}}\left(\frac{\partial E_\varphi}{\partial t}\right) - \left(\frac{g_{00}}{r^2} - \frac{2}{\ell^2}\right)B - \frac{g_{00}}{\sqrt{-g}}\left(\frac{\partial B}{\partial t}\right) \ . \tag{10.c}$$

These equation give the three terms in the expression for $I_{ECS}$.

Now, following [3] the inverse temperature as the Euclidean time period is

$$\beta = \frac{2\pi}{\alpha} \tag{11}$$

with $\alpha$ a parameter given by

$$\alpha = \frac{1}{2}\left.\frac{df(r)}{dr}\right|_{r=r_+} \quad , \quad \alpha \neq 0 \ . \tag{12}$$

Here the function $f(r)$ is equal to $g_{00}$, and is given by

$$f(r) = -8MG + \frac{r^2}{\ell^2} \tag{13}$$

where $r = r_+$, the event horizon given by

$$r = r_+ = \sqrt{8MG}\ \ell \tag{14}$$

where $M$, $G$, $\ell$ are the mass of the black hole, the gravitational constant and the cosmological constant respectively. Then the temperature, $\beta$, is given as

$$\beta = \frac{\pi\ell}{\sqrt{8MG}} \quad . \tag{15}$$

In general the temperature $T = 1/\beta$ defined in (9) coincides exactly with the Hawking's temperature for evaporation of black holes. In our case, if no consideration to topology in the Euclidean sector is given and if we put off any relation between temperature and the complex structure of the torus [3, 4, 5] the temperature is

$$T_H \sim \frac{\sqrt{M}}{2\pi\ell} \ . \tag{16}$$

The total partition function associated with Einstein-Hilbert-Chern-Simons action is

$$Z_T \simeq Z_3 \cdot Z_{\text{Simons}}^{\text{Chern}} \tag{17}$$

where $Z_3$ is the three-dimensional partition function in the saddle point approximation related to the solution (5) given in [3] by

$$Z_3 \simeq e^{\pi^2\ell^2/2G\beta} \tag{18}$$

and $Z_{ECS}$ is the dimensional partition function associated with the higher derivative Chern-Simons in a gravitational background [2] give as

$$Z_{ECS} \simeq e^{\frac{g_{00}}{\sqrt{-g}}\ BE_r - \left(\frac{g_{00}}{r^2} - \frac{2}{\ell^2}\right)B} \tag{19}$$

For simplicity only two terms from (10) are used. The total partition function is given as.

$$Z_T \sim e^{\pi^2\ell^2/2G\beta} e^{\left(\frac{\pi^2\ell^2}{\beta^2 r} - \frac{r}{\ell^2}\right)BE_r} e^{\left(\frac{\pi^2\ell^2}{\beta^2 r} - \frac{3}{\ell^2}\right)B} \ . \tag{20}$$

Now the thermodynamical formula for the average energy and the average entropy $S$ is

$$\begin{aligned} M &= -\frac{\partial}{\partial\beta}(\ln Z_T) \\ S &= \ln Z_T - \beta\partial_\beta \ln Z_T \ . \end{aligned} \tag{21}$$

The contribution to entropy is calculated from each term using (10). For instance, for the second term in (10.a) we may write the partition function $Z_T$ as

$$Z_T \sim e^{\pi^2\ell^2/2G\beta} e^{-\frac{\pi^2\ell^2}{\beta^2}\left(\frac{\partial E_r}{\partial r}\right)B + \frac{r^2}{\ell^2}\left(\frac{\partial E_r}{\partial r}\right)B} \tag{22}$$

Then the average entropy is

$$S \sim \frac{\pi^2\ell^2}{G\beta} - \frac{3\pi^2\ell^2}{\beta^2}\left(\frac{\partial E_r}{\partial r}\right)B + \frac{r^2}{\ell^2}\left(\frac{\partial E_r}{\partial r}\right)B \ . \tag{23}$$

One approaching the event horizon $r \rightarrow r_+$ the entropy is

$$S \sim \frac{\pi^2\ell^2}{G\beta} - \frac{2\pi^2\ell^2}{\beta^2}\left(\frac{\partial E_r}{\partial r}\right)_{r=r_+} \cdot B(r = r_+) \tag{24}$$

where the first part comes from Einstein-Hilbert action, together with eq. (6) and the second part comes from extension of Chern-Simons action in a gravitational background.

Again for simplicity the contribution to entropy only for static configuration, is considered in eq. (10). The other terms have a non zero contribution for entropy, in particular a contribution as given by eq. (20) and eq. (24). The reason why we have a non zero contribution for entropy in the present case is because in constrast with the abelian Chern-Simons theory for electromagnetic theory or Chern-Simons for gravitational theory where the energy momentum tensor is zero, here we find the energy-momentum tensor is not zero and is given as

$$T^{\mu\nu}_{ECS} = \frac{2}{\sqrt{-g}}\frac{\delta I_{ECS}}{\delta g_{\mu\nu}} \ . \tag{25}$$

The result is written as [2]

$$T^{\mu\nu}_{ECS} = -m^{-1}\left[\left(\varepsilon^{\mu\alpha\beta}f^\nu + \varepsilon^{\nu\alpha\beta}f^\mu\right)\partial_\alpha f_\beta - g^{\mu\nu}\varepsilon^{\alpha\beta\gamma}f_\alpha\partial_\beta f_\gamma\right] \ , \tag{26}$$

It's interesting to note that if we take the limit such that $g_{\mu\nu} \longrightarrow \eta_{\mu\nu}$; the equation (4) becomes

$$f^{\alpha} = \frac{1}{2}\,\varepsilon^{\alpha\mu\nu} F_{\mu\nu}\ , \tag{27}$$

In accordance with [2], this cannot be done here since our metric (5) is a particular case of the anti de-Sitter space.

## Conclusions and Look out:

In contrast to the abelian Chern-Simons term for the electromagnetic theory or the Chern-Simons term associated with the gravitational theory in $D = 2 + 1$ dimensions there is a contribution to the entropy of black holes due to higher derivative Chern-Simons extensions in a gravitational background.

Appropriate vector $f^{\alpha}$ for an extension of a topologically term such as Chern-Simons [2, 6] is defined and we have shown that the source of entropy for black holes in $D = 2 + 1$ dimension is the stress tensor which is not zero ($T^{\mu\nu}_{ECS} \neq 0$).

The entropy using (21) in combination with (17) is different than $S = \dfrac{A}{4}$ where $A = 2\pi r_{+}$, the area of horizon, since here the "topological contribution" is not included.

Now, we are considering the contribution to entropy of black holes in $D = 2 + 1$ but due to non abelian Chern-Simons term such as

$$S = \frac{k}{4\pi}\int d^3x\varepsilon^{\mu\nu\rho}\left(\frac{1}{3}\ f^a_{\mu}\partial_{\nu}f^a_{\rho} + \left(\frac{1}{3!}\right) f^{abc}f^a_{\mu}f^b_{\nu}f^c_{\rho}\right)$$

where

$$f^a_{\mu} = (-g)^{1/2} g_{\mu\alpha}\varepsilon^{\alpha\lambda\gamma}A^a_{\gamma}\ .$$

This goal will be hopefully realised in the next letter.


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