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Effective Lagrangian for Electrodynamics and Avoidance of the Singular Origin of the Universe

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Abstract

In the extremely condensed primordial era of our universe the structure of classical fields must be implemented by quantum corrections. In a semiclassical analysis one deals with quantum matter phenomena in a classical geometry that obeys Einstein field equations of general relativity. Among all natural process one can envisage in a universe endowed with a huge number of photons the most natural ones are those related to electromagnetic fields. The corrections to Maxwell electrodynamics arising from the quantum domain were calculated by Heisenberg and Euler [1]. We show here that the net consequence of applying such corrections to a spatially homogeneous and isotropic metric structure is to forbid the appearance of a primordial singularity in the FRW geometry. We conclude that the presence of such singularity is incompatible with the quantum laws of physics in the semiclassical regime.

Key-words: Effective Lagrangian; Non-singular cosmology.

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1 Introduction

The standard description of the universe in terms of an ultrarelativistic gas of photons can be extrapolated back up to the beginning of primordial nucleosynthesis, about $10^{-2}\,sec.$ We prove that for smaller times one should take into account the contributions from the quantum effects that allow a classical electromagnetic field to create pairs of particles. We shall see that this modification is nothing but a direct consequence of combining quantum electrodynamics with classical Einstein general relativity. There is no simpler way to prove this statement than to take into account the work of Heisenberg and Euler [1, 2], in which it was evaluated the correction to the equations of motion of classical electrodynamics in order to describe electron-positron pairs production. Indeed, working in a gauge invariant way they obtained an effective action that describes this phenomenon. In a cosmological framework, near the very high temperature epochs in the early universe, one cannot neglect such corrections. For different reasons, this modification of classical electrodynamics was not taken into account in the standard model. The present article intends to go a step further by considering the semiclassical correction.

Our aim is to provide a framework by means of which one can examine the cosmological consequences of the quantum treatment of the matter content of the universe. In the case of electrodynamics this can be analyzed through the use of an effective Lagrangian. We will show here that a consequence of dealing with such correction for electrodynamics is to forbid a primordial singularity to appear. The main reason for the avoidance of an infinite compression comes from the fact that the quantum correction allows the appearance of a very high, but finite, negative pressure in the most condensed phase of Friedmann-Robertson-Walker (FRW) geometry. In the same vein, it inhibits the singular behavior of the energy density ρ_γ of the electromagnetic field, which scales no longer as the inverse of the fourth power A^{-4} of the scale-factor like in Maxwell classical theory, but instead as the sum of two terms: the first one represents, as in the classical regime (A^{-4}), while a new *quantum* term behaving as A^{-8} arises. One should wonder if such new term should not induce an even more dramatical singularity. We shall see in a next section that this is not the case. Near the point of maximum condensation of the universe the energy density ρ reaches its maximum value and starts to decrease. This is the true responsible mechanism for the avoidance of a singularity of the gravitational field.

After obtaining the equations of motion in the semi-classical regime we will turn to the discussion of its cosmological consequences. In a FRW scenario the electromagnetic field must be considered only in its average properties. This is the standard procedure and we will adopt it here. The resulting equation of motion for the scale factor is then integrated and the non-singular behavior is obtained. The next step is to evaluate the influence of other kinds of matter on the evolution of the universe. We show that matter in its ultrarelativistic state cannot modify the regularity of such solution. We end with the comment on the fact that this work led us to argue that the primordial singularity is incompatible with the quantum regime.

2 Einstein-Maxwell Singular Universe

Classical Maxwell electrodynamics gives origin to singular universes. In a FRW framework, this is a direct consequence of the singularity theorems that follows, in a simplified way, from the exam of the energy conservation law and Raychaudhuri equation for the expansion parameter Θ. Let us set for the line element the form

$$ds^2 = c^2\, dt^2 - A^2(t)\, d\sigma^2. \tag{1}$$

The 3-dimensional surface of homogeneity is orthogonal to a fundamental class of observers endowed with four-velocity vector field $v^\mu = \delta^\mu_o$. In terms of the scale-factor $A(t)$, the expansion parameter reads

$$\Theta = 3\, \dot{A}/A, \tag{2}$$

where 'dot' means time derivative (Lie derivative respective to v).

For a perfect fluid with energy density ρ and pressure p, the two above equations assume the form

$$\dot{\rho} + (\rho + p)\Theta = 0, \tag{3}$$

$$\dot{\Theta} + \frac{1}{3}\,\Theta^2 = -\,\frac{k}{2}(\rho + 3p), \tag{4}$$

in which k is the Einstein gravitational constant. Equations (3) and (4) do admit a first integral

$$k\,\rho = \frac{1}{3}\,\theta^2. \tag{5}$$

Since the spatial sections of FRW geometry are isotropic, electromagnetic fields can generate such universe only if an averaging procedure is performed [3]. The standard way to do this is just to set[1] for the electric field E_i and magnetic field H_i the following mean values:

$$< E_i > \;=\; 0, \tag{6}$$

$$< H_i > \;=\; 0, \tag{7}$$

$$< E_i\, E_j > \;=\; -\frac{1}{3} E^2\, g_{ij}, \tag{8}$$

$$< H_i\, H_j > \;=\; -\frac{1}{3} H^2\, g_{ij}, \tag{9}$$

$$< E_i\, H_j > \;=\; 0. \tag{10}$$

Canonical energy-momentum tensor associated with Maxwell Lagrangian is given by[2]

$$T_{\mu\nu} = F_{\mu\,\alpha}\, F^{\alpha}{}_{\nu} + \frac{1}{4}\, F\, g_{\mu\nu}, \tag{11}$$

[1]Latin indices run in the range $(1, 2, 3)$ and Greek indices run in the range $(0, 1, 2, 3)$.

[2]We use Heaviside units throughout.

in which we defined $F \doteq F_{\mu\nu}\, F^{\mu\nu}$. Using the above average values it follows that (11) reduces to a perfect fluid configuration. Indeed, the above average values (8)–(10) imply

$$< F_{\mu\,\alpha}\, F^{\alpha}{}_{\nu} >= \frac{2}{3}(E^2 + H^2) v_\mu\, v_\nu + \frac{1}{3}(E^2 - 2\, H^2)\, g_{\mu\nu}. \tag{12}$$

Using this result into the expression of $T_{\mu\nu}$, the average value $< T_{\mu\nu} >$ reduces to the form of a perfect fluid with density of energy ρ_γ and of pressure p_γ, as

$$< T_{\mu\nu} >= (\rho_\gamma + p_\gamma)\, v_\mu\, v_\nu - p_\gamma\, g_{\mu\nu}, \tag{13}$$

where

$$\rho_\gamma = \frac{1}{2}\,(E^2 + H^2) \tag{14}$$

and

$$p_\gamma = \frac{1}{3}\,\rho_\gamma. \tag{15}$$

The fact that both the energy density and the pressure are, in this case, positive definite for all time yields — using the Raychaudhuri equation (4) — the singular nature of FRW universes.

Einstein field equations for the above energy-momentum configuration (Euclidean section) yield for the scale factor

$$A(t)\ \sim\ t^{1/2}. \tag{16}$$

All this is standard and well-known. However, near the maximum point of condensation, classical Maxwell equations do not provide a correct description of electrodynamics. Instead, one should take quantum corrections into account. In order to examine precisely the effects of these modifications in cosmology, let us turn our analysis to the exam of these extra terms.

3 Quantum Corrections Generate Non-Singular Universes

The effective action for electrodynamics due to quantum corrections was calculated by Heisenberg and Euler[3]. The net consequence of the first order calculation[4] yields the effective Lagrangian density

$$L = -\frac{1}{4}\, F + \frac{\mu}{4}\, F^2 + \frac{7}{16}\,\mu\,(F^\star)^2, \tag{17}$$

in which $F^\star \doteq F^\star_{\mu\nu}\, F^{\mu\nu}$ and

$$\mu \doteq \frac{8}{45}\,\alpha^2 \left(\frac{\hbar}{m_e\, c}\right)^3 \frac{1}{m_e\, c^2}, \tag{18}$$

[3] See also the Schwinger's review [2].

[4] Higher order corrections do not modify the main results presented here. We will come back to this subject in a forthcoming paper.

where α is the fine-structure constant.

Treating such quantum correction as nothing but an effective classical field theory, the corresponding modified energy-momentum tensor reads

$$T_{\mu\nu} = -4\, L_F\, F_{\mu}{}^{\alpha} F_{\alpha\nu} + (L_{F^\star} F^\star - L)\, g_{\mu\nu}, \tag{19}$$

in which L_F represents the partial derivative of the Lagrangian with respect to the invariant F and similarly for the invariant $F^\star$.

Since we are interested mainly in the analysis of the behavior of this system in the early universe, where matter should be identified with a primordial plasma, we are led to limit all our considerations here to the case in which only the average of the squared magnetic field H^2 survives — that is, we set $E^2 = 0$ in (6)–(10). Let us see what the consequences of this result are.

3.1 Equation of Motion and Energy Distribution

Since the average procedure is independent of the equations of motion of the electromagnetic field we can use the above formulae (6)–(10) to arrive at a similar expression as (13) in Maxwell case for the average energy-momentum tensor, identified as a perfect fluid with energy density ρ_γ and pressure p_γ, which are given by

$$\rho_\gamma = \frac{1}{2}\, H^2\, (1 - 2\,\mu\, H^2), \tag{20}$$

$$p_\gamma = \frac{1}{6}\, H^2\, (1 - 10\,\mu\, H^2). \tag{21}$$

Since the effective energy-momentum tensor is not trace-free, the equation of state

$$p_\gamma = p_\gamma(\rho_\gamma) \tag{22}$$

is no longer given by the Maxwell value but has instead a new term which is proportional to the constant μ, that is

$$p_\gamma = \frac{1}{3}\, \rho_\gamma - \frac{4}{3}\, \mu\, H^4. \tag{23}$$

We note that, as μ is a positive constant, one could envisage the possibility that both the energy density and the pressure could become negative. We shall see below that this does not occur for the energy density but it is precisely the case for the pressure. Furthermore, there exists an epoch in the model of the universe presented here for which $\rho + 3\,p < 0$. This is the origin of the mechanism that avoids a primordial singularity to occurs. Let us show this.

Equations (3) and (4) encompass all dynamics. Indeed, from (20) and (21) we have[5]:

$$H^2\, \left(1 - 4\,\mu\, H^2\right) \left(\frac{\dot{H}}{H} + 2\,\frac{\dot{A}}{A}\right) = 0, \tag{24}$$

[5]We restrict ourselves to the exam of the Euclidean section case.

and

$$\frac{\ddot{A}}{A} = -\frac{k}{6}\, H^2\,(1 - 6\,\mu\, H^2). \tag{25}$$

Equation (24) furnishes[6]

$$H = \frac{H_o}{A^2}, \tag{26}$$

where H_o is a constant. With this result, (25) turns out to be a differential equation for the scale-factor:

$$\ddot{A} + \frac{k\, H_o^2}{6\, A^3}\,(1 - \frac{6\,\mu\, H_o^2}{A^4}) = 0, \tag{27}$$

whose solution is

$$A^2 = H_o\,\sqrt{\frac{2}{3}\,(k\, c^2\, t^2 + 3\,\mu)}. \tag{28}$$

Thus, from equation (26), we obtain for the magnetic fied the expression

$$H^2 = \frac{3}{2}\,\frac{1}{k\, c^2\, t^2 + 3\,\mu}. \tag{29}$$

Let us make some comments on this solution. First of all we recognize that the radius of the universe attains a minimum value A_{min} at $t = 0$, which depends on the quantum parameters, that is

$$A^2_{min} = H_o\,\sqrt{2\,\mu}. \tag{30}$$

Therefore, the values of A_{min} and the corresponding maximum of the temperature are given in terms of H_o, which turns out to be the unique free parameter of the present model.

The energy density ρ_γ attains its maximum value

$$\rho_{max} = \frac{1}{16\,\mu} \tag{31}$$

at the instant $t = t_c$, where

$$t_c = \frac{1}{c}\,\sqrt{\frac{3\,\mu}{k}}. \tag{32}$$

For lower values of t the energy density decreases, vanishing at $t = 0$. At the same time the pressure becomes highly negative (see Figure 2). Using the value of the *quantum* parameter as being given by $\mu \approx 10^{-31} cm^3/erg$, we can estimate the time after the maximum condensation of the universe ($t = 0$ in our notation) as about $t_c \approx 10^{-2}\, sec$ for the density ρ_γ to arrive from zero to its maximum value ρ_{max}. As we remarked in the abstract, only for times comparable to $\sqrt{\mu/kc^2}$ the quantum effects are important. Indeed, solution (28) fits the standard expression (16) of the Maxwell case at the limit of large times.

[6]We shall not consider here the $H^2 = constant$ case.

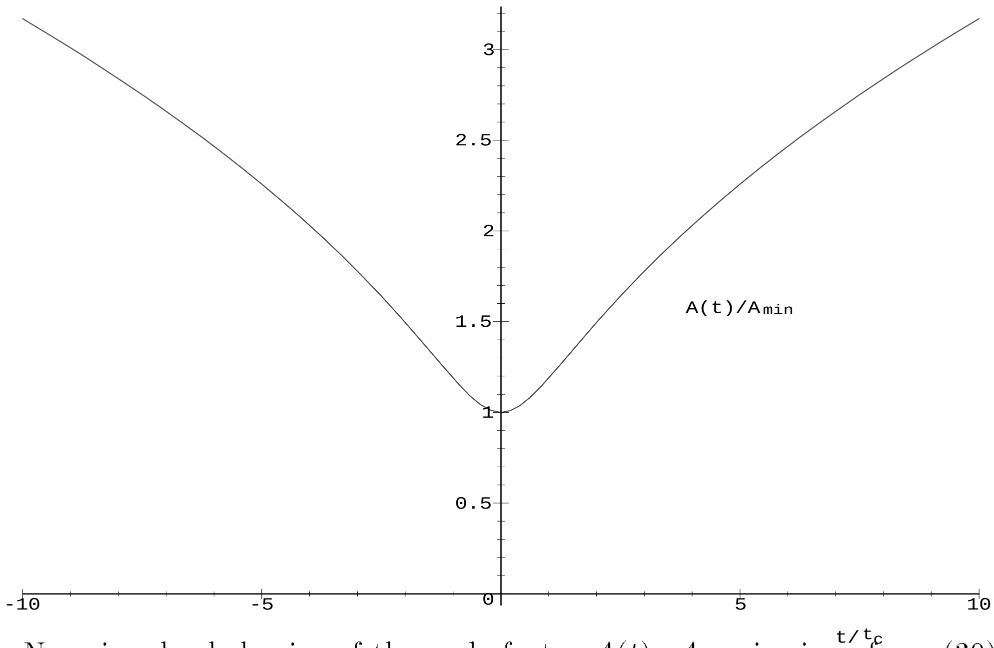

Figure 1: Non-singular behavior of the scale-factor $A(t)$. A_{min} is given from (30) and t_c from (32).

4 Conclusions

Heisenberg and Euler [1] have calculated the effective Lagrangian density to deal with the non-linear electrodynamic effects induced by virtual electron-positron pairs. This is valid for frequencies $\nu < m_e c^2/h$. One should wonder if the use of such a correction in the framework of Einstein general relativity, as we did before, belongs to the above domain of its applicability. In order to show that this is indeed the case, let us write the Heisenberg-Euler effective Lagrangian (17) in the form

$$L = -\frac{1}{2} H^2 \left(1 - 2\,\mu\, H^2\right). \tag{33}$$

The limit of validity of this expression consists in the range

$$1 - 2\,\mu\, H^2 \geq 0. \tag{34}$$

In the history of the universe we described above the scale-factor $A(t)$ attains a minimum. From equation (26) for the averaged field H it then follows that the magnetic field reaches its maximum value, for which the equality in (34) holds. This can be seen by a direct inspection on the positivity of the energy density[7] of the field. We interpret such result as a

[7]See equation (20) and Figure 2.

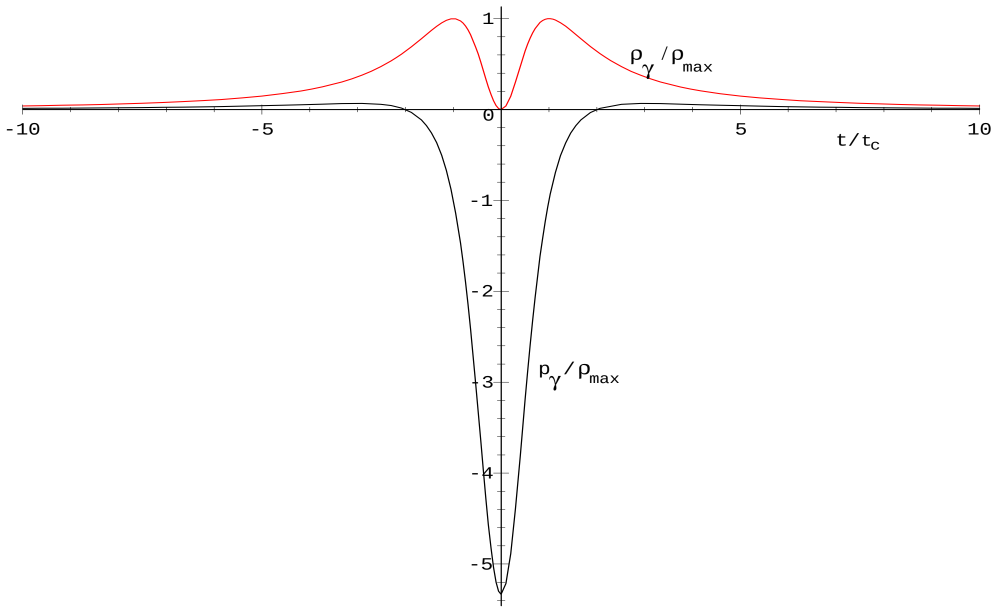

Figure 2: Time dependence of the electromagnetic energy density ρ_γ and pressure p_γ. ρ_{max} is given from (31) and t_c from (32).

conspiracy of the domain of applicability of quantum Heisenberg-Euler correction together with the existence of a minimum value for the radius of the universe, thus exhibiting the consistency of the model.

We note that the non-singular character of the cosmological solution we presented here has to be attributed to the fact that for $t < t_1$ the quantity $\rho + 3p$ becomes negative, where

$$t_1^2 = \frac{6\mu}{k\,c^2}. \tag{35}$$

Let us make another comment concerning the influence of the presence of other kinds of energy in the universe. Beside photons there are plenty of other particles. Physics of the early universe deals with various sort of matter. In the standard framework they are treated in terms of a fluid with energy density ρ_ν, which satisfies an ultrarelativistic equation of state $p_\nu = \rho_\nu/3$. Adding the contribution of this kind of matter to the average energy-momentum tensor $< T_{\mu\nu} >$ of the photons it follows that ρ_ν is proportional to the inverse of the fourth power of the scale-factor

$$\rho_\nu = \rho_\nu^{(o)} A^{-4}. \tag{36}$$

This result allows us to treat such extra matter as nothing but a reparametrization of the constants H_o and μ into $\hat{H}_o$ and $\hat{\mu}$, given by

$$\hat{H}_o^2 \;=\; H_o^2 + 2\,\rho_\nu^{(o)}, \tag{37}$$

$$\hat{\mu} = \frac{H_o^4}{(H_o^2 + 2\,\rho_\nu^{(o)})^2}\,\mu. \tag{38}$$

The net effect of this is just to diminish the value of A_{min} as

$$\hat{A}_{min} = \left(\frac{H_o}{H_o^2 + 2\rho_\nu^{(o)}}\right)^{1/4} A_{min}. \tag{39}$$

Therefore, it turns out that the phenomenon of reversing the sign of the expansion factor Θ due to the high negative pressure of the photons is not essentially modified by the ultrarelativistic gas. Only a fluid possessing density of energy ρ_{exotic} that behaves as $\rho_{exotic} = \rho_{exotic}^{(o)}\,A^n$ with $n \leq -8$ could modify the above result. However, this would be a very unrealistic case.

Let us make a final remark. Standard cosmology contains a global singularity. This is a consequence of treating matter and the gravitational field in a classical framework [4]. However, the behavior of matter should be corrected by quantum effects. The present paper has analyzed the consequences of this in the realm of a spatially homogeneous and isotropic cosmology in a semi-classical approach. The main result concerns the avoidance of a singular origin of the universe, which becomes a net consequence of the quantum modification of the evolution of the averaged electromagnetic field. Let us point out that such regular behavior allows us to extrapolate back our solution, since it is symmetric for time reversion, as one can see from (28). As a consequence of this the cosmological scenario presented here deserves further investigation.

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