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A Simple Proof of the Banach-Stone Theorem

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Abstract

We present a simple proof of the famous theorem of Banach-Stone.

1 Introduction

One of the most important result in functional analysis is the content of the celebrated theorem of Banach-Stone ([1]): My aim in this mathematical short note is to present a simple new proof of mine for the above mentioned theorem.

2 The Banach-Stone Theorem

Let X and Y denote compact topological spaces. Let me assume an isometric isomorphism between the associated space of continuous functions

$$\begin{aligned} I : C(X, R) &\to C(Y, R) \\ f &\longmapsto I(f) \end{aligned} \tag{1}$$

I have, thus, the following statement: *The topological spaces X and Y are topologically homeomorphs.*

Proof: Let me consider the following linear and multiplicative functional on $C(Y, R)$ for each $x \in X$ fixed

$$I^{-1}(g)(x) \tag{2}$$

Here I^{-1} is the inverse isometric isomorphism defined by eq. (1).

By a direct application of the Radon Theorem ([2]), there is a Dirac measure supported at a point $y \in Y$ such that for each $g \in C(Y, R)$ we have the following result as a consequence of the multiplicative property of eq. (2)

$$I^{-1}(g)(x) = \int_Y g(y) d\mu^{Dirac}(y - \bar{y}) = g(\bar{y}) \tag{3}$$

Let me define the following relation on the cartesian set $X \times Y$

$$\begin{aligned} h : X &\to Y \\ x &\longmapsto \bar{y} \end{aligned} \tag{4}$$

Since given two arbitrary points y_1 and y_2 belonging to X there is a function $\tilde{g} \in C(Y, R)$ such that $\tilde{g}(y_1) \neq \tilde{g}(y_2)$, we obtain that h is a function with domain X and image on Y.

The function $h : X \to Y$ given by eq. (4) is injective, since there is two different points $x_1 \in X$; $x_2 \in X$ and $x_1 \neq x_2$ and applied by the function h on the same point $y \in Y$, then we have that

$$I^{-1}(g)(x_1) = I^{-1}(g)(x_2) \tag{5}$$

for every $I^{-1}(g) \in C(X, R)$. Again, we have a contradiction with the fact that there is always a function $m \in C(X, R)$ such that $m(x_1) \neq m(x_2)$.

At this point we obtain that h is a *contínuos* function and sobrejective since X and Y are compact tological spaces.

I have, thus, that $h : X \to Y$ provides a homeomophism between this two topological compact spaces.

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References

[1] G.F. Simmons, “Introduction to Topology and Modern Analysis”, McGraw Hill (1977).

[2] N. Dumford and J. Schwartz, “Linear Operators (part I), Interscience, N.Y. (1958).