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A Theorem of Müntz-Szasz Type for Intervals [1,a]

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Abstract

A classical theorem of Müntz-Szasz states that the closure of all finite linear combinations of the functions $1, t^{\lambda_1}, t^{\lambda_2}, \cdots$ for $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ is $C([0,1])$ if $\sum \frac{1}{\lambda_n} = -\infty$ (see for instance [1]).

If turns out that the some result for intervals of the form $[1, a]$ $(a > 1)$ the usual given somewhat simple proofs do not work out [1], [2], [3]. However, I have the following restrict form of the above cited Müntz-Szasz theorem.

Theorem: Suppose $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ and let X be the closure in $C([1, a])$ of the set of all finite linear combinations of the functions

$$1,\ t^{\lambda_1} exp(-\lambda_1 t),\ t^{\lambda_2} exp(-\lambda_2 t), \cdots$$

We have that if $\sum 1/\lambda_n = +\infty$ then $X = C([1, a])$.

Proof: Following the usual complex variable proof of ref. [1], I need to prove that if $\sum 1/\lambda_n = +\infty$ and if μ is a complex Borel measure on $I = [1, a]$ such that

1)

$$\int_I t^{\lambda_n} e^{-td_n} \partial\mu(t) = 0 \qquad (n = 1, 2, 3, \cdots)\ , \tag{1}$$

then also

2)

$$\int_I t^k e^{-kt} d\mu(t) = 0 \qquad (k = 0, 1, 2, 3, \cdots) \tag{2}$$

or equivalently (since e^{-kt} is a analytical function on I) $\mu(t) = 0$.

So let me assume that 1 is true. Let me consider the function

3)

$$f(z) = \int_1^a t^z\ e^{-zt} \partial\mu(t) \tag{3}$$

Obviously $f(z)$ is holomorphic in the righ half plane H^+ furthermore $f(z)$ is bounded there, since

$$\sup_{0<x<\infty} |f(z)| \le \mu([1,a]) \times \sup_{\substack{0<x<\infty \\ t\in[1,a]}} |t^x e^{-xt}|$$

$$\mu([1,a]) \sup_{\substack{0<x<\infty \\ 1\le t\le a}} |e^{xlgt} e^{-xt}|$$

$$\mu([1,a]) \sup_{\substack{0<x<\infty \\ 1\le t\le a}} |e^{x(lgt-t)}| \le \mu([1,a]) \sup_{0<x<\infty} |e^{-x}| \tag{4}$$

since $lgt - t \le 0$ for $t \in [1,a] < \infty$.

Define the following function $g \in H^\infty(|z| \le 1)$ $g(z) = f\left(\frac{1+z}{(1-z)}\right)$. Then $g\left(\frac{\lambda_n-1}{\lambda_n+1}\right) = 0$ and $\sum(1 - \left|\frac{\lambda_n-1}{\lambda_n+1}\right) = +\infty$ or $\sum \frac{1}{\lambda_n} = +\infty$. Hence $g(z) \equiv 0$ or $f(z) \equiv 0$. I have thus proved my proposed result.

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References

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[2] A.F. Timan, "Theory of Approximation of Functions of a Real Variable", Dover Publications, Inc. NY., 1994.

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