Gödel Type solution with rotation, expansion and closed time-like curves

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We propose a time-varying parameter $\underline{\alpha}$ for Gödel metric and an energy momentum tensor corresponding to this geometry is found. To satisfy covariance arguments time-varying gravitational and cosmological term are introduced. The "Einstein's equation" for this special evolution for the Universe are written down where expansion, rotation and closed time-like curves appear as a combination between standard model, Gödel and steady state properties are obtained.

Key-words: General relativity; Gravitation.

Introduction

In classical cosmology it is well-known that there are two different geometries for the same source as in the case of dust or perfect fluid. The Friedmann-Robertson-Walker and Gödel solutions have a common source for different physics. In standard cosmology we have an expansion of the universe but no rotation as in the Gödel solution.

An open question is to know if rotation implies a violation of causality and if closed time-like curves imply rotation.

What happens if the $\underline{\alpha}$ parameter in Gödel solution is a time dependant function? Does the universe undergo a particular evolution to get in a special way to the present epoch?

We argue that if the α parameter is considered to be time dependent in the usual Gödel Geometry it is possible to find a special source for this new Gödel type geometry such as a "quasi-perfect fluid" or a "quasi-dust".

By quasi-dust it is understood that the state equation is not characterized by the pressure $p = 0$, but by $p \approx 0$. It is assumed that if $p \approx 0$ one can write an energy momentum tensor very similar to the energy tensor for a perfect fluid.

In general if the Einstein-Hilbert lagrangean with the cosmological term is a function of time the global covariance will be lost. But it is assumed that the universe will get a special evolution law and if both the cosmological term and the gravitational constant are time-varying under these assumptions the covariance is considered again and it is possible to find a generalization of the Einstein equations. A Gödel type solution could describe the universe with expansion like in the Friedman-Robertson-Walker model plus rotation and closed time like curves as in the case of the Gödel solution.

In accordance with [1, 2, 7] the time-varying Λ may describe the creation of matter like the steady state model [4].

In reality there are two possibilities to write the time-varying cosmological term: either one can write it in the lagrangian or one can write it on the right side of a Einstein's equations. The second way appears quite often [7], but here we choose the first possibility. Thus we obtain a special evolution law for time varying gravitational and cosmological term to save covariance.

An energy momentum tensor which describes the new "Gödel-type" geometry is proposed.

It is argued that such a universe is in reality a combination characteristics of the standard cosmology, the Gödel solution and the steady state model.

The four functions $\Lambda(t)$, $k(t)$, $\theta(t)$, $\omega(t)$ establishing the connection among the expansion from standard cosmology described by the scalar factor θ and the angular velocity $\omega(t)$ to showing the rotation and the time varying Λ and k indicating the appearence of matter inside the universe [2, 7], gives us the motivation to jutisfy a revived interest in the creation of same kind of matter in the universe.

For large scale time we can wait to find a null expansion, constant angular velocity, zero pressure, the usual energy momentum tensor for dust, and the standard Gödel solution, but all the time we have a possibility of a closed time-like curve.

Nothing can be said about the initial state of the universe as for example the singularities for the energy density. The conservation of energy is achieved only asymptotically [7].

The Gödel metric is written as

$$ds^2 = \left(dx^0 + e^{\alpha x^1} dx^2\right)^2 - (dx^1)^2 - \frac{1}{2}\, e^{2\alpha x^1}(dx^2)^2 - (dx^3)^2 \tag{1}$$

The α parameter is assumed to be a function of time.

The Einstein Hilbert lagrangean with cosmological term is written as

$$\mathcal{L} = -\frac{1}{2k^2}\, R\sqrt{-g} + \Lambda(t)\sqrt{-g}\ . \tag{2}$$

Here both the gravitational constant and the cosmological term are functions of time.

The variation principle $\dfrac{\delta S}{\delta g^{\mu\nu}}$ gives us the generalized Einstein tensor $\tilde{G}_{\mu\nu}$ as

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} + k^2\Lambda(t) g_{\mu\nu} - \left(\frac{2R}{k}\,\frac{\delta k}{\delta t} + 2k^2\,\frac{\delta\Lambda}{\delta t}\right)\,\frac{\delta t}{\delta g^{\mu\nu}} \tag{3}$$

where R is the scalar curvature and $\dfrac{\delta t}{\delta g^{\mu\nu}} \neq 0$ for all time.

The Ricci scalar in our case is given by

$$R = \alpha(t)^2 + 2x^2\alpha'(t)^2 + 2x\alpha''(t) \tag{4}$$

where first and second derivative of the α parameter are indicated by primes.

A particular evolution law for cosmological and gravitational term is proposed

$$\Lambda(t) = \sqrt{2}\ e^{-\alpha(t)x}\Lambda_0 \tag{5}$$

and

$$k^2(t) = \frac{1}{2\sqrt{2}}\ e^{\alpha(t)x}\left(\alpha(t)^2 + 2x^2\alpha'(t)^2 + 2x\alpha''(t)\right) k_0^2 \tag{6}$$

where Λ_0 and k_0 are the values of cosmological and gravitational constants at the present time.

The equation of state for a quasi perfect fluid can be written as

$$p \simeq 0\ . \tag{7}$$

The fact that the pressure can be approximately zero means that the universe could be like "dust".

The appropriate stress energy-momentum tensor $T_{\mu\nu}$ for an incoherent field matter in quasi-rest is given by

$$T^{\mu\nu} = \rho v^u v^\nu \tag{8}$$

where

$$v^u = \delta_0^u \tag{9}$$

because we are using quasi-comoving coordinates.

The energy momentum tensor for quasi perfect fluid is given as.

$$\tilde{T}_{\mu\nu} = \rho \begin{pmatrix} 1 & 0 & e^{\alpha(t)x} & 0 \\ 0 & 0 & \dfrac{1/2e^{\alpha(t)x}\left(1+2\alpha(t)x\right)\alpha'(t)}{\alpha(t)^2} & 0 \\ e^{\alpha(t)x} & \dfrac{1/2e^{\alpha(t)x}\left(1+2\alpha(t)x\right)\alpha'(t)}{\alpha(t)^2} & e^{2\alpha(t)x} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

It is easy to verify that $\lim_{t\to\infty}\tilde{T}_{\mu\nu}$ leads to the usual energy-momentum tensor of perfect fluid if the α parameter is constant asymptotically.

The Einstein's equations are written as

$$\tilde{G}_{\mu\nu} = -\frac{8\pi k(t)}{c^2}\,\tilde{T}_{\mu\nu} \tag{11}$$

with the relevant components for the Einstein tensor, $G_{\mu\nu}$, given by

$$\begin{aligned}
G_{11} &= \frac{1}{2}\left(\alpha(t)^2 + 2x^2\alpha'(t)^2 + 2x\alpha''(t)\right) \ , \\
G_{12} &= \frac{1}{2}\ e^{\alpha(t)x}\left(1 + 2\alpha(t)x\right)\alpha'(t) \ , \\
G_{10} &= \alpha(t)\alpha'(t)x \ , \\
G_{22} &= \frac{3}{4}\ e^{2\alpha(t)x}\alpha(t)^2 \ , \\
G_{20} &= \frac{1}{2}\ e^{2\alpha(t)x}\alpha(t)^2 \ , \\
G_{33} &= \frac{1}{2}\left(\alpha(t)^2 + 2x^2\alpha'(t)^2 + 2x\alpha''(t)\right) \ , \\
G_{00} &= \frac{\alpha(t)^2}{2} \ .
\end{aligned} \tag{12}$$

Then the "Einstein's" equations are

$$\frac{2R}{k}\ \frac{\delta k}{\delta t} + 2k^2\ \frac{\delta\Lambda}{\delta t} = 0 \ , \tag{.13a}$$

$$\frac{8\pi k_0}{c^2}\ \rho = \frac{\alpha(t)^2 + \ x^2\alpha'(t)^2 + \ x\alpha''(t)}{\sqrt{\frac{1}{2\sqrt{2}}\ e^{\alpha(t)x}\left(\alpha(t)^2 + 2x^2\alpha'(t)^2 + 2x\alpha''(t)\right)}} \ , \tag{.13b}$$

$$\alpha(t)\alpha'(t) = 0 \ , \tag{.13c}$$

$$\frac{8\pi k_0}{c^2}\ \rho = \frac{\alpha(t)^2 + \frac{1}{2}x^2\alpha'(t)^2 + \frac{1}{2}\ x\alpha''(t)}{\sqrt{\frac{1}{2\sqrt{2}}\ e^{\alpha(t)x}\left(\alpha(t)^2 + 2x^2\alpha'(t)^2 + 2x\alpha''(t)\right)}} \ , \tag{.13d}$$

$$\frac{8\pi k_0}{c^2}\ \rho = \frac{\alpha(t)^2 + x^2\alpha'(t)^2 + x\alpha''(t)}{\sqrt{\frac{1}{2\sqrt{2}}\ e^{\alpha(t)x}\left(\alpha(t)^2 + 2x^2\alpha'(t)^2 + 2x\alpha''(t)\right)}} \ , \tag{.13e}$$

$$\frac{8\pi k_0}{c^2}\ \rho = \frac{\alpha(t)^2}{\sqrt{\frac{1}{2\sqrt{2}}\ e^{\alpha(t)x}\left(\alpha(t)^2 + 2x^2\alpha'(t)^2 + 2x\alpha''(t)\right)}} \ . \tag{.13f}$$

A strong constrant is indicated by eq. (13.a) arising from the special form of the energy-mometum tensor (10).

An asymptotic solution for "Einstein's" equation, which describes the energy density under special conditions is found such that

$$\begin{aligned}
\lim_{t\to\infty}\alpha(t) &\ \rightarrow \text{constant.} \ , \\
\lim_{t\to\infty}\alpha'(t) &\ \rightarrow 0 \ , \\
\lim_{t\to\infty}\alpha''(t) &\ \rightarrow 0 \ .
\end{aligned} \tag{.14}$$

A particular solution which satisfies (14) is shown as

$$\alpha(t) = \text{tagh}\ (t) = \frac{e^t - e^{-t}}{e^t + e^{-t}} \tag{.15}$$

At $t = 0$ nothing can be said about the initial state of the universe. Differently from the standard cosmology nothing is known about the initial singularity of the universe. The initial state for the energy density is not determined. At $t = \infty$ the energy density assumes a constant value like a perfect fluid.

The expansion factor for this case is

$$\theta = V^{\alpha}_{;\alpha} = \frac{1}{\sqrt{-g}}\ \left(\sqrt{-g}V^{\alpha}\right)_{,\alpha} \tag{.16}$$

where semi-colon is a covariant derivative, coma means ordinary derivative and V^{α} is the quadri-velocity given by (9).

For this case the expansion factor can be shown to be

$$\theta \sim \frac{1}{2}\ \alpha'(t) > 0 \qquad (.17)$$

Again, at $t = 0$ there is no expansion of the universe, but for $0 < t < \infty$ the size of the universe is continually increasing similar to the usual expansion in the standard cosmology. At $t = \infty$ the expansion stops again and the α parameter is a constant like the usual Gödel solution.

The solution (1) has rotation with angular velocity ω. If we assume that the angular velocity as in Gödel solution [4] is given by

$$\omega = \sqrt{\pi k \rho} \qquad (.18)$$

where k is the gravitational constant and ρ is the energy density, the angular velocity in our case becomes.

$$w^2 \approx \alpha(t)^2 + \alpha'(t)^2 + \alpha''(t)\ . \qquad (.19)$$

The angular velocity is zero at the beginning but it increases continuously for $0 < t < \infty$ and finally it goes to a constant value for $t = \infty$ exactly the same way as (18).

It may be verified that $\dot{\Lambda}(t)/\Lambda(t)$ is proportional to $-\alpha'(t)$ which means that the cosmological term will be decreasing in time so that it reaches a constant value. For large scale of time there is no more expansion for the universe but only a constant rotation.

It is important to remember that Gödel solution, besides having a constant angular velocity, has closed time-like curves. For the usual Gödel solution (1) with constant parameter, α, the coordinate system may be changed from (x^0, x^1, x^2, x^3) to another system such as (r, θ, ϕ, z) and the metric is written as

$$ds^2 = 4a^2\left[dt^2 + 2H(r)d\phi dt + G(r)d\phi^2 - dz^2 - dr^2\right]\ , \qquad (.20)$$

where

$$G(r) = \mathrm{senh}^4(r) - \sinh^2(r)\ ,$$
$$\text{and } H(r) = 2\sqrt{2}\sinh^2(r)\ .$$

The Gödel solution has in fact closed time like curves as we consider t, z and r as constant [4, 6] since $d\phi^2$ varies between zero and 2π.

Here arbitrary time dependence of functions is considered as the usual Gödel solution to be the $\underline{\alpha}$ parameter. With an arbitrary time dependent function we get

$$ds^2 = 4a^2dt^2 + 8a^2H(r)d\phi dt + 4a^2G(r)m(t)d\phi^2 - 4a^2dz^2 - 4a^2dr^2\ , \qquad (.21)$$

the same form as for t, z, r given by constants we may find closed time like-curves. The line element is written now as

$$ds^2 = 4a^2G(r)m(t)d\phi^2 \qquad (.22)$$

where a is a constant, and $m(t)$ some positive function.

We can change from (x^0, x^1, x^2, x^3) to a new coordinate system (r, θ, ϕ, z) since the covariance is guaranteed by the two appropriate choices of cosmological term and gravitational time dependent function.

The proposed solution carries the possibility of expansion like the Friedmann model but it contains rotation and closed time-like curves as in Gödel solution.

The universe could get a special way of evolution in accordance with eq. (5,6) and to continue today its course with slow expansion plus increasing rotation and closed time like curves.

The time dependent gravitational term could mean that some kind of matter appears in the universe exactly the same way as in the steady state model [4].

We may be living in an universe which is a combination of the standard model (expansion), steady state (appearence of matter) and finally, Gödel type solution (rotation). Asymptotically in time the α parameter is constant, energy density as is usual for dust source, zero expansion, non null rotation and a gravitational constant.

The conservation of energy will be possible only for large scale time. This fact is in accordance with [1, 3, 7] and with our time-varying Λ and k it becomes justifiable to revive the interest in creation of matter.

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