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Algebraic Renormalization Perturbative twisted considerations on topological Yang-Mills theory and on N=2 supersymmetric gauge theories

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Abstract

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1 Introduction

The aim of these notes is to provide a simple and pedagogical (as much as possible) introduction to what is nowadays commonly called *Algebraic Renormalization* [1]. As the name itself let it understand, the *Algebraic Renormalization* gives a systematic set up in order to analyse the quantum extension of a given set of classical symmetries. The framework is purely algebraic, yielding a complete characterization of all possible anomalies and invariant counterterms without making use of any explicit computation of the Feynman diagrams. This goal is achieved by collecting, with the introduction of suitable ghost fields, all the symmetries into a unique operation summarized by a generalized Slavnov-Taylor (or master equation) identity which is the starting point for the quantum analysis. The Slavnov-Taylor identity allows to define a nilpotent operator whose cohomology classes in the space of the integrated local polynomials in the fields and their derivatives with dimensions bounded by power counting give all nontrivial anomalies and counterterms. In other words, the proof of the renormalizability is reduced to the computation of some cohomology classes.

However, before going any further, let us make some necessary remarks on the limitations of the method. The *Algebraic Renormalization* applies basically to the perturbative regime, meaning that the quantum extension of the theory is constructed order by order in the loop parameter expansion $\hbar$. Aspects regarding the convergence and the resummation of the perturbative series are not considered and cannot, in general[1], be handled within this algebraic setup. In spite of its perturbative character the *Algebraic Renormalization*, being based only on the locality and power counting properties of the renormalization theory, does not rely on the existence of any regularization preserving the symmetries. This means that the algebraic proofs of the renormalizability extend to all orders of perturbation theory and are independent from the regularization scheme. This very important feature gives to the *Algebraic Renormalization* a very large domain of applicability. In practice, one can include almost all known power counting renormalizable theories in flat space-time covering, in particular, those for which no invariant regularization is known. It is also worthwhile to mention that, besides the pure characterization of the anomalies and of the invariant counterterms, the *Algebraic Renormalization* plays a quite useful role in the study of other aspects of field theory which, although not touched in these lectures, are object of a fruitful research activity. Let us mention, for instance:

- **The nonrenormalization theorems and the ultraviolet finiteness**

The aim here is to establish nonrenormalization theorems for anomalies (as the Adler-Bardeen nonrenormalization theorem of the gauge anomaly [2]) and to provide a classification of models which have vanishing β-function to all orders of perturbation theory. Examples of the latters are given by some four dimensional gauge theories with $N = 1, 2, 4$ supersymmetry [3, 4] and by some topological field theories [5, 6].

- **The characterization of new symmetries**

This aspect, deeply related to the previous one, consists of the search of additional unknown symmetries (eventually linearly broken) which may be responsible for the finiteness

[1] With the exception of few two dimensional models.

properties displayed by a particular model. This problem, related to the existence of cohomology classes[2] *with negative ghost number, allows for a cohomological (re)interpretation of the Noether theorem [7]. Examples of such additional symmetries are provided by the so called vector supersymmetry of the topological theories and by the Landau ghost equation responsible for the ultraviolet finiteness of a wide class of invariant local field polynomials in the Yang-Mills theories [8].*

- **The geometrical aspects**

As one can easily expect, the number of applications related to the geometrical aspects of the BRST transformations and of the anomalies is very large. We shall limit ourselves only to mention a particular feature which is focusing our attention since a few years, namely the possibility of encoding all the transformations of all the fields and antifields (or BRST external sources) into a unique equation which takes the form of a generalized zero curvature condition [9]. The zero curvature formalism allows us to obtain in a very simple way the BRST cohomology classes and improves our understanding of the role of the antifileds.

Having (hopefully) motivated the usefulness of the *Algebraic Renormalization,* let us now briefly describe the plan of these notes. We shall adopt here the point of view of not entering into the technical aspects concerning the antifields formulation and the computation of the BRST cohomology classes, limiting ourselves only to state the main results and reminding the reader to the several reviews and books appeared recently in the literature *[1, 10]*. Rather, we shall work out in detail the renormalization of a model rich enough to cover all the main aspects of the algebraic method. The example we will refer to is the four dimensional euclidean topological Yang-Mills (TYM) theory proposed by E. Witten *[11]* at the end of the eighty's. Besides the mere fact that TYM is a continuous source of investigations, we shall see that this model possesses a very interesting structure, requiring a highly nontrivial quantization and displaying peculiar cohomological properties. This is due to the deep relationship with the $N = 2$ euclidean supersymmetric Yang-Mills theory. In fact TYM in flat space-time can be actually seen as the twisted version of the $N = 2$ Yang-Mills theory in the Wess-Zumino gauge *[12]*, the Witten's fermionic symmetry being identified with the singlet generator of the twisted $N = 2$ supersymmetric algebra. Furthermore, by means of the introduction of appropriate constant ghosts associated to the twisted $N = 2$ generators, we shall be able to quantize the model by taking into account both the gauge invariance and the $N = 2$ supersymmetry, overcoming the well known difficulties of the $N = 2$ susy algebra in the Wess-Zumino gauge *[4, 13]*. Concerning now the BRST cohomology, we will have the opportunity of checking how the twisted $N = 2$ susy algebra can be used to obtain in a straightforward way the relevant cohomology classes. In particular, it will turn out that the origin of the TYM action can be traced back to the invariant polynomial $tr\,(\phi^2)$, ϕ being one of the scalar fields of the model. This relation has a very appealing meaning. Needless to say, the $N = 2$ susy YM theory is indeed the corner stone of the duality mechanism recently discussed by N. Seiberg and E. Witten *[14]*, who used in fact $tr\,(\phi^2)$ in order to label the

[2]It turns out that in some cases also certain exact cocycles with negative ghost number become relevant *[8]*.

different vacua of the theory. Finally, we will show that the requirement of analyticity *[15]* in the constant ghosts of the twisted $N = 2$ supersymmetry can be deeply related to the so called *equivariant cohomology* proposed by R. Stora et al.*[16, 17]* in order to deal with the topological theories of the cohomological type.

2 Generalities on the Slavnov-Taylor identity and on Cohomology

2.1 Classical action and symmetry content

The starting point of our analysis consists of assigning a set of fields $(A_\mu, \{\lambda\})$, A_μ and $\{\lambda\}$ being respectively a gauge field and a set of spinor and scalar matter fields, and a classical gauge invariant action S_{inv}

$$
\begin{aligned}
S_{inv} &= \int d^4x \mathcal{L}(A,\lambda) , \\
\delta^g_\epsilon S_{inv} &= 0 ,
\end{aligned}
\tag{2.1}
$$

with

$$
\begin{aligned}
\delta^g_\epsilon A_\mu &= -D_\mu \epsilon = -\left(\partial_\mu \epsilon + [A_\mu, \epsilon]\right) , \\
\delta^g_\epsilon \lambda &= [\epsilon, \lambda] ,
\end{aligned}
\tag{2.2}
$$

where $\mathcal{L}(A,\lambda)$ is a power counting renormalizable local polynomial in the fields and their derivatives and δ^g_ϵ is the generator of the gauge transformations with local infinitesimal parameter $\epsilon(x)$. All the fields are Lie algebra valued, *i.e.* $A_\mu = A^a_\mu T^a$ and $\lambda = \lambda^a T^a$, the generators of the corresponding gauge group G being chosen to be antihermitians $\left[T^a, T^b\right] = f^{ab}_{\ \ c} T^c$, with $f^{ab}_{\ \ c}$ the structure constants.

Remark 1 *The action* (2.1) *refers to the standard case of a model containing only gauge and matter fields, implying in particular that the only degeneracy in order to compute the propagators is the one associated with the transverse quadratic term in the gauge fields following from the usual Yang-Mills term $tr(F_{\mu\nu}F^{\mu\nu})$. Although the algebraic method can be applied to more sophisticated cases (p-forms,...), the field content of the expression* (2.1) *covers a very large class of models, including the TYM theory.*

Besides the gauge invariance, the classical action (2.1) will be assumed to be left invariant by a set of additional global transformations whose corresponding generators $\{\delta_A, \ A = 1, ...\}$

$$
\delta_A S_{inv} = 0 , \tag{2.3}
$$

give rise, together with the gauge generator δ^g_ϵ, to the following algebraic relations

$$
\begin{aligned}
[\delta_A, \delta_B] &= -C^C_{AB}\delta_C + (\text{matter eqs. of motion}) + (\text{gauge transf.}) , \\
[\delta_A, \delta^g_\epsilon] &= 0 ,
\end{aligned}
\tag{2.4}
$$

where $[\cdot,\cdot]$ denotes the graded commutator and C^C_{AB} are appropriate constant coefficients. We do not specify further the nature of the indices A, B of the generators δ_A which, according to the particular model considered, may refer to spinor indices, to Lorentz indices, to group indices, etc... The generators δ_A may act nonlinearly on the fields. The fact that they are related to global invariances means that, unlike the gauge parameter $\epsilon(x)$ of eq.(2.2), the corresponding infinitesimal parameters entering the δ_A-transformations do not depend on space-time. The algebraic structure (2.4) is typical of supersymmetric gauge theories in the Wess-Zumino gauge *[4, 13, 18, 19]* and of many topological theories including, in particular, TYM.

The first step towards the construction of a classical Slavnov-Taylor identity consists of turning the infinitesimal parameters associated to the generators $(\delta^g_\epsilon,\ \delta_A)$ into suitable ghosts. The local gauge parameter $\epsilon(x)$ will be thus replaced by the Faddeev-Popov ghost $c(x)$ and δ^g_ϵ will give rise to the well known nilpotent operator s corresponding to the gauge transformations

$$
\begin{aligned}
sA_\mu &= -D_\mu c \,, \\
s\lambda &= [c,\lambda] \,, \\
sc &= c^2 \,, \\
s^2 &= 0 \,.
\end{aligned} \tag{2.5}
$$

Concerning now the infinitesimal parameters associated to the δ_A 's, they will be replaced by global constant ghosts ε^A which will be taken as commuting or anticommuting according to the bosonic or fermionic character of the corresponding generator. In addition, it can be shown *[4, 13, 18, 19]* that one may define the action of s and of the δ_A's on the Faddeev-Popov ghost c and on the global ghosts ε^A in such a way that the extendend BRST operator given by

$$
Q := s + \varepsilon^A \delta_A + \frac{1}{2} C^C_{AB} \varepsilon^A \varepsilon^B \frac{\partial}{\partial \varepsilon^C} \,, \tag{2.6}
$$

has ghost number one and enjoys the following important property

$$
Q^2 = 0 \quad (\mathrm{mod\,.\ matter\ eqs.\ of\ motion}) \,. \tag{2.7}
$$

Of course,

$$
QS_{inv} = 0 \,, \tag{2.8}
$$

showing that the operator Q collects together all the symmetries of the action (2.1).

The second step in the construction of the Slavnov-Taylor identity is the introduction of a gauge fixing term S_{gf}. To this purpose we introduce an antighost $\overline{c}$ and a Lagrangian multiplier b transforming as

$$
\begin{aligned}
Q\overline{c} &= b + (\varepsilon - \text{dependent terms}) \,, \\
Qb &= 0 + (\varepsilon - \text{dependent terms}) \,,
\end{aligned} \tag{2.9}
$$

where the $\varepsilon-$dependent terms are chosen in such a way that

$$Q^2\overline{c} = Q^2 b = 0 \ . \tag{2.10}$$

Therefore, recalling from eq.(2.7) that $Q^2 A_\mu = 0$, it follows that a linear covariant Landau type gauge fixing term

$$\begin{aligned} S_{gf} &= Q \int d^4x \ tr(\overline{c}\partial A) & (2.11) \\ &= tr \int d^4x \ \left(b\partial A + \overline{c}\partial^\mu D_\mu c + (\varepsilon - \text{dep. terms})\right) \ , & (2.12) \end{aligned}$$

provides a gauge fixed action $(S_{inv} + S_{gf})$ which is invariant under Q, *i.e.*

$$Q(S_{inv} + S_{gf}) = 0 \ . \tag{2.13}$$

The above equation means that the gauge fixing procedure has been carried out in a way which is compatible with all the additional global symmetries δ_A of the classical action (2.1). We also remark that the expression (2.11) belongs to a class of linear covariant gauge conditions which can be proven to be renormalizable *[1, 13]*.

We are now ready to write down the Slavnov-Taylor identity. Denoting with $\{\varphi^i\} = (A_\mu, \{\lambda\}, c, \overline{c}, b)$ all the local fields of $(S_{inv} + S_{gf})$, we associate to each field φ^i of ghost number N_{φ^i} and dimension d_{φ^i} the corresponding antifield φ^{*i} with ghost number $-(1 + N_{\varphi^i})$ and dimension $(4 - d_{\varphi^i})$, and we introduce the antifield dependent action

$$S_{ext} = tr \int d^4x \left(\varphi^{*i} Q \varphi^i + \omega_{ij} \varphi^{*i} \varphi^{*j} \right) \ . \tag{2.14}$$

The first term in the expression (2.14) is needed in order to define the nonlinear Q-variations of the fields φ^i as composite operators. The second term, quadratic in the antifields, allows to take care of the fact that the extended operator Q of eq.(2.6) is nilpotent only modulo the matter equations of motion. The coefficients ω_{ij}, depending in general from both fields φ^i and global ghosts ε^A, are fixed by requiring that the following identity holds

$$\mathcal{S}(\Sigma) = 0 \ , \tag{2.15}$$

with

$$\mathcal{S}(\Sigma) = \int d^4x \frac{\delta\Sigma}{\delta\varphi^i} \frac{\delta\Sigma}{\delta\varphi^{*i}} \ + \ \frac{1}{2} C^C_{AB} \varepsilon^A \varepsilon^B \frac{\partial\Sigma}{\partial\varepsilon^C} \ , \tag{2.16}$$

and Σ being the complete action

$$\Sigma = S_{inv} \ + \ S_{gf} \ + \ S_{ext} \ . \tag{2.17}$$

The equation (2.15) is called the Slavnov-Taylor identity and will be the starting point for the quantum analysis.

Remark 2 *Although higher order terms (cubic, etc.,..) in the antifields* φ^{*i} *may be required in the action* (2.14) *in order to obtain the Slavnov-Taylor identity, they will not be needed in the example considered here.*

2.2 Cohomology and renormalizability: anomalies and stability of the classical action

We face now the problem of the quantum extension of the classical Slavnov-Taylor identity (2.15), *i.e.* of the perturbative construction of a renormalized vertex functional[3] Γ

$$\Gamma = \Sigma + O(\hbar) , \tag{2.18}$$

fulfilling the quantum version of eq.(2.15), *i.e.*

$$\mathcal{S}(\Gamma) = 0 , \tag{2.19}$$

which would imply that all the classical symmetries, *i.e.* the gauge and the δ_A-invariances, can be implemented at the quantum level without anomalies. In order to detect the presence of possible anomalies, let us suppose that eq.(2.19) breaks down at a certain order $\hbar^n$, $(n \geq 1)$,

$$\mathcal{S}(\Gamma) = \hbar^n \Delta + O(\hbar^{n+1}) , \tag{2.20}$$

where, from the power counting and locality properties of the renormalized perturbation theory, the breaking Δ is an integrated local polynomial in the fields, antifields and global ghosts with ghost number one.

The breaking Δ is easily seen to be constrained by a consistency condition. In fact, defining the linearized operator $\mathcal{B}_{\mathcal{F}}$

$$\mathcal{B}_{\mathcal{F}} = \int d^4x \left(\frac{\delta \mathcal{F}}{\delta \varphi^i} \frac{\delta}{\delta \varphi^{*i}} + \frac{\delta \mathcal{F}}{\delta \varphi^{*i}} \frac{\delta}{\delta \varphi^i} \right) + \frac{1}{2} C^C_{AB} \varepsilon^A \varepsilon^B \frac{\partial}{\partial \varepsilon^C} , \tag{2.21}$$

$\mathcal{F}$ being an arbitrary functional with even ghost number, we have the following exact algebraic relation

$$\mathcal{B}_{\mathcal{F}} \, \mathcal{S}(\mathcal{F}) = 0 . \tag{2.22}$$

In addition, if $\mathcal{F}$ satisfies the Slavnov-Taylor identity

$$\mathcal{S}(\mathcal{F}) = 0 , \tag{2.23}$$

it follows that the operator $\mathcal{B}_{\mathcal{F}}$ is nilpotent,

$$\mathcal{B}_{\mathcal{F}} \mathcal{B}_{\mathcal{F}} = 0 . \tag{2.24}$$

In particular, from the classical Slavnov-Taylor identity (2.15) it follows that the linearized operator $\mathcal{B}_\Sigma$ corresponding to the classical action Σ is nilpotent

$$\begin{aligned} \mathcal{B}_\Sigma &= \int d^4x \left(\frac{\delta \Sigma}{\delta \varphi^i} \frac{\delta}{\delta \varphi^{*i}} + \frac{\delta \Sigma}{\delta \varphi^{*i}} \frac{\delta}{\delta \varphi^i} \right) + \frac{1}{2} C^C_{AB} \varepsilon^A \varepsilon^B \frac{\partial}{\partial \varepsilon^C} , \\ \mathcal{B}_\Sigma \mathcal{B}_\Sigma &= 0 . \end{aligned} \tag{2.25}$$

[3] Γ is the generator of the 1PI Green's functions.

Definition 1 *The cohomology classes of the operator $\mathcal{B}_\Sigma$ in the space of the local integrated polynomials in the fields φ^i, antifields φ^{*i}, global ghosts ε^A and their space-time derivatives, are defined as the solutions Ξ of the consistency condition*

$$\mathcal{B}_\Sigma \Xi = 0 \ , \tag{2.26}$$

which are not of the form

$$\Xi = \mathcal{B}_\Sigma \widehat{\Xi} \ , \tag{2.27}$$

with $\left(\Xi, \widehat{\Xi}\right)$ local integrated polynomials in the fields, antifields and global ghosts. Solutions of the type (2.27) *are called exact and can be proven to be physically irrelevant. The cohomology of $\mathcal{B}_\Sigma$ is called empty if all solutions of eq.*(2.26) *are of the type* (2.27).

Acting now on both sides of eq.(2.20) with the operator $\mathcal{B}_\Gamma$

$$\mathcal{B}_\Gamma = \int d^4x \left(\frac{\delta\Gamma}{\delta\varphi^i}\frac{\delta}{\delta\varphi^{*i}} + \frac{\delta\Gamma}{\delta\varphi^{*i}}\frac{\delta}{\delta\varphi^i} \right) + \frac{1}{2} C^C_{\ AB}\varepsilon^A\varepsilon^B \frac{\partial}{\partial\varepsilon^C} = \mathcal{B}_\Sigma + O(\hbar) \ , \tag{2.28}$$

and making use of

$$\mathcal{B}_\Gamma \mathcal{S}(\Gamma) = 0 \ , \tag{2.29}$$

we get, to the lowest order in $\hbar$, the consistency condition

$$\mathcal{B}_\Sigma \Delta = 0 \ . \tag{2.30}$$

The latter is nothing but a cohomology problem for the operator $\mathcal{B}_\Sigma$ in the sector of the integrated local field polynomials with ghost number one. Let us now suppose that the most general solution of the consistency condition (2.30) can be written in the exact form

$$\Delta = \mathcal{B}_\Sigma \widehat{\Delta} \ , \tag{2.31}$$

for some local integrated polynomial $\widehat{\Delta}$ with ghost number zero. Therefore, the redefined vertex functional

$$\overline{\Gamma} = \Gamma - \hbar^n \widehat{\Delta} \ , \tag{2.32}$$

obeys the Slavnov-Taylor identity

$$\mathcal{S}(\overline{\Gamma}) = O(\hbar^{n+1}) \ . \tag{2.33}$$

This equation means that if the breaking term Δ is cohomologically trivial, one can always extend the Slavnov-Taylor identity to the order $\hbar^n$. The procedure can be iterated, allowing us to conclude that if the cohomology of $\mathcal{B}_\Sigma$ is empty in the sector of ghost number one it is always possible to implement at the quantum level the classical Slavnov-Taylor identity (2.15). In this case the model is said to be anomaly free. On the contrary, when the cohomology of $\mathcal{B}_\Sigma$ is not empty, *i.e.*

$$\begin{aligned} \Delta &= r\mathcal{A} + \mathcal{B}_\Sigma \widehat{\Delta} \ , \\ \mathcal{A} &\neq \mathcal{B}_\Sigma \, \widehat{\mathcal{A}} \ , \end{aligned} \tag{2.34}$$

with r an arbitrary coefficient and $\widehat{\mathcal{A}}$ some local field polynomial, it is not possible to compensate the breaking term by adding suitable local terms to the vertex functional. The best that one can do is just to reabsorb the trivial part $\mathcal{B}_\Sigma\widehat{\Delta}$ of eq.(2.34),

$$\mathcal{S}(\Gamma) = r\hbar^n \mathcal{A} + O(\hbar^{n+1}) \ . \tag{2.35}$$

In this case one speaks of an anomaly, meaning that the classical symmetries cannot be implemented at the quantum level.

Remark 3 *It is important here to underline that the algebraic method does not provide the numerical value of the coefficient r. This means that the anomaly $\mathcal{A}$ appearing in the left hand side of eq.(2.35) is only a potential anomaly, whose existence has to be confirmed with an explicit computation of r. Moreover, the vanishing of the coefficient r does not imply the absence of the anomaly. It only means that the anomaly is absent at the order $\hbar^n$. Possible anomalous contributions are expected at higher orders, unless one is able to establish a nonrenormalization theorem. This is the case, for instance, of the Adler-Bardeen nonrenormalization theorem of the gauge anomaly which states that if the coefficient r is vanishing at the one loop order, it will vanish at all orders [2].*

Having discussed the characterization of the possible anomalous terms, let us turn now to the analysis of the invariant counterterms, *i.e.* of the local ambiguities which affect the Slavnov-Taylor identity. In fact, if Γ is a vertex functional which satisfies the Slavnov-Taylor identity to the order $\hbar^n$

$$\mathcal{S}(\Gamma) = O(\hbar^{n+1}) \ , \tag{2.36}$$

then adding to Γ any local invariant field polynomial $\hbar^n\Theta$ with the same quantum numbers and dimension of the classical action Σ

$$\mathcal{B}_\Sigma \Theta = 0 \ , \tag{2.37}$$

the resulting vertex functional still satisfies eq.(2.36), *i.e.*

$$\begin{aligned} \Gamma_\Theta &= \Gamma + \hbar^n \Theta \ , \\ \mathcal{S}(\Gamma_\Theta) &= O(\hbar^{n+1}) \ . \end{aligned} \tag{2.38}$$

In other words, the Slavnov-Taylor identity characterizes the vertex functional Γ only up to local invariant polynomials Θ which can be freely added to each order of perturbation theory. The vertex functional Γ will be uniquely fixed only once the most general solution of eq.(2.37) has been given and a suitable set of renormalization conditions has been imposed. Again, eq.(2.37) shows that the search of the invariant counterterms is a problem of cohomology of $\mathcal{B}_\Sigma$ in the space of the integrated local field polynomials with ghost number zero. In general, Θ will be of the form

$$\Theta = \Theta^{coh} + \mathcal{B}_\Sigma \widehat{\Theta} \ , \tag{2.39}$$

with Θ^{coh} identifying the nontrivial cohomology sectors of $\mathcal{B}_\Sigma$.

Let us now introduce the notion of *stability* of the classical action. The complete action Σ of eq.(2.17) is said to be *stable* if the most general local invariant counterterm can be reabsorbed by means of a redefinition of the fields and of the parameters, *i.e.* denoting with $\{g\}$ the parameters of Σ (coupling constants, masses, gauge parameters, etc....) we have

$$\Sigma(\varphi^i, \varphi^{*i}, \varepsilon^A, g) + \hbar^n \Theta = \Sigma(\varphi_0^i, \varphi_0^{*i}, \varepsilon_0^A, g_0) + O(\hbar^{n+1}) \ , \tag{2.40}$$

with

$$\begin{aligned} \varphi_0^i &= \varphi^i(1 + \hbar^n \varsigma_\varphi) + O(\hbar^{n+1}) \ , \\ \varphi_0^{*i} &= \varphi^{*i}(1 + \hbar^n \varsigma_{\varphi^*}) + O(\hbar^{n+1}) \ , \\ \varepsilon_0^A &= \varepsilon^A(1 + \hbar^n \varsigma_\varepsilon) + O(\hbar^{n+1}) \ , \\ g_0 &= g(1 + \hbar^n \varsigma_g) + O(\hbar^{n+1}) \ , \end{aligned} \tag{2.41}$$

$(\varsigma_\varphi \varsigma_{\varphi^*} \varsigma_\varepsilon \varsigma_g)$ being *renormalization constants*. Let us also remark that the knowledge of the nontrivial counterterms Θ^{coh} has a very important meaning. One can show indeed that the elements of Θ^{coh} correspond to the renormalization of the physical parameters of Σ, *i.e.* of the coupling constants and of the masses, while the trivial term $\mathcal{B}_\Sigma \widehat{\Theta}$ turns out to be related to the unphysical renormalization of the field amplitudes and of the gauge parameters. In addition, the renormalized Green's functions with the insertion of local gauge invariant composite operators can be proven to be independent from the parameters belonging to the trivial part $\mathcal{B}_\Sigma \widehat{\Theta}$.

Definition 2 *The classical action Σ satisfying the classical Slavnov-Taylor identity* (2.15) *is said to be renormalizable if the following two items are fulfilled, namely*
i) there are no anomalies, i.e.

$$\begin{aligned} \Sigma &\rightarrow \Gamma = \Sigma + O(\hbar) \ , \\ \mathcal{S}(\Gamma) &= 0 \ , \end{aligned} \tag{2.42}$$

ii) the action is stable.
The absence of anomalies ensures that the classical symmetries can be implemented at the quantum level, while the stability means that all possible local countertems compatible with the symmetry content can be reabsorbed by redefining the fields and the parameters of the original action Σ.

In summary, we have seen that the renormalizability of a given classical model can be established by looking at the cohomology of the operator $\mathcal{B}_\Sigma$ in the sector of the integrated local field polynomials with ghost number respectively one (anomalies) and zero (counterterms).

Remark 4 *We should also mention that in the case in which one (or more) of the global generators δ_A acts linearly on the quantum fields φ^i, the dependence of the quantum action Γ from the corresponding global ghost turns out to be uniquely fixed already at the classical level. This means that, denoting with δ^l_C the linearly realized global generator, for the corresponding global ghost ε^{lC} we may write the following classical identity*

$$\frac{\partial\Sigma}{\partial\varepsilon^{lC}} = \Delta^l_C \ = \int d^4x \mathcal{M}^{li}_C \varphi^i \ , \tag{2.43}$$

where $\mathcal{M}^{li}_C$ denote a set of generalized coefficients depending only on the antifields φ^{*i}, on the global ghosts ε^A and on their space-time derivatives. The breaking Δ^l_C, being linear in the quantum fields φ^i, is thus a classical breaking and will be not affected by the quantum corrections [1]. Therefore the equation (2.43) *has the meaning of a linearly broken Ward identity which, once extended at the quantum level, will imply that the higher order terms of the renormalized vertex functional Γ*

$$\Gamma = \Sigma + \sum_{j=1}^{\infty} \hbar^j \Gamma^j \ , \tag{2.44}$$

do not depend from the global ghost ε^{lC}. In fact, from

$$\frac{\partial\Gamma}{\partial\varepsilon^{lC}} = \Delta^l_C \ , \tag{2.45}$$

it follows that

$$\frac{\partial\Gamma^j}{\partial\varepsilon^{lC}} = 0 \ , \qquad j \geq 1 \ , \tag{2.46}$$

due to equation (2.43)*. This result shows that the dependence of the theory from the global ghosts corresponding to linearly realized symmetries is completely fixed by the classical equation* (2.43)*. As we shall see later on in the analysis of TYM, this will be the case of the global ghost associated to the space-time translation invariance.*

2.3 Some useful result on cohomology

Let us conclude this short introduction to the *Algebraic Renormalization* by stating some useful result on the computation of the cohomology of the operator $\mathcal{B}_\Sigma$. Let us begin by underlining the important role played by the functional space the operator $\mathcal{B}_\Sigma$ acts upon. Different functional spaces yield, in general, different cohomology classes for $\mathcal{B}_\Sigma$. In the previous Subsection we have adopted as basic functional space for the operator $\mathcal{B}_\Sigma$ the space of the integrated local polynomials in the fields φ, antifields φ^*, global ghosts ε and their space-time derivatives. We emphasize here that the choice of this functional space follows directly from the locality properties of the renormalized perturbation theory.

Remark 5 *Concerning in particular the global ghosts ε^A it will be very easy to check that the Feynman rules stemming from the quantized TYM action yield a perturbative expansion which is analytic in the ε^A's. This analyticity property, whose precise mathematical meaning is that of a formal power series, will be of great importance in order to understand the BRST cohomology classes of TYM.*

On the space of the integrated local field polynomials the operator $\mathcal{B}_\Sigma$ has a natural decomposition as

$$\begin{aligned} \mathcal{B}_\Sigma &= b_0 + b_R \,, \\ b_0^2 &= 0 \,, \end{aligned} \tag{2.47}$$

b_0 being the so called abelian approximation.

Example 1 *Let s be the nilpotent operator of the eq.(2.5) acting on the space of the local polynomials in the variables $(A_\mu, \{\lambda\}, c)$. Therefore*

$$s = s_0 + s_R \,, \tag{2.48}$$

with

$$s_0 A_\mu = -\partial_\mu c \,, \qquad s_0 c = 0 \,, \qquad s_0 \lambda = 0 \,, \qquad s_0^2 = 0 \,, \tag{2.49}$$

and

$$s_R A_\mu = [c, A_\mu] \,, \qquad s_R c = c^2 \,, \qquad s_R \lambda = [c, \lambda] \,. \tag{2.50}$$

One sees thus that s_0 corresponds in fact to the abelian approximation in which all the commutators $[\cdot,\cdot]$ have been ignored.

The usefulness of the decomposition (2.47) relies on a very general theorem stating that the cohomology of the complete operator $\mathcal{B}_\Sigma$ is isomorphic to a subspace of the cohomology of the operator b_0. In most cases this result allows to obtain a large amount of informations on the cohomology of $\mathcal{B}_\Sigma$ by analysing that of the simpler operator b_0. Let us also remark that the aforementioned theorem, although referred here to the abelian approximation, is valid for other kinds of decomposition of the operator $\mathcal{B}_\Sigma$.

Let us give now a second important result, known as the doublets theorem. A pair of fields (u, v) is called a BRST doublet if

$$b_0 u = v \,, \qquad b_0 v = 0 \,. \tag{2.51}$$

It can be shown that if two fields appear in a BRST doublet, then the cohomology of b_0, and therefore that of $\mathcal{B}_\Sigma$, does not depend on these fields. This second result allows to eliminate from the game all the fields appearing as BRST doublets, greatly simplifying the computation of the cohomology classes.

More details on cohomology can be found in Chapter 5 of ref. *[1]*.

3 Witten's topological Yang-Mills theory

3.1 The action and its fermionic symmetry

Topological Yang-Mills theory has been proposed at the end of the eighty's with the aim of providing a field theory framework for the topological invariants of euclidean four manifolds *[11]*. The model allows in fact to define a set of observables, *i.e.* local field polynomials integrated over suitable homology cycles, whose correlation functions turn out to be deeply related with the so called Donaldson invariants *[20]*.

Although TYM can be formulated on smooth four manifolds, we shall consider here the case of the flat euclidean space-time[4]. In fact our attitude in these lectures is to interpret TYM theory as a twisted version of the conventional $N = 2$ supersymmetric euclidean Yang-Mills theory, as it will become clear in the next Sections where the relationship with the cohomological formulations of Labastida-Pernici *[24]* and Baulieu-Singer *[25]* will be also discussed. Following the original Witten's work, the TYM classical action is given by

$$\begin{aligned}\mathcal{S}_{TYM} \;=\; & \frac{1}{g^2} tr \int d^4x \; \Big(\frac{1}{2} F^{+}_{\mu\nu} F^{+\mu\nu} \; - \chi^{\mu\nu} (D_\mu \psi_\nu - D_\nu \psi_\mu)^{+} \\ & + \eta D_\mu \psi^\mu \; - \frac{1}{2} \overline{\phi} D_\mu D^\mu \phi \; + \frac{1}{2} \overline{\phi} \left\{ \psi^\mu, \psi_\mu \right\} \\ & - \frac{1}{2} \phi \left\{ \chi^{\mu\nu}, \chi_{\mu\nu} \right\} \; - \frac{1}{8} \left[\phi, \eta \right] \eta - \frac{1}{32} \left[\phi, \overline{\phi} \right] \left[\phi, \overline{\phi} \right] \Big) \;, \end{aligned} \tag{3.52}$$

where g is the *unique* coupling constant and $F^{+}_{\mu\nu}$ is the self-dual part of the Yang-Mills field strength

$$\begin{aligned} F^{+}_{\mu\nu} &= F_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \;, \qquad \widetilde{F}^{+}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{+\rho\sigma} = F^{+}_{\mu\nu} \;, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \;, \end{aligned} \tag{3.53}$$

$\varepsilon_{\mu\nu\rho\sigma}$ being the totally antisymmetric Levi-Civita tensor

$$\varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\rho\sigma\tau\lambda} = 2(\delta^\tau_\mu \delta^\lambda_\nu - \delta^\tau_\nu \delta^\lambda_\mu) \;. \tag{3.54}$$

The three fields $(\chi_{\mu\nu}, \psi_\mu, \eta)$ in the expression (3.52) are anticommuting with $\chi_{\mu\nu}$ self-dual

$$\widetilde{\chi}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \chi^{\rho\sigma} = \chi_{\mu\nu} = -\chi_{\nu\mu} \;. \tag{3.55}$$

Accordingly, the term $(D_\mu \psi_\nu - D_\nu \psi_\mu)^{+}$ in (3.52) has to be understood as

$$(D_\mu \psi_\nu - D_\nu \psi_\mu)^{+} = (D_\mu \psi_\nu - D_\nu \psi_\mu) + \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} (D^\rho \psi^\sigma - D^\sigma \psi^\rho) \;, \tag{3.56}$$

[4] See *[21]* for the computation of some topological invariant associated to submanifolds of R^4.

($D_\mu \cdot = \partial_\mu \cdot + [A_\mu, \cdot]$) being the covariant gauge derivative. Finally $\left(\phi, \overline{\phi}\right)$ are commuting complex scalar fields, $\overline{\phi}$ being assumed to be the complex conjugate[5] of ϕ.

Of course, TYM being a gauge theory, is left invariant by the gauge transformations

$$\begin{aligned} \delta^g_\epsilon A_\mu &= -D_\mu \epsilon \ , \\ \delta^g_\epsilon \lambda &= [\epsilon, \lambda] \ , \qquad \lambda = \chi, \psi, \eta, \phi, \overline{\phi} \ , \end{aligned} \tag{3.57}$$

$$\delta^g_\epsilon \mathcal{S}_{TYM} = 0 \ , \tag{3.58}$$

Remark 6 *It is easily checked that the kinetic terms in the action* (3.52) *corresponding to the fields* $\left(\chi, \psi, \eta, \phi, \overline{\phi}\right)$ *are nondegenerate, so that these fields have well defined propagators. The only degeneracy is that related to the pure Yang-Mills term* $F^+_{\mu\nu} F^{+\mu\nu}$*. Therefore, from eq.*(3.57) *one is led to interpret the fields* $\left(\chi, \psi, \eta, \phi, \overline{\phi}\right)$ *as ordinary matter fields, in spite of the unconventional tensorial character of* $(\chi_{\mu\nu}, \psi_\mu)$*. This point will become more clear later on, once the relationship between TYM and the* $N = 2$ *euclidean gauge theories will be established.*

In addition to the gauge invariance, the action (3.52) turns out to be left invariant by the following nonlinear transformations *[11]*

$$\begin{aligned} \delta_{\mathcal{W}} A_\mu &= \psi_\mu \ , \\ \delta_{\mathcal{W}} \psi_\mu &= -D_\mu \phi \ , \\ \delta_{\mathcal{W}} \phi &= 0 \ , \\ \delta_{\mathcal{W}} \chi_{\mu\nu} &= F^+_{\mu\nu} \ , \\ \delta_{\mathcal{W}} \overline{\phi} &= 2\eta \ , \\ \delta_{\mathcal{W}} \eta &= \frac{1}{2} \left[\phi, \overline{\phi}\right] \ , \end{aligned} \tag{3.59}$$

and

$$\delta_{\mathcal{W}} \mathcal{S}_{TYM} = 0 \ . \tag{3.60}$$

The operator $\delta_{\mathcal{W}}$ is of fermionic type and obeys the relation

$$\delta^2_{\mathcal{W}} = \delta^g_\phi \ + \ (\chi\text{-eq. of motion}) \ , \tag{3.61}$$

δ^g_ϕ denoting a gauge transformation with gauge parameter ϕ. More precisely

$$\begin{aligned} \delta^2_{\mathcal{W}} A_\mu &= -D_\mu \phi \ , \\ \delta^2_{\mathcal{W}} \psi_\mu &= [\phi, \psi_\mu] \ , \end{aligned} \tag{3.62}$$

[5] See also Witten's remark on this point given at the end of the Subsect. 2.2 of *[11]*.

$$\begin{aligned}
\delta^2_{\mathcal{W}}\phi &= 0\,, \\
\delta^2_{\mathcal{W}}\overline{\phi} &= \left[\phi,\overline{\phi}\right]\,, \\
\delta^2_{\mathcal{W}}\eta &= [\phi,\eta]\,, \\
\delta^2_{\mathcal{W}}\chi_{\mu\nu} &= [\phi,\chi_{\mu\nu}] - \frac{g^2}{2}\frac{\delta\mathcal{S}_{TYM}}{\delta\chi^{\mu\nu}}\,.
\end{aligned}$$

The equations (3.61) , (3.62) mean essentially that the operator $\delta_{\mathcal{W}}$ becomes nilpotent when acting on the space of the gauge invariant functionals. This property turns out to play an important role in the construction of the Witten's observables. In particular, eq.(3.61) shows that the operator $\delta_{\mathcal{W}}$ belongs to the class of the operators $\{\delta_A\}$ considered in the previous Section (see eq.(2.4)).

Remark 7 *One should remark that the relative coefficients of the various terms of the TYM action* (3.52) *are not completely fixed by the fermionic symmetry $\delta_{\mathcal{W}}$. In other words, the action* (3.52) *is not the most general gauge invariant action compatible with the $\delta_{\mathcal{W}}$-invariance. Nevertheless, we shall see that $\mathcal{S}_{TYM}$ turns out to possess additional nonlinear invariances which fix completely the relative numerical coefficients of the various terms of expression* (3.52) *and allow for a unique coupling constant. Moreover, these additional nonlinear symmetries give rise together with the $\delta_{\mathcal{W}}$-symmetry to a twisted version of the $N=2$ susy algebra in the Wess-Zumino gauge.*

Following Witten, it is also easily seen that assigning to $\left(A,\chi,\psi,\eta,\phi,\overline{\phi}\right)$ the following $\mathcal{R}$-charges $(0,-1,1,-1,2,-2)$, the TYM action (3.52) has vanishing total $\mathcal{R}$-charge.

Let us display, finally, the quantum numbers of all the fields and of $\delta_{\mathcal{W}}$.

Dim.and $\mathcal{R}-$charges

	A_μ	$\chi_{\mu\nu}$	ψ_μ	η	ϕ	$\overline{\phi}$	$\delta_{\mathcal{W}}$
dim.	1	3/2	3/2	3/2	1	1	1/2
$\mathcal{R}-$ charg.	0	-1	1	-1	2	-2	1
nature	*comm.*	*ant.*	*ant.*	*ant.*	*comm.*	*comm.*	*ant.*

Table 1.

3.2 Twisting the N=2 supersymmetric algebra

For a better understanding of the TYM action (3.52) and of its fermionic symmetry (3.60), let us present now the twisting procedure of the $N=2$ supersymmetric algebra in flat euclidean space-time. We shall follow here mainly the detailed analysis done by M. Mariño[6] *[12]*. In the absence of central extension[7] the $N=2$ supersymmetry is characterized by 8 charges $\left(\mathcal{Q}^i{}_\alpha,\overline{\mathcal{Q}}^j{}_{\dot\alpha}\right)$ obeying the following relations

$$\begin{aligned}
\left\{\mathcal{Q}^i{}_\alpha,\overline{\mathcal{Q}}_{j\dot\alpha}\right\} &= \delta^i_j\,(\sigma^\mu)_{\alpha\dot\alpha}\,\partial_\mu\,, \\
\left\{\mathcal{Q}^i{}_\alpha,\mathcal{Q}^j{}_\beta\right\} &= \left\{\overline{\mathcal{Q}}^i{}_{\dot\alpha},\overline{\mathcal{Q}}^j{}_{\dot\beta}\right\} = 0\,,
\end{aligned} \tag{3.63}$$

[6]This work is an uptodate reference on this subject, including a discussion of the twisting procedure in the presence of matter multiplets and central charges, as well as a study of the relationship between topological field theories and gauge models with extended supersymmetry.

[7]This is the case for instance of theories involving only massless fields.

where $(\alpha, \dot{\alpha}) = 1,2$ are the spinor indices, $(i,j) = 1,2$ the internal $SU(2)$ indices labelling the different charges of $N=2$, and $(\sigma^\mu)_{\alpha\dot{\alpha}} = (1, i\vec{\sigma})$, $\vec{\sigma}$ being the Pauli matrices (see App.A for the euclidean susy conventions). The special feature of $N=2$ is that both spinor and internal indices run from 1 to 2, making possible to identify the index i with one of the two spinor indices $(\alpha, \dot{\alpha})$. It is precisely this identification which defines the twisting procedure. As explained in *[12]*, this is equivalent to a redefinition of the action of the four dimensional rotation group. Indeed, in the four dimensional flat euclidean space-time the global symmetry group of $N=2$ supersymmetry is $SU(2)_L \times SU(2)_R \times SU(2)_I \times U(1)_{\mathcal{R}}$ where $SU(2)_L \times SU(2)_R$ is the rotation group, and $SU(2)_I$ and $U(1)_{\mathcal{R}}$ are the symmetry groups corresponding respectively to $SU(2)$-transformations of the internal index i and to the $\mathcal{R}$-symmetry, the $\mathcal{R}$-charges of $\left(\mathcal{Q}^i{}_\alpha, \overline{\mathcal{Q}}^j{}_{\dot{\alpha}}\right)$ being respectively $(1,-1)$. The twisting procedure consists thus of replacing the rotation group $SU(2)_L \times SU(2)_R$ by $SU(2)_L \times SU(2)'_R$ where $SU(2)'_R$ is the diagonal sum of $SU(2)_R$ and of $SU(2)_I$. Identifying therefore the internal index i with the spinor index α, the $N=2$ susy algebra (3.63) becomes

$$
\begin{aligned}
\left\{\mathcal{Q}^\beta{}_\alpha, \overline{\mathcal{Q}}_{\gamma\dot{\alpha}}\right\} &= \delta^\beta_\gamma \, (\sigma^\mu)_{\alpha\dot{\alpha}} \, \partial_\mu \, , \\
\left\{\mathcal{Q}^\beta{}_\alpha, \mathcal{Q}^\delta{}_\gamma\right\} &= \left\{\overline{\mathcal{Q}}_{\alpha\dot{\alpha}}, \overline{\mathcal{Q}}_{\beta\dot{\beta}}\right\} = 0 \, .
\end{aligned} \tag{3.64}
$$

Let us define now the following generators $(\delta, \delta_\mu, \delta_{\mu\nu})$ with $\mathcal{R}$-charge $(1,-1,1)$ respectively;

$$
\begin{aligned}
\delta &= \frac{1}{\sqrt{2}} \varepsilon^{\alpha\beta} \mathcal{Q}_{\beta\alpha} \, , \\
\delta_\mu &= \frac{1}{\sqrt{2}} \overline{\mathcal{Q}}_{\alpha\dot{\alpha}} \, (\overline{\sigma}_\mu)^{\dot{\alpha}\alpha} \, , \\
\delta_{\mu\nu} &= \frac{1}{\sqrt{2}} (\sigma_{\mu\nu})^{\alpha\beta} \mathcal{Q}_{\beta\alpha} = -\delta_{\nu\mu} \, ,
\end{aligned} \tag{3.65}
$$

where, as usual,

$$
\sigma_{\mu\nu} = \frac{1}{2} (\sigma_\mu \overline{\sigma}_\nu - \sigma_\nu \overline{\sigma}_\mu) \, . \tag{3.66}
$$

Notice that the generators $\delta_{\mu\nu}$ are self-dual

$$
\delta_{\mu\nu} = \widetilde{\delta}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \delta^{\rho\sigma} \, , \tag{3.67}
$$

due to the fact that the matrices $\sigma_{\mu\nu}$ are self-dual in euclidean space-time (see App.A). Equations (3.65) show that we can replace the spinorial charges $\left(\mathcal{Q}^i{}_\alpha, \overline{\mathcal{Q}}^j{}_{\dot{\alpha}}\right)$ with the 8 generators $(\delta, \delta_\mu, \delta_{\mu\nu})$, respectively a scalar δ, a vector δ_μ and a self-dual tensor $\delta_{\mu\nu}$. In terms of these generators, the $N=2$ susy algebra reads now

$$
\begin{aligned}
\delta^2 &= 0 \, , \\
\{\delta, \delta_\mu\} &= \partial_\mu \, , \\
\{\delta_\mu, \delta_\nu\} &= 0 \, ,
\end{aligned} \tag{3.68}
$$

and

$$
\begin{aligned}
\{\delta_\mu, \delta_{\rho\sigma}\} &= -(\varepsilon_{\mu\rho\sigma\nu}\partial^\nu + g_{\mu\rho}\partial_\sigma - g_{\mu\sigma}\partial_\rho) \,, \\
\{\delta, \delta_{\mu\nu}\} &= \{\delta_{\mu\nu}, \delta_{\rho\sigma}\} = 0 \,,
\end{aligned} \tag{3.69}
$$

where $g_{\mu\nu} = diag(+,+,+,+)$ is the flat euclidean metric and where use has been made of the relations (see App.A)

$$
\begin{aligned}
tr(\sigma^\mu \overline{\sigma}^\nu) &= 2g^{\mu\nu}, \\
\sigma_{\mu\nu}\sigma_\rho &= -(\varepsilon_{\mu\nu\rho\tau}\sigma^\tau + g_{\mu\rho}\sigma_\nu - g_{\nu\rho}\sigma_\mu) \,.
\end{aligned} \tag{3.70}
$$

The charges $(\delta, \delta_\mu, \delta_{\mu\nu})$ are called twisted generators and the eqs.(3.68), (3.69) define the twisted version of the $N = 2$ susy algebra (3.63). We see, in particular, that the singlet operator δ, although rather different from the operator s of the gauge transformations (2.5), is nilpotent, allowing thus for a BRST-charge like interpretation. It should also be remarked from eq.(3.68) that the singlet and the vector generators (δ, δ_μ) give rise to an algebraic structure which is typical of the topological models *[6]*. In this case the vector charge δ_μ, usually called vector supersymmetry, is known to play an important role in the derivation of the ultraviolet finiteness properties of the topological models and in the construction of their observables *[6]*. Therefore the $N = 2$ susy algebra, when rewritten in terms of the twisted generators, displays the same algebraic structure of the topological models. In what follows we shall check that this feature is more than a simple analogy. Let us emphasize in fact that, since a few years, the relationship between topological field theories and models with extended supesrymmetry is becoming more and more apparent *[12]*.

Remark 8 *It is useful to remark here that the supersymmetric charges* $\left(\mathcal{Q}^i{}_\alpha, \overline{\mathcal{Q}}^j{}_{\dot\alpha}\right)$ *of eq.(3.63) have been implicitly referred to a linear realization of supersymmetry, meaning that they act linearly on the components of the $N = 2$ multiplets. Instead, in what follows we shall deal with a different situation in which supersymmetry in nonlinearly realized, due to the use of the Wess-Zumino gauge [4, 13]. As it is well known, the Wess-Zumino gauge allows to reduce the number of fields component, simplifying considerably the full analysis. There is however a price to pay. The supersymmetric transformations are now nonlinear and the algebra between the supersymmetric charges closes on the translations only modulo gauge transformations and equations of motion. Accordingly, the $N = 2$ algebra in the Wess-Zumino gauge will read then*

$$
\begin{aligned}
\left\{\mathcal{Q}^i{}_\alpha, \overline{\mathcal{Q}}_{j\dot\alpha}\right\} &= \delta^i_j \left(\sigma^\mu\right)_{\alpha\dot\alpha} \partial_\mu + (\text{gauge transf.}) \;+\; (\text{eqs. of mot.}) \,, \\
\left\{\mathcal{Q}^i{}_\alpha, \mathcal{Q}^j{}_\beta\right\} &= \left\{\overline{\mathcal{Q}}^i_{\dot\alpha}, \overline{\mathcal{Q}}^j_{\dot\beta}\right\} = (\text{gauge transf.}) \;+\; (\text{eqs. of mot.}) \,.
\end{aligned} \tag{3.71}
$$

Analogously, for the twisted version

$$
\begin{aligned}
\delta^2 &= (\text{gauge transf.}) \;+\; (\text{eqs. of mot.}) \,, \\
\{\delta, \delta_\mu\} &= \partial_\mu \;+ (\text{gauge transf.}) \;+\; (\text{eqs. of mot.}) \,, \\
\{\delta_\mu, \delta_\nu\} &= (\text{gauge transf.}) \;+\; (\text{eqs. of mot.}) \,,
\end{aligned} \tag{3.72}
$$

and

$$\begin{aligned}\{\delta, \delta_{\mu\nu}\} &= \{\delta_{\mu\nu}, \delta_{\rho\sigma}\} = (\text{gauge transf.}) + (\text{eqs. of mot.}) \ , \\ \{\delta_\mu, \delta_{\rho\sigma}\} &= -(\varepsilon_{\mu\rho\sigma\nu}\partial^\nu + g_{\mu\rho}\partial_\sigma - g_{\mu\sigma}\partial_\rho) + (\text{g. tr.}) + (\text{eqs. mot.}) \ ,\end{aligned} \tag{3.73}$$

As we shall see in the next subsections, it is precisely this twisted version of the $N = 2$ susy algebra which shall be found in the TYM theory.

3.3 Relationship between TYM and N=2 pure Yang-Mills

Having discussed the twisting procedure, let us now turn to the relationship between Witten's TYM and the $N = 2$ Yang-Mills theory. Let us show, in particular, that TYM has the same field content of the $N = 2$ Yang-Mills theory in the Wess-Zumino gauge. The minimal $N = 2$ supersymmetric pure Yang-Mills theory is described by a gauge multiplet which, in the Wess-Zumino gauge, contains *[13, 12]*: a gauge field A_μ, two spinors ψ^i_α $i = 1, 2$, and their conjugate $\overline{\psi}^i_{\dot\alpha}$, two scalars $\phi, \overline{\phi}$ ($\overline{\phi}$ being the complex conjugate of ϕ). All these fields are in the adjoint representations of the gauge group. We also recall that in the Wess-Zumino gauge the generators $\left(\mathcal{Q}^i{}_\alpha, \overline{\mathcal{Q}}^j{}_{\dot\alpha}\right)$ of $N = 2$ act nonlinearly and the supersymmetry algebra is that of eqs.(3.71).

Let us proceed by applying the previous twisting procedure to the $N = 2$ Wess-Zumino gauge multiplet $(A_\mu, \psi^i_\alpha, \overline{\psi}^i_{\dot\alpha}, \phi, \overline{\phi})$. Identifying then the internal index i with the spinor index α, it is very easy to see that the spinor $\overline{\psi}^i_{\dot\alpha}$ can be related to an anticommuting vector ψ_μ, i.e

$$\overline{\psi}^i_{\dot\alpha} \xrightarrow{twist} \overline{\psi}_{\alpha\dot\alpha} \longrightarrow \psi_\mu = (\overline{\sigma}_\mu)^{\dot\alpha\alpha}\overline{\psi}_{\alpha\dot\alpha} \ . \tag{3.74}$$

Concerning now the fields ψ^i_β we have

$$\psi^i_\beta \xrightarrow{twist} \psi_{\alpha\beta} = \psi_{(\alpha\beta)} + \psi_{[\alpha\beta]} \ , \tag{3.75}$$

$\psi_{(\alpha\beta)}$ and $\psi_{[\alpha\beta]}$ being respectively symmetric and antisymmetric in the spinor indices α, β. To the antisymmetric component $\psi_{[\alpha\beta]}$ we associate an anticommuting scalar field η

$$\psi_{[\alpha\beta]} \longrightarrow \eta = \varepsilon^{\alpha\beta}\psi_{[\alpha\beta]} \ , \tag{3.76}$$

while the symmetric part $\psi_{(\alpha\beta)}$ turns out to be related to an antisymmetric self-dual field $\chi_{\mu\nu}$ through

$$\psi_{(\alpha\beta)} \longrightarrow \chi_{\mu\nu} = \widetilde{\chi}_{\mu\nu} = (\sigma_{\mu\nu})^{\alpha\beta}\psi_{(\alpha\beta)} \ . \tag{3.77}$$

Therefore, the twisting procedure allows to replace the $N = 2$ Wess-Zumino multiplet $(A_\mu, \psi^i_\alpha, \overline{\psi}^i_{\dot\alpha}, \phi, \overline{\phi})$ by the twisted multiplet $(A_\mu, \psi_\mu, \chi_{\mu\nu}, \eta, \phi, \overline{\phi})$ whose field content is precisely that of the TYM action (3.52).

As one can now easily guess, the same property holds for the $N = 2$ pure Yang-Mills action, as it has been detailed analysed in the Chapters 3 and 6 of *[12]*, i.e.

$$\mathcal{S}^{N=2}_{YM}(A_\mu, \psi^i_\alpha, \overline{\psi}^i_{\dot\alpha}, \phi, \overline{\phi}) \xrightarrow{twist} \mathcal{S}_{TYM}(A_\mu, \psi_\mu, \chi_{\mu\nu}, \eta, \phi, \overline{\phi}) \ , \tag{3.78}$$

showing thus that the TYM is in fact the twisted version of the ordinary $N = 2$ Yang-Mills in the Wess-Zumino gauge. This important point, already underlined by Witten in its original work *[11]*, deserves a few clarifying remarks in order to make contact with the results on topological field theories obtained in the recent years.

Remark 9 *The first observation is naturally related to the existence of further symmetries of the TYM action* (3.52). *In fact, from eq.*(3.78) *one can immediately infer that the TYM should possess the same symmetry content of the $N = 2$ Yang-Mills. We expect therefore that, according to the previous analysis, the TYM will be left invariant by a set of additional transformations whose generators $(\delta, \delta_\mu, \delta_{\mu\nu})$ correspond to the twisted $N = 2$ supersymmetric charges ${\mathcal{Q}^i}_\alpha, {\overline{\mathcal{Q}}^j}_{\dot\alpha}$ and fulfill the Wess-Zumino susy algebra* (3.72), (3.73). *It is easy to check that Witten's fermionic symmetry $\delta_{\mathcal{W}}$ of eq.*(3.59) *corresponds to the twisted scalar generator δ. Concerning now the vector and the tensor invariances $\delta_\mu, \delta_{\mu\nu}$, we shall postpone their detailed analysis to the next Subsect., limiting here to confirm their existence and their relevance (especially that of δ_μ) for the quantum analysis.*

Remark 10 *The second remark is related to the standard perturbative Feynman diagram computations. From eq.*(3.78) *it is very tempting to argue that the values of quantities like the β-function should be the same when computed in the ordinary $N = 2$ Yang-Mills and in the twisted version. After all, at least at the perturbative level, the twisting procedure has the effect of a linear change of variables on the fields. The computation of the one loop β-function for TYM has indeed been performed by R.Brooks et al.[22]. As expected, the result agrees with that of the pure $N = 2$ Yang-Mills. We recall here that the $N = 2$ Yang-Mills β-function receives contributions only of one loop order [23]. It is also wortwhile to mention that, recently, the algebraic renormalization analysis of $N = 2$ Yang-Mills theory in the Wess-Zumino gauge has been carried out by N. Maggiore [13] who has shown that the model is anomaly free and that there is only one possible nontrivial local invariant counterterm, corresponding to a possible renormalization of the unique gauge coupling constant. As we shall see later, the same conclusion will be reached in the case of TYM.*

Remark 11 *The third point is related to the important and intriguing issue of the cohomological triviality of the TYM theory. Witten's TYM theory is commonly classified as a topological theory of the cohomological type [5], meaning that the TYM action can be expressed as a pure BRST variation. Of course, this property seems to be in disagreement with the interpretation of TYM following from the relation* (3.78). *Nevertheless, we shall prove that there is a way out, allowing us to clearly establish to what extent we can consider TYM as a cohomological theory. The point here is that the quantization of the TYM action has to be done by taking into account the full $N = 2$ twisted supersymmetric algebra. This step can be easily handled by following the same procedure proposed by P. White [4] in the proof of the ultraviolet finiteness of $N = 4$ Yang-Mills in the Wess-Zumino gauge. As already remarked, one introduces constant ghosts associated to the nonlinear supersymmetric transformations. These constant ghosts allow to define an extended BRST operator which turns out to be nilpotent modulo the (matter) equations of motion, as the operator of eq.*(2.6) *. Then, the analysis of the renormalizability can be performed along the* *lines of the previous Section. Moreover, the perturbative Feynman expansion is easily seen to be analytic in these constant ghosts, meaning that the functional space which is acted upon by the BRST operator is that of the formal power series in the global parameters. It is precisely the requirement of analyticity in the constant ghosts which yields nontrivial cohomology classes, as it has been already established by P.White and N. Maggiore in the cases of the conventional untwisted $N = 2, 4$ theories [4, 13]. This means that, as long as the analyticity requirement is preserved, the theory is nontrivial and can be viewed as an ordinary supersymmetric field theory, the fields $(\chi_{\mu\nu}, \psi_\mu, \eta, \phi, \overline{\phi})$ being interpreted as matter fields. On the other hand, if the analyticity requirement is relaxed, the theory becomes cohomologically trivial and we fall into the well known Labastida-Pernici [24] and Baulieu-Singer [25] formulations. It is useful to notice here that in these formulations the fields $(\chi_{\mu\nu}, \psi_\mu, \eta, \phi, \overline{\phi})$ carry a nonvanishing ghost number and are no longer considered as matter fields. Rather, they are interpreted as ghost fields. In particular, ψ_μ corresponds to the ghost of the so called topological shift symmetry [24, 25]. It is an important fact, however, that TYM possesses a nontrivial content also when considered as a cohomological BRST trivial theory. Indeed, as shown by R. Stora et al.[16, 17], the relavant cohomology which characterizes the TYM in the cohomological version is the so called equivariant cohomology which, unlike the BRST cohomology, is found to be not empty, allowing to recover the original observables proposed by Witten. Remarkably in half, the two points of view can be shown to be equivalent, as proven by the authors [18, 15], who have been able to establish that the equivariant cohomology coincides in fact with the cohomology of an extended operator, provided analyticity in a suitable global parameter is required. In the present case, the role of this global parameter is palyed by the constant global ghosts of $N = 2$ supersymmetry, the analyticity requirement following from perturbation theory. Summarizing, the TYM theory can be seen either as a conventional field theory (analyticity in the global ghosts is here demanded) or as a topological theory of the cohomologycal type. In this latter case one has to remember that the Witten's observables should belong to the equivariant cohomology. We will have the opportunity of showing the explicit equivalence of both points of view in the analysis of the invariant field polynomial $tr(\phi^2)$. Let us conclude this remark by emphasizing that the analyticity requirement, being naturally related to perturbation theory, is more closed to the present discussion.*

3.4 The vector supersymmetry

Let us now complete the previous analysis by showing that the TYM action (3.52) possesses indeed further nonlinear symmetries whose anticommutation relations with the Witten's fermionic symmetry $\delta_{\mathcal{W}}$ yield precisely the twisted $N = 2$ susy algebra of eqs.(3.72) , (3.73). Let us first focus on the vector invariance δ_μ. To this aim we introduce the following nonlinear transformations

$$
\begin{aligned}
\delta_\mu A_\nu &= \frac{1}{2}\chi_{\mu\nu} + \frac{1}{8}g_{\mu\nu}\eta \ , \qquad (3.79) \\
\delta_\mu \psi_\nu &= F_{\mu\nu} - \frac{1}{2}F^+_{\mu\nu} - \frac{1}{16}g_{\mu\nu}[\phi, \overline{\phi}] \ ,
\end{aligned}
$$

$$\begin{aligned}
\delta_\mu \eta &= \frac{1}{2} D_\mu \overline{\phi} \ , \\
\delta_\mu \chi_{\sigma\tau} &= \frac{1}{8}(\varepsilon_{\mu\sigma\tau\nu} D^\nu \overline{\phi} + g_{\mu\sigma} D_\tau \overline{\phi} - g_{\mu\tau} D_\sigma \overline{\phi}) \ , \\
\delta_\mu \phi &= -\psi_\mu \ , \\
\delta_\mu \overline{\phi} &= 0 \ ,
\end{aligned}$$

and

Dim.and $\mathcal{R}$−charges

	δ_μ
dim.	$1/2$
$\mathcal{R}$ − charg.	-1
nature	*ant.*

Table 2.

Transformations (3.79) are found to leave the TYM action (3.52) invariant

$$\delta_\mu \mathcal{S}_{TYM} = 0 \ . \tag{3.80}$$

In addition, it is easily verified that the vector generator δ_μ gives rise, together with the operator $\delta_{\mathcal{W}}$, to the following algebraic relations

$$\begin{aligned}
\{\delta_{\mathcal{W}}, \delta_\mu\} &= \partial_\mu + \delta^g_{A_\mu} + (\text{matt. eqs. of motion}) \ , \qquad (3.81) \\
\{\delta_\mu, \delta_\nu\} &= -\frac{1}{8} g_{\mu\nu} \delta^g_{\overline{\phi}} + (\text{matt. eqs. of motion}) \ ,
\end{aligned}$$

where $\delta^g_{A_\mu}$ and $\delta^g_{\overline{\phi}}$ are gauge transformations with field dependent parameters A_μ and $\overline{\phi}$, respectively. It is apparent thus from eqs.(3.61) , (3.81) that, as expected, the operators $\delta_{\mathcal{W}}$ and δ_μ obey the twisted $N = 2$ supersymmetric algebra (3.72) .

Let us also write down, for further use, the explicit form of eqs.(3.81), *i.e.*

$$\begin{aligned}
\{\delta_\mu, \delta_\nu\} A_\sigma &= \frac{1}{8} g_{\mu\nu} D_\sigma \overline{\phi} \ , \qquad (3.82) \\
\{\delta_{\mathcal{W}}, \delta_\mu\} A_\sigma &= \partial_\mu A_\sigma - D_\sigma A_\mu \ ,
\end{aligned}$$

$$\begin{aligned}
\{\delta_{\mathcal{W}}, \delta_\mu\} \psi_\sigma &= \partial_\mu \psi_\sigma + [A_\mu, \psi_\sigma] + \frac{g^2}{4} \frac{\delta \mathcal{S}_{TYM}}{\delta \chi^{\mu\sigma}} \ , \qquad (3.83) \\
\{\delta_\mu, \delta_\nu\} \psi_\sigma &= -\frac{1}{8} g_{\mu\nu} [\overline{\phi}, \psi_\sigma] + \frac{g^2}{16} (g_{\mu\sigma} g_{\nu\tau} + g_{\nu\sigma} g_{\mu\tau} - 2 g_{\mu\nu} g_{\sigma\tau}) \frac{\delta \mathcal{S}_{TYM}}{\delta \psi_\tau} \ ,
\end{aligned}$$

$$\begin{aligned}
\{\delta_\mu, \delta_\nu\} \eta &= -\frac{1}{8} g_{\mu\nu} [\overline{\phi}, \eta] \ , \qquad (3.84) \\
\{\delta_{\mathcal{W}}, \delta_\mu\} \eta &= \partial_\mu \eta + [A_\mu, \eta] \ ,
\end{aligned}$$

$$
\begin{aligned}
\{\delta_\mu, \delta_\nu\}\, \chi_{\sigma\tau} &= -\frac{1}{8} g_{\mu\nu} [\overline{\phi}, \chi_{\sigma\tau}] \,, \\
\{\delta_{\mathcal{W}}, \delta_\mu\}\, \chi_{\sigma\tau} &= \partial_\mu \chi_{\sigma\tau} + [A_\mu, \chi_{\sigma\tau}] + \frac{g^2}{8} \left(\varepsilon_{\mu\sigma\tau\nu} + g_{\mu\sigma} g_{\tau\nu} - g_{\mu\tau} g_{\sigma\nu}\right) \frac{\delta \mathcal{S}_{TYM}}{\delta \psi_\nu} \,,
\end{aligned}
\tag{3.85}
$$

$$
\begin{aligned}
\{\delta_\mu, \delta_\nu\}\, \phi &= -\frac{1}{8} g_{\mu\nu} [\overline{\phi}, \phi] \,, \\
\{\delta_{\mathcal{W}}, \delta_\mu\}\, \phi &= \partial_\mu \phi + [A_\mu, \phi] \,,
\end{aligned}
\tag{3.86}
$$

and

$$
\begin{aligned}
\{\delta_\mu, \delta_\nu\}\, \overline{\phi} &= 0 \,, \\
\{\delta_{\mathcal{W}}, \delta_\mu\}\, \overline{\phi} &= \partial_\mu \overline{\phi} + [A_\mu, \overline{\phi}] \,.
\end{aligned}
\tag{3.87}
$$

Remark 12 *We underline here that the form of the TYM action* (3.52)*, not completely specified by the fermionic symmetry $\delta_{\mathcal{W}}$, turns out to be uniquely characterized by the vector invariance δ_μ. In other words, eqs.*(3.80) *and* (3.81) *fix all the relative numerical coefficients of the Witten's action* (3.52) *allowing, in particular, for a single coupling constant. This feature will be of great importance for the renormalizability analysis of the model.*

Concerning now the existence of a self-dual symmetry $\delta_{\mu\nu}$, we shall remind the reader to the App.B, where the explicit form of the self-dual transformations will be given. Needless to say, the self-dual generator $\delta_{\mu\nu}$ will reproduce, together with the operators $\delta_{\mathcal{W}}, \delta_\mu$, the complete $N = 2$ susy algebra (3.72) , (3.73). The reasons why we do not actually take in further account the self-dual transformations $\delta_{\mu\nu}$ are due partly to the fact that, as previously remarked, the TYM action is already uniquely fixed by the ($\delta_{\mathcal{W}}$, δ_μ)$-$symmetries and partly to the fact that the generator $\delta_{\mu\nu}$ turns out to be almost trivially realized on the fields, as one can easily infer from the App.B. Looking for instance at the $\delta_{\mu\nu}$-transformations of the fields A_σ and ψ_σ, it is apparent to check that they can be rewritten as

$$
\begin{aligned}
\delta_{\mu\nu} A_\sigma &= -(\varepsilon_{\mu\nu\sigma\tau} + g_{\mu\sigma} g_{\nu\tau} - g_{\nu\sigma} g_{\mu\tau}) \delta_{\mathcal{W}} A^\tau \,, \\
\delta_{\mu\nu} \psi_\sigma &= (\varepsilon_{\mu\nu\sigma\tau} + g_{\mu\sigma} g_{\nu\tau} - g_{\nu\sigma} g_{\mu\tau}) \delta_{\mathcal{W}} \psi^\tau \,,
\end{aligned}
\tag{3.88}
$$

showing in fact that the $\delta_{\mu\nu}$-transformations can be trivially realized in terms of $\delta_{\mathcal{W}}$-transformations. This means that the subalgebra (3.72) carries essentially al the relevant informations concerning the $N = 2$ supersymmetric structure ot the TYM.

We can turn now to the quantization of the model. This will be the task of the next Section.

Remark 13 *Of course, all the nonlinear $(\delta_{\mathcal{W}}, \delta_\mu, \delta_{\mu\nu})$-transformations of the fields of TYM can be obtained by performing the twist of the conventional $(\mathcal{Q}^i{}_\alpha, \overline{\mathcal{Q}}^j{}_{\dot\alpha})$-transformations of the untwisted N=2 susy Wess-Zumino multiplet $(A_\mu, \psi^i_\alpha, \overline{\psi}^i_{\dot\alpha}, \phi, \overline{\phi})$.*

4 Quantizing topological Yang-Mills

4.1 Extendend BRST operator and gauge fixing

As we have seen in the previous Section, the TYM action $\mathcal{S}_{TYM}$ is left invariant by a set of nonlinear symmetries whose generators $\delta_{\mathcal{W}}$, δ_μ, give rise to an algebra (3.72) of the type of that considered in eq.(2.4). Following therefore the discussion of Sect.2., we shall begin by looking at an extended BRST operator $\mathcal{Q}$ which turns out to be nilpotent on shell. To this purpose we first introduce the Faddeev-Popov ghost field c corresponding to the local gauge invariance (3.58) of the action (3.52),

$$\epsilon \to c \ , \ \delta^g_\epsilon \to s \ , \tag{4.89}$$

with

$$\begin{aligned} sA_\mu &= -D_\mu c \ , \qquad\qquad (4.90) \\ s\psi_\mu &= \{c, \psi_\mu\} \ , \\ s\chi_{\mu\nu} &= \{c, \chi_{\mu\nu}\} \ , \\ s\eta &= \{c, \eta\} \ , \\ s\phi &= [c, \phi] \ , \\ s\overline{\phi} &= \left[c, \overline{\phi}\right] \ , \\ sc &= c^2 \ , \qquad\qquad (4.91) \\ s^2 &= 0 \ , \end{aligned}$$

and

$$s\mathcal{S}_{TYM} = 0 \ . \tag{4.92}$$

We associate now to each generator entering the algebra (3.72), namely $\delta_{\mathcal{W}}$, δ_μ and ∂_μ, the corresponding constant ghost parameters $(\omega, \varepsilon^\mu, v^\mu)$

$$\omega \to \delta_{\mathcal{W}} \ , \quad \varepsilon^\mu \to \delta_\mu \ , \quad v^\mu \to \partial_\mu \ . \tag{4.93}$$

with

Dim., $\mathcal{R}$−charges and gh−numbers

	c	ω	ε^μ	v^μ
dim.	0	−1/2	−1/2	−1
$\mathcal{R}$−charge	0	−1	1	0
gh − number	1	1	1	1
nature	*ant.*	*comm.*	*comm.*	*ant.*

Table 3.

Therefore, the extended BRST operator

$$\mathcal{Q} = s + \omega\delta_{\mathcal{W}} + \varepsilon^\mu\delta_\mu + v^\mu\partial_\mu - \omega\varepsilon^\mu\frac{\partial}{\partial v^\mu} \ , \tag{4.94}$$

has ghost number one, vanishing $\mathcal{R}$-charge, and

$$
\begin{aligned}
\mathcal{Q}\mathcal{S}_{TYM} &= 0 , \\
\mathcal{Q}^2 &= (\text{matter eqs. of mot.}) .
\end{aligned}
\tag{4.95}
$$

Remark 14 *We underline here that the fields $(A_\mu, \psi_\mu, \eta, \chi_{\mu\nu}, \phi, \overline{\phi})$ do not carry any ghost number. In particular $(\psi_\mu, \eta, \chi_{\mu\nu}, \phi, \overline{\phi})$ are considered as (twisted) matter fields.*

While the first condition of the above equation follows from the simple observation that the TYM action (3.52) does not depend from the ghosts and that it is invariant under ordinary space-time translations, the second one requires some care and follows from defining in a suitable way the action of the four generators $s, \delta_{\mathcal{W}}, \delta_\mu$ and ∂_μ on the ghosts $(c, \omega, \varepsilon^\mu, v^\mu)$. To have a more precise idea of how this is done, let us work out in detail the case of the two operators s and $\delta_{\mathcal{W}}$. Recalling that

$$
\delta^2_{\mathcal{W}} = \delta^g_\phi \; + \; (\chi\text{-eq. of motion}) , \tag{4.96}
$$

one looks then for an operator $(s + \omega\delta_{\mathcal{W}})$ such that

$$
\begin{aligned}
(s + \omega\delta_{\mathcal{W}})^2 &= 0 \quad \text{on} \quad (A_\mu, \psi_\mu, \eta, \phi, \overline{\phi}, c, \omega) , \\
(s + \omega\delta_{\mathcal{W}})^2 \chi_{\mu\nu} &= (\chi\text{-eq. of motion}) .
\end{aligned}
\tag{4.97}
$$

After a little experiment, it is not difficult to convince oneself that the above conditions are indeed verified by defining the action of s and $\delta_{\mathcal{W}}$ on the ghost (c, ω) as

$$
s\omega = 0 , \quad \delta_{\mathcal{W}}\omega = 0 , \quad \delta_{\mathcal{W}} c = -\omega\phi . \tag{4.98}
$$

Notice that the only nontrivial extension is that of the operator $\delta_{\mathcal{W}}$ on the Faddeev-Popov ghost c. It is precisely this transformation which compensates the gauge transformation δ^g_ϕ in the right hand side of eq.(4.96), ensuring then the on shell nilpotency of the operator $(s + \omega\delta_{\mathcal{W}})$.

The above procedure can be now easily repeated in order to include in the game also the operators δ_μ and ∂_μ. The final result is that the extension of the operator $\mathcal{Q}$ on the ghosts $(c, \omega, \varepsilon^\mu, v^\mu)$ is found to be

$$
\begin{aligned}
\mathcal{Q}c &= c^2 - \omega^2\phi - \omega\varepsilon^\mu A_\mu + \frac{\varepsilon^2}{16}\overline{\phi} + v^\mu\partial_\mu c , \\
\mathcal{Q}\omega &= 0 , \quad \mathcal{Q}\varepsilon^\mu = 0 , \\
\mathcal{Q}v^\mu &= -\omega\varepsilon^\mu .
\end{aligned}
\tag{4.99}
$$

The construction of the gauge fixing term is now almost trivial. We introduce an antighost $\overline{c}$ and a Lagrangian multiplier b transforming as *[4, 13]*

$$
\begin{aligned}
\mathcal{Q}\overline{c} &= b + v^\mu\partial_\mu\overline{c} , \\
\mathcal{Q}b &= \omega\varepsilon^\mu\partial_\mu\overline{c} + v^\mu\partial_\mu b ,
\end{aligned}
\tag{4.100}
$$

with

$$\mathcal{Q}^2\bar{c} = \mathcal{Q}^2 b = 0 \ . \tag{4.101}$$

Dim., $\mathcal{R}-$charges and gh$-$numbers

	$\bar{c}$	b
dim	2	2
$\mathcal{R}-$ch arg e	0	0
gh $-$ number	-1	0
nature	*ant.*	*comm.*

Table 4.

According to the eq.(2.11), for the gauge fixing action we thus get

$$\begin{aligned} S_{gf} &= Q\int d^4x \ tr(\bar{c}\partial A) \\ &= tr\int d^4x \ \left(b\partial^\mu A_\mu + \bar{c}\partial^\mu D_\mu c - \omega\bar{c}\partial^\mu\psi_\mu - \frac{\varepsilon^\nu}{2}\bar{c}\partial^\mu\chi_{\nu\mu} - \frac{\varepsilon^\mu}{8}\bar{c}\partial_\mu\eta \right) \ , \end{aligned} \tag{4.102}$$

so that the gauge fixed action $(\mathcal{S}_{TYM} + S_{gf})$ is $\mathcal{Q}$-invariant,

$$\mathcal{Q}\left(\mathcal{S}_{TYM} + S_{gf}\right) = 0 \ . \tag{4.103}$$

The above equation means that the gauge fixing procedure has been worked out by taking into account not only the pure local gauge symmetry but also the additional nonlinear invariances $\delta_{\mathcal{W}}$ and δ_μ, as one can easily deduce from the explicit dependence of the gauge fixing term (4.102) from the global ghosts ω, ε^μ. Of course, the absence of the ghost v^μ is due to the space-time translation invariance of expression (4.102).

Let us conclude this paragraph by summarizing all the properties of the extended operator $\mathcal{Q}$, *i.e.*

$$\begin{aligned} \mathcal{Q}A_\mu &= -D_\mu c + \omega\psi_\mu + \frac{\varepsilon^\nu}{2}\chi_{\nu\mu} + \frac{\varepsilon_\mu}{8}\eta + v^\nu\partial_\nu A_\mu \ , \\ \mathcal{Q}\psi_\mu &= \{c,\psi_\mu\} - \omega D_\mu\phi + \varepsilon^\nu\left(F_{\nu\mu} - \frac{1}{2}F^+_{\nu\mu}\right) - \frac{\varepsilon_\mu}{16}[\phi,\bar{\phi}] \\ &\quad + v^\nu\partial_\nu\psi_\mu \ , \\ \mathcal{Q}\chi_{\sigma\tau} &= \{c,\chi_{\sigma\tau}\} + \omega F^+_{\sigma\tau} + \frac{\varepsilon^\mu}{8}(\varepsilon_{\mu\sigma\tau\nu} + g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma})D^\nu\bar{\phi} \\ &\quad + v^\nu\partial_\nu\chi_{\sigma\tau} \ , \\ \mathcal{Q}\eta &= \{c,\eta\} + \frac{\omega}{2}[\phi,\bar{\phi}] + \frac{\varepsilon^\mu}{2}D_\mu\bar{\phi} + v^\nu\partial_\nu\eta \ , \\ \mathcal{Q}\phi &= [c,\phi] - \varepsilon^\mu\psi_\mu + v^\nu\partial_\nu\phi \ , \\ \mathcal{Q}\bar{\phi} &= \left[c,\bar{\phi}\right] + 2\omega\eta + v^\nu\partial_\nu\bar{\phi} \ , \\ \mathcal{Q}c &= c^2 - \omega^2\phi - \omega\varepsilon^\mu A_\mu + \frac{\varepsilon^2}{16}\bar{\phi} + v^\nu\partial_\nu c \ , \\ \mathcal{Q}\omega &= 0 \ , \end{aligned} \tag{4.104}$$

$$
\begin{aligned}
\mathcal{Q}\varepsilon^{\mu} &= 0 , \\
\mathcal{Q}v^{\mu} &= -\omega\varepsilon^{\mu} , \\
\mathcal{Q}\overline{c} &= b + v^{\mu}\partial_{\mu}\overline{c} , \\
\mathcal{Q}b &= \omega\varepsilon^{\mu}\partial_{\mu}\overline{c} + v^{\mu}\partial_{\mu}b ,
\end{aligned}
$$

with

$$
\mathcal{Q}^2 = 0 \quad \text{on} \quad \left(A, \phi, \overline{\phi}, \eta, c, \omega, \varepsilon, v, \overline{c}, b\right) , \tag{4.105}
$$

and

$$
\begin{aligned}
\mathcal{Q}^2\psi_{\sigma} &= \frac{g^2}{4}\omega\varepsilon^{\mu}\frac{\delta\mathcal{S}_{TYM}}{\delta\chi^{\mu\sigma}} \\
&\quad +\frac{g^2}{32}\varepsilon^{\mu}\varepsilon^{\nu}\left(g_{\mu\sigma}\frac{\delta\mathcal{S}_{TYM}}{\delta\psi^{\nu}} + g_{\nu\sigma}\frac{\delta\mathcal{S}_{TYM}}{\delta\psi^{\mu}} - 2g_{\mu\nu}\frac{\delta\mathcal{S}_{TYM}}{\delta\psi^{\sigma}}\right) ,
\end{aligned} \tag{4.106}
$$

$$
\begin{aligned}
\mathcal{Q}^2\chi_{\sigma\tau} &= -\frac{g^2}{2}\omega^2\frac{\delta\mathcal{S}_{TYM}}{\delta\chi^{\sigma\tau}} \\
&\quad +\frac{g^2}{8}\omega\varepsilon^{\mu}\left(\varepsilon_{\mu\sigma\tau\nu}\frac{\delta\mathcal{S}_{TYM}}{\delta\psi_{\nu}} + g_{\mu\sigma}\frac{\delta\mathcal{S}_{TYM}}{\delta\psi^{\tau}} - g_{\mu\tau}\frac{\delta\mathcal{S}_{TYM}}{\delta\psi^{\sigma}}\right) .
\end{aligned} \tag{4.107}
$$

Remark 15 *Notice that the $\mathcal{Q}$-transformation of the Faddeev-Popov ghost c contains terms quadratic in the global parameters $\omega, \varepsilon^{\mu}$. The presence of these terms (in particular of $\omega^2\phi$) in the transformation of the ghost c has been shown to be of great importance by the authors of ref. [18] in order to identify the relevant nontrivial cohomology classes of TYM.*

4.2 The Slavnov-Taylor identity

As explained in Sect.2, in order to obtain the Slavnov-Taylor identity we first couple the nonlinear $\mathcal{Q}$-transformations of the fields in eqs.(4.104) to a set of antifields $(L, D, \Omega^{\mu}, \xi^{\mu}, \rho, \tau, B_{\mu\nu})$,

$$
\begin{aligned}
\mathcal{S}_{ext} &= tr\int d^4x \; (\; L\mathcal{Q}c + D\mathcal{Q}\phi + \Omega^{\mu}\mathcal{Q}A_{\mu} + \xi^{\mu}\mathcal{Q}\psi_{\mu} \\
&\quad +\rho\mathcal{Q}\overline{\phi} + \tau\mathcal{Q}\eta + \frac{1}{2}B^{\mu\nu}\mathcal{Q}\chi_{\mu\nu} \;) ,
\end{aligned} \tag{4.108}
$$

with

Dim., $\mathcal{R}-$charges and gh$-$numbers

	L	D	Ω^{μ}	ξ^{μ}	ρ	τ	$B^{\mu\nu}$
dim.	4	3	3	5/2	3	5/2	5/2
$\mathcal{R}-$charge	0	-2	0	-1	2	1	1
gh $-$ number	-2	-1	-1	-1	-1	-1	-1
nature	*comm.*	*ant.*	*ant.*	*comm.*	*ant.*	*comm.*	*comm.*

Table 5.

Moreover, taking into account that the extended operator $\mathcal{Q}$ is nilpotent only modulo the equations of motion of the fields ψ_μ and $\chi_{\mu\nu}$, we also introduce a term quadratic in the corresponding antifields ξ^μ, $B^{\mu\nu}$, *i.e.*

$$\mathcal{S}_{quad} = tr\int d^4x \left(\frac{\alpha}{4}\omega^2 B^{\mu\nu}B_{\mu\nu} + \frac{\beta}{2}\omega B^{\mu\nu}\varepsilon_\mu\xi_\nu + \frac{\lambda}{2}\varepsilon^\mu\varepsilon^\nu\xi_\mu\xi_\nu + \frac{\gamma}{2}\varepsilon^2\xi^2 \right) \ , \tag{4.109}$$

where the coefficients $(\alpha, \beta, \lambda, \gamma)$ are fixed by requiring that the complete action

$$\Sigma = \mathcal{S}_{TYM} + S_{gf} + \mathcal{S}_{ext} + \mathcal{S}_{quad} \ , \tag{4.110}$$

obeys the following identity

$$\mathcal{S}(\Sigma) = 0 \ , \tag{4.111}$$

with

$$\begin{aligned}\mathcal{S}(\Sigma) \ = \ tr\int d^4x \Bigg(& \frac{\delta\Sigma}{\delta A^\mu}\frac{\delta\Sigma}{\delta\Omega_\mu} + \frac{\delta\Sigma}{\delta\xi^\mu}\frac{\delta\Sigma}{\delta\psi_\mu} + \frac{\delta\Sigma}{\delta L}\frac{\delta\Sigma}{\delta c} + \frac{\delta\Sigma}{\delta D}\frac{\delta\Sigma}{\delta\phi} + \frac{\delta\Sigma}{\delta\rho}\frac{\delta\Sigma}{\delta\overline{\phi}} \\ & + \frac{\delta\Sigma}{\delta\tau}\frac{\delta\Sigma}{\delta\eta} + \frac{1}{2}\frac{\delta\Sigma}{\delta B^{\mu\nu}}\frac{\delta\Sigma}{\delta\chi_{\mu\nu}} + (b + v^\mu\partial_\mu\overline{c})\frac{\delta\Sigma}{\delta\overline{c}} \\ & + (\omega\varepsilon^\mu\partial_\mu\overline{c} + v^\mu\partial_\mu b)\frac{\delta\Sigma}{\delta b}\Bigg) - \omega\varepsilon^\mu\frac{\partial\Sigma}{\partial v^\mu} \ . \end{aligned} \tag{4.112}$$

The condition (4.111) is easily worked out, yielding for the coefficients α, β, λ and γ the following values

$$\alpha = \frac{g^2}{2}, \quad \beta = -\frac{g^2}{2}, \quad \lambda = -\frac{g^2}{16}, \quad \gamma = \frac{g^2}{16} \ . \tag{4.113}$$

The equation (4.111) yields thus the classical Slavnov-Taylor identity for the TYM and will be the starting point for the analysis of the renormalizability of the model. However, before entering into the quantum aspects, let us make some further useful considerations which allow to cast the Slavnov-Taylor identity (4.111) in a simplified form which is more suitable for the quantum discussion.

4.3 Analysis of the classical Slavnov-Taylor identity

In order to obtain a simplified version of the Slavnov-Taylor identity, we shall make use of the fact that the complete action Σ is invariant under space-time translations, as expressed by

$$\mathcal{P}_\mu\Sigma \ = \ \sum_i \int d^4x \left(\partial_\mu\varphi^i\frac{\delta\Sigma}{\delta\varphi^i} + \partial_\mu\varphi^{*i}\frac{\delta\Sigma}{\delta\varphi^{*i}} \right) = \ 0 \ , \tag{4.114}$$

$$\begin{aligned} \varphi^i \ &= \ \text{all the fields } (A, \psi, \phi, \overline{\phi}, \eta, \chi, c, \overline{c}, b) \ , \\ \varphi^{*i} \ &= \ \text{all the antifields } (\Omega, \xi, L, D, \rho, \tau, B) \ . \end{aligned} \tag{4.115}$$

Let us now observe that, as a consequence of the fact that $\mathcal{P}_\mu$ acts linearly on the fields and antifields, the dependence of the complete action Σ from the corresponding translation constant ghost v^μ turns out to be fixed by the following linearly broken Ward identity, namely

$$\frac{\partial\Sigma}{\partial v^\mu} = \Delta^{cl}_\mu \ , \tag{4.116}$$

where

$$\begin{aligned}\Delta^{cl}_\mu \ &= \ tr\int d^4x(\ L\partial_\mu c - D\partial_\mu\phi - \Omega^\nu\partial_\mu A_\nu + \xi^\nu\partial_\mu\psi_\nu \\ &\qquad -\rho\partial_\mu\overline{\phi} + \tau\partial_\mu\eta + \frac{1}{2}B^{\nu\sigma}\partial_\mu\chi_{\nu\sigma} \ ) \ ,\end{aligned} \tag{4.117}$$

is a classical breaking, being linear in the quantum fields. We fall thus in the situation described in the **Remark 5** of Sect.2, meaning that we can completely eliminate the global constant ghost v^μ without any further consequence. Introducing in fact the action $\widehat{\Sigma}$ through

$$\begin{aligned}\Sigma \ &= \ \widehat{\Sigma} + v^\mu\Delta^{cl}_\mu \ , \\ \frac{\partial\widehat{\Sigma}}{\partial v^\mu} \ &= \ 0 \ ,\end{aligned} \tag{4.118}$$

it is easily verified from (4.111) that $\widehat{\Sigma}$ obeys the modified Slavnov-Taylor identity

$$\mathcal{S}(\widehat{\Sigma}) = \omega\varepsilon^\mu\Delta^{cl}_\mu \ \ , \tag{4.119}$$

with

$$\begin{aligned}\mathcal{S}(\widehat{\Sigma}) \ &= \ tr\int d^4x\left(\frac{\delta\widehat{\Sigma}}{\delta A^\mu}\frac{\delta\widehat{\Sigma}}{\delta\Omega_\mu} + \frac{\delta\widehat{\Sigma}}{\delta\xi^\mu}\frac{\delta\widehat{\Sigma}}{\delta\psi_\mu} + \frac{\delta\widehat{\Sigma}}{\delta L}\frac{\delta\widehat{\Sigma}}{\delta c} + \frac{\delta\widehat{\Sigma}}{\delta D}\frac{\delta\widehat{\Sigma}}{\delta\phi} \ + \frac{\delta\widehat{\Sigma}}{\delta\rho}\frac{\delta\widehat{\Sigma}}{\delta\overline{\phi}}\right. \\ &\qquad \left. + \frac{\delta\widehat{\Sigma}}{\delta\tau}\frac{\delta\widehat{\Sigma}}{\delta\eta} + \frac{1}{2}\frac{\delta\widehat{\Sigma}}{\delta B^{\mu\nu}}\frac{\delta\widehat{\Sigma}}{\delta\chi_{\mu\nu}} \ + b\frac{\delta\widehat{\Sigma}}{\delta\overline{c}} + \omega\varepsilon^\mu\partial_\mu\overline{c}\frac{\delta\widehat{\Sigma}}{\delta b}\right) \ ,\end{aligned} \tag{4.120}$$

and Δ^{cl}_μ as in eq.(4.117). The equation (4.119) represents the final form of the Slavnov-Taylor identity which will be taken as the starting point for the quantum analysis of the model. It is interesting to observe that, due to the elimination of the ghost parameter v^μ, the classical breaking Δ^{cl}_μ appears now on the right hand side of the identity (4.119), yielding thus a linearly broken Slavnov-Taylor identity. As a consequence the linearized Slavnov-Taylor operator $\mathcal{B}_{\widehat{\Sigma}}$ defined as

$$\mathcal{B}_{\widehat{\Sigma}} \ = \ tr\int d^4x\left(\frac{\delta\widehat{\Sigma}}{\delta A^\mu}\frac{\delta}{\delta\Omega_\mu} + \frac{\delta\widehat{\Sigma}}{\delta\Omega_\mu}\frac{\delta}{\delta A^\mu} + \frac{\delta\widehat{\Sigma}}{\delta\psi_\mu}\frac{\delta}{\delta\xi^\mu} + \frac{\delta\widehat{\Sigma}}{\delta\xi^\mu}\frac{\delta}{\delta\psi_\mu} + \frac{\delta\widehat{\Sigma}}{\delta L}\frac{\delta}{\delta c}\right)$$

$$
+\frac{\delta\widehat{\Sigma}}{\delta c}\frac{\delta}{\delta L}+\frac{\delta\widehat{\Sigma}}{\delta\phi}\frac{\delta}{\delta D}+\frac{\delta\widehat{\Sigma}}{\delta D}\frac{\delta}{\delta\phi}+\frac{\delta\widehat{\Sigma}}{\delta\overline{\phi}}\frac{\delta}{\delta\rho}+\frac{\delta\widehat{\Sigma}}{\delta\rho}\frac{\delta}{\delta\overline{\phi}}+\frac{\delta\widehat{\Sigma}}{\delta\eta}\frac{\delta}{\delta\tau}
+\frac{\delta\widehat{\Sigma}}{\delta\tau}\frac{\delta}{\delta\eta}+\frac{1}{2}\frac{\delta\widehat{\Sigma}}{\delta\chi_{\mu\nu}}\frac{\delta}{\delta B^{\mu\nu}}+\frac{1}{2}\frac{\delta\widehat{\Sigma}}{\delta B^{\mu\nu}}\frac{\delta}{\delta\chi_{\mu\nu}}+b\frac{\delta}{\delta\overline{c}}+\omega\varepsilon^{\mu}\partial_{\mu}\overline{c}\frac{\delta}{\delta b}\Bigg)\ , \tag{4.121}
$$

is not strictly nilpotent. Instead, we have

$$
\mathcal{B}_{\widehat{\Sigma}}\mathcal{B}_{\widehat{\Sigma}}=\omega\varepsilon^{\mu}\mathcal{P}_{\mu}\ , \tag{4.122}
$$

meaning that $\mathcal{B}_{\widehat{\Sigma}}$ is nilpotent only modulo a total derivative. It follows then that $\mathcal{B}_{\widehat{\Sigma}}$ becomes a nilpotent operator when acting on the space of the integrated local polynomials in the fields and antifields. This is the case, for instance, of the invariant counterterms and of the anomalies.

Besides the Slavnov-Taylor identity (4.119), the classical action $\widehat{\Sigma}$ turns out to be characterized by further additional constraints *[1]*, namely

- the Landau gauge fixing condition

$$
\frac{\delta\widehat{\Sigma}}{\delta b}=\partial A\ , \tag{4.123}
$$

- the antighost equation

$$
\frac{\delta\widehat{\Sigma}}{\delta\overline{c}}+\partial_{\mu}\frac{\delta\widehat{\Sigma}}{\delta\Omega_{\mu}}=0\ , \tag{4.124}
$$

- the linearly broken ghost Ward identity, typical of the Landau gauge

$$
\int d^4x\left(\frac{\delta\widehat{\Sigma}}{\delta c}+\left[\overline{c},\frac{\delta\widehat{\Sigma}}{\delta b}\right]\right)=\Delta_c^{cl}\ , \tag{4.125}
$$

with Δ_c^{cl} a linear classical breaking

$$
\Delta_c^{cl}=\int d^4x\left(\,[c,L]-[A,\Omega]-[\phi,D]+[\psi,\xi]-[\overline{\phi},\rho]+[\eta,\tau]+\frac{1}{2}[\chi,B]\right)\ . \tag{4.126}
$$

As usual, commuting the ghost equation (4.125) with the Slavnov-Taylor identity (4.119) one obtains the Ward identity for the rigid gauge invariance *[1]*, expressing the fact that all the fields and antifields belong to the adjoint representation of the gauge group.

4.4 Classical approximation: the reduced action

Following the standard procedure, let us introduce, for further use, the so called reduced action $\widetilde{\mathcal{S}}$ *[1]* defined through the gauge fixing condition (4.123) as

$$
\widehat{\Sigma}=\widetilde{\mathcal{S}}\ +\ tr\int d^4x\ b\partial A\ , \tag{4.127}
$$

so that $\widetilde{\mathcal{S}}$ is independent from the lagrangian multiplier b. Moreover, from the antighost equation (4.124) it follows that $\widetilde{\mathcal{S}}$ depends from the antighost $\overline{c}$ only through the combination[8] γ^μ

$$\gamma_\mu = \Omega_\mu + \partial_\mu \overline{c} \ . \tag{4.128}$$

Therefore, for $\widetilde{\mathcal{S}}$ we have

$$\begin{aligned}
\widetilde{\mathcal{S}} \ \ = \ \ & \frac{1}{g^2} tr \int d^4x \ \left(\frac{1}{2} F^+ F^+ - \chi^{\mu\nu}(D_\mu\psi_\nu - D_\nu\psi_\mu)^+ + \eta D_\mu \psi^\mu - \frac{1}{2}\overline{\phi} D_\mu D^\mu \phi \right. \\
& \qquad \left. + \frac{1}{2}\overline{\phi}\left\{\psi^\mu, \psi_\mu\right\} - \frac{1}{2}\phi\left\{\chi^{\mu\nu}, \chi_{\mu\nu}\right\} \ - \frac{1}{8}\left[\phi, \eta\right]\eta - \frac{1}{32}\left[\phi, \overline{\phi}\right]^2 \right) \\
& + tr \int d^4x \ \left(L(c^2 - \omega^2\phi - \omega\varepsilon^\mu A_\mu + \frac{\varepsilon^2}{16}\overline{\phi}) + D([c,\phi] - \varepsilon^\mu \psi_\mu) \right. \\
& \qquad + \gamma^\mu(-D_\mu c + \omega\psi_\mu + \frac{\varepsilon^\nu}{2}\chi_{\nu\mu} + \frac{\varepsilon_\mu}{8}\eta) \ + \ \rho(\left[c, \overline{\phi}\right] + 2\omega\eta) \\
& \qquad + \xi^\mu(\{c, \psi_\mu\} - \omega D_\mu \phi + \varepsilon^\nu F_{\nu\mu} - \frac{\varepsilon^\nu}{2} F^+_{\nu\mu} - \frac{\varepsilon_\mu}{16}[\phi, \overline{\phi}]) \\
& \qquad + \tau(\{c, \eta\} + \frac{\omega}{2}[\phi, \overline{\phi}] + \frac{\varepsilon^\mu}{2} D_\mu \overline{\phi}) \\
& \qquad \left. + \frac{1}{2} B^{\sigma\tau}(\{c, \chi_{\sigma\tau}\} + \omega F^+_{\sigma\tau} + \frac{1}{8}(\varepsilon^\mu \varepsilon_{\mu\sigma\tau\nu} + \varepsilon_\sigma g_{\nu\tau} - \varepsilon_\tau g_{\nu\sigma}) D^\nu \overline{\phi} \ ) \right) \\
& + tr \int d^4x \left(\frac{g^2}{8}\omega^2 B^{\mu\nu} B_{\mu\nu} - \frac{g^2}{4}\omega B^{\mu\nu}\varepsilon_\mu \xi_\nu - \frac{g^2}{32}\varepsilon^\mu \varepsilon^\nu \xi_\mu \xi_\nu + \frac{g^2}{32}\varepsilon^2 \xi^2 \right) \ .
\end{aligned} \tag{4.129}$$

Accordingly, for the Slavnov-Taylor identity (4.119) we get

$$\mathcal{S}(\widetilde{\mathcal{S}}) = \omega \varepsilon^\mu \widetilde{\Delta}^{cl}_\mu \ \ , \tag{4.130}$$

where $\mathcal{S}(\widetilde{\mathcal{S}})$ denotes now the homogeneous operator

$$\begin{aligned}
\mathcal{S}(\widetilde{\mathcal{S}}) \ \ = \ \ & tr \int d^4x \left(\frac{\delta\widetilde{\mathcal{S}}}{\delta A^\mu}\frac{\delta\widetilde{\mathcal{S}}}{\delta\gamma_\mu} + \frac{\delta\widetilde{\mathcal{S}}}{\delta\xi^\mu}\frac{\delta\widetilde{\mathcal{S}}}{\delta\psi_\mu} + \frac{\delta\widetilde{\mathcal{S}}}{\delta L}\frac{\delta\widetilde{\mathcal{S}}}{\delta c} + \frac{\delta\widetilde{\mathcal{S}}}{\delta D}\frac{\delta\widetilde{\mathcal{S}}}{\delta\phi} \right. \\
& \qquad \left. + \frac{\delta\widetilde{\mathcal{S}}}{\delta\rho}\frac{\delta\widetilde{\mathcal{S}}}{\delta\overline{\phi}} \ \ + \frac{\delta\widetilde{\mathcal{S}}}{\delta\tau}\frac{\delta\widetilde{\mathcal{S}}}{\delta\eta} + \frac{1}{2}\frac{\delta\widetilde{\mathcal{S}}}{\delta B^{\mu\nu}}\frac{\delta\widetilde{\mathcal{S}}}{\delta\chi_{\mu\nu}} \right) \ \ ,
\end{aligned} \tag{4.131}$$

and $\widetilde{\Delta}^{cl}_\mu$ is given by the expression (4.117) with Ω_μ replaced by γ_μ, *i.e.*

$$\begin{aligned}
\widetilde{\Delta}^{cl}_\mu \ \ = \ \ & tr \int d^4x(\ L\partial_\mu c - D\partial_\mu\phi - \gamma^\nu \partial_\mu A_\nu + \xi^\nu \partial_\mu \psi_\nu \\
& \qquad \qquad - \rho\partial_\mu\overline{\phi} + \tau\partial_\mu\eta + \frac{1}{2}B^{\nu\sigma}\partial_\mu\chi_{\nu\sigma} \ ) \ .
\end{aligned} \tag{4.132}$$

[8]The variable γ_μ is also called shifted antifield.

For the linearized Slavnov-Taylor operator $\mathcal{B}_{\widetilde{\mathcal{S}}}$ we obtain now

$$\begin{aligned} \mathcal{B}_{\widetilde{\mathcal{S}}} \;=\; tr \int d^4x \biggl(& \frac{\delta \widetilde{\mathcal{S}}}{\delta A^\mu} \frac{\delta}{\delta \gamma_\mu} + \frac{\delta \widetilde{\mathcal{S}}}{\delta \gamma_\mu} \frac{\delta}{\delta A^\mu} + \frac{\delta \widetilde{\mathcal{S}}}{\delta \psi_\mu} \frac{\delta}{\delta \xi^\mu} + \frac{\delta \widetilde{\mathcal{S}}}{\delta \xi^\mu} \frac{\delta}{\delta \psi_\mu} + \frac{\delta \widetilde{\mathcal{S}}}{\delta L} \frac{\delta}{\delta c} \\ & + \frac{\delta \widetilde{\mathcal{S}}}{\delta c} \frac{\delta}{\delta L} + \frac{\delta \widetilde{\mathcal{S}}}{\delta \phi} \frac{\delta}{\delta D} + \frac{\delta \widetilde{\mathcal{S}}}{\delta D} \frac{\delta}{\delta \phi} + \frac{\delta \widetilde{\mathcal{S}}}{\delta \overline{\phi}} \frac{\delta}{\delta \rho} + \frac{\delta \widetilde{\mathcal{S}}}{\delta \rho} \frac{\delta}{\delta \overline{\phi}} + \frac{\delta \widetilde{\mathcal{S}}}{\delta \eta} \frac{\delta}{\delta \tau} \\ & + \frac{\delta \widetilde{\mathcal{S}}}{\delta \tau} \frac{\delta}{\delta \eta} + \frac{1}{2} \frac{\delta \widetilde{\mathcal{S}}}{\delta \chi_{\mu\nu}} \frac{\delta}{\delta B^{\mu\nu}} + \frac{1}{2} \frac{\delta \widetilde{\mathcal{S}}}{\delta B^{\mu\nu}} \frac{\delta}{\delta \chi_{\mu\nu}} \biggr) \;, \end{aligned} \tag{4.133}$$

and

$$\mathcal{B}_{\widetilde{\mathcal{S}}} \mathcal{B}_{\widetilde{\mathcal{S}}} = \omega \varepsilon^\mu \mathcal{P}_\mu \; . \tag{4.134}$$

The usefulness of working with the reduced action relies on the fact that $\widetilde{\mathcal{S}}$ depends only on those variables which are really relevant for the quantum analysis. It is apparent, for instance, that the Landau gauge fixing condition (4.123) can be regarded indeed as a linearly broken local Ward identity, implying thus that the Lagrangian multiplier b cannot appear in the expression of the invariant counterterms and of the possible anomalies. Of course, the same holds for the antighost $\overline{c}$ which, due to the equation[9] (4.124), can enter only through the combination γ_μ.

Let us also recall, finally, the quantum numbers of all the fields and antifields entering the expression of the reduced action (4.129).

Quantum numbers

	A_μ	$\chi_{\mu\nu}$	ψ_μ	η	ϕ	$\overline{\phi}$
dim.	1	3/2	3/2	3/2	1	1
$\mathcal{R}-$ charge	0	-1	1	-1	2	-2
gh $-$ number	0	0	0	0	0	0
nature	*comm.*	*ant.*	*ant.*	*ant.*	*comm.*	*comm.*

Table 6. Fields (4.135)

Quantum numbers

	c	ω	ε_μ
dim.	0	$-1/2$	$-1/2$
$\mathcal{R}-$ charge	0	-1	1
gh $-$ number	1	1	1
nature	*ant.*	*comm.*	*comm.*

Table 7. Ghosts (4.136)

[9]Both the Landau gauge-fixing condition and the antighost equation can be proven to be renormalizable *[1]*.

Quantum numbers

	L	D	γ^{μ}	ξ^{μ}	ρ	τ	$B^{\mu\nu}$
dim.	4	3	3	5/2	3	5/2	5/2
$\mathcal{R}-$charge	0	-2	0	-1	2	1	1
gh $-$ number	-2	-1	-1	-1	-1	-1	-1
nature	*comm.*	*ant.*	*ant.*	*comm.*	*ant.*	*comm.*	*comm.*

(4.137)

Table 8. Antifileds

5 Renormalization of topological Yang-Mills

5.1 Cohomology of the linearized Slavnov-Taylor operator $\mathcal{B}_{\widetilde{\mathcal{S}}}$

We are now ready to discuss the renormalization of the TYM. As already underlined in Sect.2, the first task is that of characterizing the cohomology classes of the linearized Slavnov-Taylor operator which turn out to be relevant for the anomalies and the invariant counterterms. Let us recall that both anomalies and invariant counterterms are integrated local polynomials Δ^G in the fields $(A, \psi, \chi, \eta, \phi, \overline{\phi}, c)$, in the antifields $(L, D, \gamma, \xi, \rho, \tau, B)$, and in the global ghosts (ω, ε), with dimension four, vanishing $\mathcal{R}$-charge and ghost number G respectively one and zero. In addition, they are constrained by the consistency condition

$$\mathcal{B}_{\widetilde{\mathcal{S}}}\Delta^G = 0 \ , \qquad G = 0, 1 \ , \tag{5.138}$$

$\mathcal{B}_{\widetilde{\mathcal{S}}}$ being the linearized Slavnov-Taylor operator of eq.(4.133). From the relation (4.134), *i.e.*

$$\mathcal{B}_{\widetilde{\mathcal{S}}}\mathcal{B}_{\widetilde{\mathcal{S}}} = \omega\varepsilon^\mu\mathcal{P}_\mu \ , \tag{5.139}$$

one sees that $\mathcal{B}_{\widetilde{\mathcal{S}}}$ is in fact a nilpotent operator when acting on the space of the integrated functionals which are invariant under space-time translations. It follows therefore that the relevant solutions of the eq.(5.138) identify nontrivial elements of the integrated cohomology of $\mathcal{B}_{\widetilde{\mathcal{S}}}$. In order to characterize the integrated cohomology of $\mathcal{B}_{\widetilde{\mathcal{S}}}$ we introduce the operator

$$\mathcal{N}_\varepsilon = \varepsilon^\mu\frac{\partial}{\partial\varepsilon^\mu} \ , \tag{5.140}$$

which counts the number of global ghosts ε^μ contained in a given field polynomial. Accordingly, the functional operator $\mathcal{B}_{\widetilde{\mathcal{S}}}$ displays the following ε-expansion

$$\mathcal{B}_{\widetilde{\mathcal{S}}} = b_{\widetilde{\mathcal{S}}} + \varepsilon^\mu\mathcal{W}_\mu + \frac{1}{2}\varepsilon^\mu\varepsilon^\nu\mathcal{W}_{\mu\nu} \ , \tag{5.141}$$

where, from eq.(5.139) the operators $b_{\widetilde{\mathcal{S}}}$, $\mathcal{W}_\mu$, $\mathcal{W}_{\mu\nu}$ are easily seen to obey the following algebraic relations

$$b_{\widetilde{\mathcal{S}}}b_{\widetilde{\mathcal{S}}} = 0 \ , \tag{5.142}$$

$$\left\{b_{\widetilde{\mathcal{S}}}, \mathcal{W}_\mu\right\} = \omega\mathcal{P}_\mu \ , \tag{5.143}$$

and

$$
\begin{aligned}
\{\mathcal{W}_\mu, \mathcal{W}_\nu\} + \left\{b_{\widetilde{S}}, \mathcal{W}_{\mu\nu}\right\} &= 0 \, , \\
\{\mathcal{W}_\mu, \mathcal{W}_{\nu\rho}\} + \{\mathcal{W}_\nu, \mathcal{W}_{\rho\mu}\} + \{\mathcal{W}_\rho, \mathcal{W}_{\mu\nu}\} &= 0 \, , \\
\{\mathcal{W}_{\mu\nu}, \mathcal{W}_{\rho\sigma}\} + \{\mathcal{W}_{\mu\rho}, \mathcal{W}_{\nu\sigma}\} + \{\mathcal{W}_{\mu\sigma}, \mathcal{W}_{\nu\rho}\} &= 0 \, .
\end{aligned}
\tag{5.144}
$$

with

Quantum numbers

	$b_{\widetilde{S}}$	$\mathcal{W}_\mu$	$\mathcal{W}_{\mu\nu}$
dim.	0	1/2	1
$\mathcal{R}-$ charge	0	-1	-2
gh $-$ number	1	0	-1
nature	*ant.*	*ant.*	*ant.*

Table 9.

In particular, from the eqs.(5.142), (5.143) we observe that the operator $b_{\widetilde{S}}$ is strictly nilpotent and that the vector operator $\mathcal{W}_\mu$ allows to decompose the space-time translations $\mathcal{P}_\mu$ as a $b_{\widetilde{S}}-$anticommutator, providing thus an off-shell realization of the algebra (3.72). Furthermore, we shall check that the decomposition formula (5.143) will be of great usefulness in order to obtain the nontrivial expression of the invariant counterterms of the complete operator $\mathcal{B}_{\widetilde{S}}$. Let us also give, for further use, the explicit form of the operators $b_{\widetilde{S}}$, $\mathcal{W}_\mu$, $\mathcal{W}_{\mu\nu}$, namely

$$
\begin{aligned}
b_{\widetilde{S}} A_\mu &= -D_\mu c + \omega \psi_\mu \, , \\
b_{\widetilde{S}} \psi_\mu &= \{c, \psi_\mu\} - \omega D_\mu \phi \, , \\
b_{\widetilde{S}} c &= c^2 - \omega^2 \phi \, , \\
b_{\widetilde{S}} \phi &= [c, \phi] \, , \\
b_{\widetilde{S}} \overline{\phi} &= \left[c, \overline{\phi}\right] + 2\omega\eta \, , \\
b_{\widetilde{S}} \eta &= \{c, \eta\} + \frac{\omega}{2}\left[\phi, \overline{\phi}\right] \, , \\
b_{\widetilde{S}} \chi_{\sigma\tau} &= \{c, \chi_{\sigma\tau}\} + \omega F^+_{\sigma\tau} + \frac{g^2}{2}\omega^2 B_{\sigma\tau} \, ,
\end{aligned}
\tag{5.145}
$$

and

$$
\begin{aligned}
b_{\widetilde{S}} \gamma_\mu &= \frac{1}{g^2}\left(4D^\nu F_{\mu\nu} + 4\{\psi^\nu, \chi_{\mu\nu}\} - \{\psi_\mu, \eta\} + \frac{1}{2}\left[\overline{\phi}, D_\mu \phi\right] - \frac{1}{2}\left[D_\mu \overline{\phi}, \phi\right]\right) \\
&\quad -\omega[\phi, \xi_\mu] + 2\omega D^\nu B_{\mu\nu} + \{c, \gamma_\mu\} \, , \\
b_{\widetilde{S}} \xi_\mu &= \frac{1}{g^2}\left(4D^\nu \chi_{\mu\nu} + D_\mu \eta - \left[\overline{\phi}, \psi_\mu\right]\right) - \omega\gamma_\mu + [c, \xi_\mu] \, , \\
b_{\widetilde{S}} L &= [c, L] - [\phi, D] - D^\mu \gamma_\mu + [\psi_\mu, \xi^\mu] - \left[\overline{\phi}, \rho\right] + [\eta, \tau] + \frac{1}{2}[\chi^{\sigma\tau}, B_{\sigma\tau}] \, ,
\end{aligned}
$$

$$
\begin{aligned}
b_{\widetilde{S}} D &= \frac{1}{g^2}\left(-\frac{1}{2}D^\mu D_\mu \overline{\phi} - \frac{1}{2}\left\{\chi^{\mu\nu}, \chi_{\mu\nu}\right\} - \frac{1}{8}\left\{\eta, \eta\right\} - \frac{1}{16}\left[\overline{\phi}, \left[\phi, \overline{\phi}\right]\right]\right) \\
&\quad + \omega D^\mu \xi_\mu - \omega^2 L + \frac{\omega}{2}\left[\overline{\phi}, \tau\right] + \left\{\, c, D\right\} \ , \\
b_{\widetilde{S}} \rho &= \frac{1}{2g^2}\left(-D^\mu D_\mu \phi + \left\{\psi^\mu, \psi_\mu\right\} + \frac{1}{8}\left[\phi, \left[\phi, \overline{\phi}\right]\right]\right) - \frac{\omega}{2}\left[\phi, \tau\right] + \left\{c, \rho\right\} \ , \\
b_{\widetilde{S}} \tau &= \frac{1}{g^2}\left(D^\mu \psi_\mu + \frac{1}{4}\left[\phi, \eta\right]\right) - 2\omega\rho + \left[c, \tau\right] \ , \qquad (5.146) \\
b_{\widetilde{S}} B_{\mu\nu} &= \frac{1}{g^2}\left(-2(D_\mu \psi_\nu - D_\nu \psi_\mu)^+ + 2\left[\phi, \chi_{\mu\nu}\right]\right) + \left[c, B_{\mu\nu}\right] \ .
\end{aligned}
$$

For the operator $\mathcal{W}_\mu$ we get

$$
\begin{aligned}
\mathcal{W}_\mu A_\nu &= \frac{1}{2}\chi_{\mu\nu} + \frac{1}{8}g_{\mu\nu}\eta \ , \\
\mathcal{W}_\mu \psi_\nu &= F_{\mu\nu} - \frac{1}{2}F^+_{\mu\nu} - \frac{1}{16}g_{\mu\nu}\left[\phi, \overline{\phi}\right] - \frac{g^2}{4}\omega B_{\mu\nu} \ , \\
\mathcal{W}_\mu c &= -\omega A_\mu \ , \\
\mathcal{W}_\mu \phi &= -\psi_\mu \ , \\
\mathcal{W}_\mu \overline{\phi} &= 0 \ , \\
\mathcal{W}_\mu \eta &= \frac{1}{2}D_\mu \overline{\phi} \ , \\
\mathcal{W}_\mu \chi_{\sigma\tau} &= \frac{1}{8}\left(\varepsilon_{\mu\sigma\tau\nu}D^\nu \overline{\phi} + g_{\mu\sigma}D_\tau \overline{\phi} - g_{\mu\tau}D_\sigma \overline{\phi}\right) \\
&\quad - \frac{g^2}{8}\omega\left(\varepsilon_{\mu\sigma\tau\nu}\xi^\nu + \eta_{\mu\sigma}\xi_\tau - \eta_{\mu\tau}\xi_\sigma\right) \ , \qquad (5.147)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{W}_\mu \gamma_\sigma &= \left(-\omega g_{\mu\sigma} L - \frac{1}{2}D_\mu \xi_\sigma + \frac{1}{2}g_{\mu\sigma}D^\tau \xi_\tau + \frac{1}{2}g_{\mu\sigma}\left[\overline{\phi}, \tau\right]\right. \\
&\quad \left. - \frac{1}{4}\left[B_{\mu\sigma}, \overline{\phi}\right] - \frac{1}{2}\varepsilon_{\mu\sigma\nu\rho}D^\nu \xi^\rho \ ,\right) \\
\mathcal{W}_\mu \xi_\sigma &= g_{\mu\sigma} D \ , \\
\mathcal{W}_\mu L &= 0 \ , \\
\mathcal{W}_\mu D &= -\frac{1}{16}\left[\overline{\phi}, \xi_\mu\right] \ , \\
\mathcal{W}_\mu \rho &= \frac{1}{16}\left[\phi, \xi_\mu\right] - \frac{1}{2}D_\mu \tau - \frac{1}{4}D^\nu B_{\mu\nu} \ , \\
\mathcal{W}_\mu \tau &= -\frac{1}{8}\gamma_\mu \ , \qquad (5.148) \\
\mathcal{W}_\mu B_{\sigma\tau} &= -\frac{1}{4}\left(\varepsilon_{\mu\sigma\tau\nu}\gamma^\nu + g_{\mu\sigma}\gamma_\tau - g_{\mu\tau}\gamma_\sigma\right) \ .
\end{aligned}
$$

Finally, for $\mathcal{W}_{\mu\nu}$ one obtains

$$
\begin{aligned}
\mathcal{W}_{\mu\nu} A_\sigma &= \mathcal{W}_{\mu\nu}\phi = \mathcal{W}_{\mu\nu}\overline{\phi} = \mathcal{W}_{\mu\nu}\eta = \mathcal{W}_{\mu\nu}\chi_{\sigma\tau} = 0 \ , \\
\mathcal{W}_{\mu\nu}\psi_\sigma &= \frac{g^2}{16}\left(2g_{\mu\nu}\xi_\sigma - g_{\mu\sigma}\xi_\nu - g_{\nu\sigma}\xi_\mu\right) \ , \\
\mathcal{W}_{\mu\nu} c &= \frac{1}{8} g_{\mu\nu}\overline{\phi} \ ,
\end{aligned} \tag{5.149}
$$

and, for the antifields

$$
\begin{aligned}
\mathcal{W}_{\mu\nu}\gamma_\sigma &= \mathcal{W}_{\mu\nu}\xi_\sigma = \mathcal{W}_{\mu\nu}L = \mathcal{W}_{\mu\nu}D = \mathcal{W}_{\mu\nu}\tau = \mathcal{W}_{\mu\nu}B_{\sigma\tau} = 0 \ , \\
\mathcal{W}_{\mu\nu}\rho &= \frac{1}{8} g_{\mu\nu} L \ .
\end{aligned} \tag{5.150}
$$

According to the general results on cohomology given in the Subsect. 2.3, the integrated cohomology of $\mathcal{B}_{\widetilde{\mathcal{S}}}$ is isomorphic to a subspace of the integrated cohomology of $b_{\widetilde{\mathcal{S}}}$. Let us therefore first focus on the operator $b_{\widetilde{\mathcal{S}}}$. We have now to mention that one of the advantages of having decomposed the operator $\mathcal{B}_{\widetilde{\mathcal{S}}}$ according to the counting operator (5.140) is due to the fact that the (nonintegrated) cohomology classes of $b_{\widetilde{\mathcal{S}}}$ have been already identified by the authors of ref. *[18]*, who computed in fact the cohomology of $b_{\widetilde{\mathcal{S}}}$ in terms of the so called *invariant* or *constrained* cohomology, the name invariant cohomology standing for the computation of the cohomology in the space of the gauge invariant polynomials. Their result can be immediately adapted to our present case, being stated as follows:

- *The cohomology classes of the operator $b_{\widetilde{\mathcal{S}}}$ in the space of the nonintegrated local polynomials in the fields and antifields which are analytic in the global ghosts are given by invariant polynomials in the undifferentiated field ϕ built up with monomials $\mathcal{P}_n(\phi)$ of the type*

$$
\mathcal{P}_n(\phi) = tr\left(\frac{\phi^n}{n}\right) \ , \qquad n \geq 2 \ . \tag{5.151}
$$

Remark 16 *Although the formal proof of the above result can be found in the original work [18], let us present here a very simple and intuitive argument for a better understanding of eq.*(5.151)*. Following [18] we further decompose the operator $b_{\widetilde{\mathcal{S}}}$ according to the counting operator $\mathcal{N} = \omega\partial/\partial\omega$, i.e.*

$$
b_{\widetilde{\mathcal{S}}} = b^0_{\widetilde{\mathcal{S}}} + \omega b^1_{\widetilde{\mathcal{S}}} + \omega^2 b^2_{\widetilde{\mathcal{S}}} \ . \tag{5.152}
$$

From eqs.(5.145)*,* (5.146) *it is apparent to see that the first term $b^0_{\widetilde{\mathcal{S}}}$ of the decomposition* (5.152) *picks up the part of the operator $b_{\widetilde{\mathcal{S}}}$ corresponding to the pure gauge transformations, while the second and the third term $(b^1_{\widetilde{\mathcal{S}}}, b^2_{\widetilde{\mathcal{S}}})$ have basically the effect of a shift transformation. This is particularly evident if one looks at the first set of transformations* (5.145) *concerning only the fields $(A, \psi, \chi, \eta, \phi, \overline{\phi}, c)$. It should also be remarked that among the fields $(A, \psi, \chi, \eta, \phi, \overline{\phi}, c)$ the scalar component ϕ is the* *only field whose transformation does not contain the global ghost ω, so that the action of the operator $b_{\widetilde{\mathcal{S}}}$ on ϕ reduces to a simple pure gauge transformation. Recalling now that the cohomology of $b_{\widetilde{\mathcal{S}}}$ is, in turn, isomorphic to a subspace of the cohomology of $b^0_{\widetilde{\mathcal{S}}}$, we can easily understand that the characterization of the cohomology of $b_{\widetilde{\mathcal{S}}}$ can be reduced in fact to a computation of the so called invariant cohomology, i.e. to the computation of the cohomology in the space of the gauge invariant polynomials which are left unchanged by the shift transformations corresponding to $(b^1_{\widetilde{\mathcal{S}}}, b^2_{\widetilde{\mathcal{S}}})$. It follows therefore that the only invariants which survive are exactely the polynomials in the undifferentiated field ϕ, justifying thus the result (5.151). We also remark that the polynomials $\mathcal{P}_n(\phi)$ can be eventually multiplied by appropriate powers in the constant ghosts $(\varepsilon_\mu, \omega)$ in order to obtain invariants with the right $\mathcal{R}$-charge, ghost number and dimension. Finally let us underline that, due to the commuting nature of the field ϕ, the expression $tr(\phi^n)$ (for n sufficiently large) is related to higher order invariant Casimir tensors whose existence relies on the choice of the gauge group G.*

5.2 Analyticity in the constant ghosts and triviality of the cohomology of $b_{\widetilde{\mathcal{S}}}$

Before analysing the consequences which follow from the result (5.151) on the cohomology of the complete operator $\mathcal{B}_{\widetilde{\mathcal{S}}}$, let us discuss here the important issue of the analyticity in the constant global ghosts. In fact, as repeatedly mentioned in the previous Sections, the requirement of analyticity in the ghosts $(\varepsilon_\mu, \omega)$, stemming from pure perturbative considerations, is one of the most important ingredient in order to interpret the TYM in terms of a standard field theory which can be characterized by a nonvanishing BRST cohomology. In particular, we have already emphasized that the nonemptiness of the BRST cohomology relies exactly on the analyticity requirement. For a better understanding of this point, let us consider in detail the case of the simplest invariant polynomial $\mathcal{P}_2(\phi)$

$$\mathcal{P}_2(\phi) \ = \frac{1}{2} tr\phi^2 \ . \tag{5.153}$$

It is an almost trivial exercise to show that $tr\phi^2$ can be expressed indeed as a pure $b_{\widetilde{\mathcal{S}}}-$variation, namely

$$tr\phi^2 = b_{\widetilde{\mathcal{S}}} tr\left(-\frac{1}{\omega^2}c\phi + \frac{1}{3\omega^4}c^3\right) \ . \tag{5.154}$$

This formula illustrates in a very clear way the relevance of the analyticity requirement. It is apparent from the eq.(5.154) that the price to be payed in order to write $tr\phi^2$ as a pure $b_{\widetilde{\mathcal{S}}}-$variation is in fact the loss of analyticity in the global ghost ω.

In other words, as long as one works in a functional space whose elements are power series in the global ghosts, the cohomology of $b_{\widetilde{\mathcal{S}}}$ is not empty. As we shall see later on, this will imply that also the cohomology of $\mathcal{B}_{\widetilde{\mathcal{S}}}$ will be nontrivial, meaning that TYM can be regarded as a standard supersymmetric gauge theory of the Yang-Mills type. On the other hand, if the analyticity requirement is relaxed, the cohomology of $b_{\widetilde{\mathcal{S}}}$, and therefore that of the complete operator $\mathcal{B}_{\widetilde{S}}$, becomes trivial, leading thus to the cohomological interpretation of Baulieu-Singer *[25]* and Labastida-Pernici *[24]*. We see therefore that the analyticity requirement in the global ghosts is the property which intertwines the two possible interpretation of TYM. One goes from the standard field theory point of view to the cohomological one by simply setting $\omega = 1$, which of course implies that analyticity is lost. In addition, it is rather simple to convince oneself that setting $\omega = 1$ has the meaning of identifying the $\mathcal{R}$-charge with the ghost number, so that the fields $(\chi, \psi, \eta, \phi, \overline{\phi})$ aquire a nonvanishing ghost number given respectively by $(-1, 1, -1, 2, -2)$. They correspond now to the so called topological ghosts of the cohomological interpretation.

Remark 17 *It is also interesting to point out that there is a deep relationship between the analyticity in the global ghosts and the so called equivariant cohomology proposed by R. Stora et al. [16, 17] in order to recover the Witten's observables in the case in which TYM is considered as a cohomological theory with vanishing BRST cohomology. Roughly speaking, the equivariant cohomology can be defined as the restriction of the BRST cohomology to the space of the gauge invariant polynomials which cannot be written as the BRST variation of local quantities which are independent from the Faddeev-Popov ghost c. In other words, a gauge invariant cocycle ϑ is called nontrivial in the equivariant cohomology if ϑ cannot be written as the $b_{\widetilde{S}}$-variation of a local polynomial $\widetilde{\vartheta}$ which is independent from the ghost c, i.e. if*

$$\vartheta = b_{\widetilde{S}} \widetilde{\vartheta} \ , \qquad (5.155)$$

with $\widetilde{\vartheta}$ containing necessarily c, then ϑ identifies a nontrivial element of the equivariant cohomology[10]. *Considering now the polynomial $tr\phi^2$, we see that it yields a nontrivial equivariant cocycle in the cohomological interpretation (i.e. $\omega = 1$), due to the unavoidable presence of the Faddeev-Popov ghost c on the right hand side of eq.(5.154). Moreover, keeping the standard point of view (i.e. $\omega \neq 1$), it is apparent that in order to write $tr\phi^2$ as a trivial cocycle use has to be done of the effective variable c/ω, meaning that the loss of anlyticity is accompanied by the presence of the Faddeev-Popov ghost c. This shows that, in the case of the invariant polynomials $\mathcal{P}_n(\phi)$, the analyticity requirement and the equivariant cohomology identify indeed the same class of invariants [15].*

5.3 The integrated cohomology of $b_{\widetilde{S}}$ and the absence of anomalies.

Having characterized the cohomology of $b_{\widetilde{S}}$, let us now briefly analyse the integrated cohomology classes or, equivalently, the local cocycles which belong to the cohomology of $b_{\widetilde{S}}$ modulo a total space-time derivative. Let us begin first with the integrated cohomology classes which have the same quantum numbers of the invariant counterterms, *i.e.* dimension four and vanishing $\mathcal{R}-$charge and ghost number. Combining the result (5.151) with the technique of the descent equations[11], it turns out that the integrated cohomology

[10] See ref.*[17]* for a geometrical construction of the equivariant cohomology classes.
[11] See Chapt.5 of ref.*[1]* for a self contained illustration of the method.

classes of $b_{\widetilde{S}}$ with dimension four and vanishing ghost number can be identified, modulo exact cocycles, with the following three elements,

$$\int d^4x \; tr(\phi^4) \; , \qquad \int d^4x \; \left(tr\phi^2\right)^2 \; , \tag{5.156}$$

and

$$\varepsilon^{\mu\nu\rho\tau}\mathcal{W}_\mu\mathcal{W}_\nu\mathcal{W}_\rho\mathcal{W}_\tau \int d^4x \; tr\left(\frac{\phi^2}{2}\right) \; , \tag{5.157}$$

with $\mathcal{W}_\mu$ given in eqs.(5.141) , (5.147) , (5.148). The first two terms in eq. (5.156) possess $\mathcal{R}$-charge 8 and therefore have to be ruled out. The only term with the correct quantum numbers (see Tables 6-8 of eqs.(4.135)-(4.137)) is thus that of eq.(5.157) . The invariance of the term (5.157) under the action of $b_{\widetilde{S}}$ easily follows from the decomposition (5.143), which implies that $b_{\widetilde{S}}$ and $\mathcal{W}_\mu$ can be regarded as anticommuting operators when acting on the space of the integrated local functionals. Its nontriviality is easily seen to be a consequence of the nontriviality of $(tr\phi^2)$, according to eq.(5.151) .

Remark 18 *It is worth to underline here that expressions of the type of eq.*(5.157) *are not a novelty, as they appear rather naturally in the study of the BRST cohomology of the topological models. Their occurrence lies precisely on the existence of a vector operator* $\mathcal{W}_\mu$ *which allows to decompose the space-time derivatives as a BRST anticommutator, as in the equation* (5.143)*. As shown in [26], the decomposition formula* (5.143) *turns out to be of great importance in order to obtain the integrated cohomology classes from the nonintegrated ones. The operator* $\mathcal{W}_\mu$ *plays in fact the role of a climbing operator which allows to solve in a very straightforward and elegant way the descent equations corresponding to the integrated cohomology. It is a simple exercise, for instance, to verify that the following cocycles, respectively a one, two and a three form,*

$$\begin{aligned} &\mathcal{W}_\mu \left(tr\frac{\phi^2}{2}\right) dx^\mu \; , \\ &\mathcal{W}_\mu\mathcal{W}_\nu \left(tr\frac{\phi^2}{2}\right) dx^\mu \wedge dx^\nu \; , \\ &\mathcal{W}_\mu\mathcal{W}_\nu\mathcal{W}_\rho \left(tr\frac{\phi^2}{2}\right) dx^\mu \wedge dx^\nu \wedge dx^\rho \; , \end{aligned} \tag{5.158}$$

belong indeed to the cohomology of $b_{\widetilde{S}}$ *modulo a total derivative. The expression* (5.158) *are seen to reproduce (modulo trivial terms) the Witten observables of the TYM theory [11], giving a more direct idea of the usefulness of the operator* $\mathcal{W}_\mu$*.*

Turning now to the integrated cohomology classes in the sector of dimension four, $\mathcal{R}$-charge zero and ghost number one, it can be proven that the result (5.151) implies that the only integrated invariants which can be defined are those which are $b_{\widetilde{S}}$-exact, meaning that the integrated cohomology of $b_{\widetilde{S}}$ in the sector of the anomalies is empty. Therefore the integrated cohomology of the complete operator $\mathcal{B}_{\widetilde{S}}$ turns out to be empty as well, so that the classical Slavnov-Taylor identity (4.119) can be extended at the quantum level without anomalies. It is important to mention that this result, already obtained by N. Maggiore *[13]* in the analysis of the $N=2$ untwisted gauge theories, means, in particular, that there is no possible extension of the nonabelian Adler-Bardeen gauge anomaly compatible with $N=2$ supersymmetry.

In summary, we have seen that the operator $b_{\widetilde{\mathcal{S}}}$ has a nonvanishing integrated cohomology only in the sector of the invariant counterterms, in which the unique nontrivial element is given by the expression (5.157). We have now to recall that the computation of the cohomology of $b_{\widetilde{\mathcal{S}}}$ is only a first step towards the characterization of the nontrivial classes of the complete operator $\mathcal{B}_{\widetilde{\mathcal{S}}}$. Moreover, from the previous results, we can infer that the cohomology of $\mathcal{B}_{\widetilde{\mathcal{S}}}$ in the sector of the invariant counterterms can contain at most a unique element. The task of the next final section will be that of providing the expression of the unique nontrivial invariant counterterm of TYM.

5.4 The invariant counterterm of TYM

Let us face now the problem of the characterization of the most general local invariant counterterm Δ^{count} which can be freely added, to each order of perturbation theory, to the vertex functional Γ which fulfils the quantum version of the Slavnov-Taylor identity[12] (4.119),

$$
\begin{aligned}
\mathcal{S}(\Gamma) &= \omega\varepsilon^{\mu}\Delta^{cl}_{\mu} \ , \\
\Gamma &= \widehat{\Sigma} + O(\hbar) \ .
\end{aligned}
\tag{5.159}
$$

We look then for an integrated local polynomial in the fields, antifields and global ghosts with dimension four and vanishing $\mathcal{R}$-charge and ghost number, which is a nontrivial solution of the consistency condition

$$
\mathcal{B}_{\widetilde{\mathcal{S}}}\Delta^{count} = 0 \ . \tag{5.160}
$$

In order to find a candidate for Δ^{count} we observe that the classical breaking term in the right-hand side of eqs.(4.119) and (5.159) does not depend from the gauge coupling constant g of TYM, according to eq.(4.117). Therefore, acting with $\partial/\partial g$ on both side of eq.(4.130) we get

$$
\mathcal{B}_{\widetilde{\mathcal{S}}}\frac{\partial\widetilde{\mathcal{S}}}{\partial g} = 0 \ . \tag{5.161}
$$

We see that $\partial\widetilde{\mathcal{S}}/\partial g$ yields a solution of the consistency condition (5.160). Of course, it remains to prove that $\partial\widetilde{\mathcal{S}}/\partial g$ identifies a nontrivial element of the cohomology of $\mathcal{B}_{\widetilde{\mathcal{S}}}$. In order to prove the nontriviality of $\partial\widetilde{\mathcal{S}}/\partial g$ we proceed, following a well known standard cohomology argument, by assuming the converse, *i.e.* that $\partial\widetilde{\mathcal{S}}/\partial g$ can be written as an exact cocycle:

[12] We remind here that, due to the absence of anomalies, the classical Slavnov-Taylor identity (4.119) can be always extended at the quantum level.

$$\frac{\partial \widetilde{\mathcal{S}}}{\partial g} = \mathcal{B}_{\widetilde{\mathcal{S}}} \Xi^{-1} \,, \tag{5.162}$$

for some integrated local polynomial Ξ^{-1} in the fields, antifields and constant ghosts with negative ghost number. Decomposing now the equation (5.162) according to the counting operator $\mathcal{N}_\varepsilon = \varepsilon^\mu \partial / \partial \varepsilon^\mu$ of eq.(5.140) we get, to zeroth order in ε^μ,

$$\left(\frac{\partial \widetilde{\mathcal{S}}}{\partial g} \right)_{\varepsilon=0} = b_{\widetilde{\mathcal{S}}} \Xi^{-1}_{\varepsilon=0} \,, \tag{5.163}$$

with $\left(\partial \widetilde{\mathcal{S}} / \partial g \right)_{\varepsilon=0}$ given by

$$\left(\frac{\partial \widetilde{\mathcal{S}}}{\partial g} \right)_{\varepsilon=0} = -\frac{2}{g} \mathcal{S}_{TYM} + \frac{g \omega^2}{4} \int d^4x \; tr B^{\mu\nu} B_{\mu\nu} \,, \tag{5.164}$$

and $\mathcal{S}_{TYM}$ being the original TYM action of eq.(3.52) . However, it is very simple to check that the expression (5.164) can be rewritten as

$$\begin{aligned} \left(g \frac{\partial \widetilde{\mathcal{S}}}{\partial g} \right)_{\varepsilon=0} &= \frac{2}{3g^3} \varepsilon^{\mu\nu\rho\tau} \mathcal{W}_\mu \mathcal{W}_\nu \mathcal{W}_\rho \mathcal{W}_\tau \int d^4x \; tr \left(\frac{\phi^2}{2} \right) \\ &\quad + b_{\widetilde{\mathcal{S}}} \int d^4x \; tr \left(\phi D - \xi^\mu \psi_\mu \right) \;. \end{aligned} \tag{5.165}$$

Therefore, according to the analysis of the previous section, it follows that the term $\left(\partial \widetilde{\mathcal{S}} / \partial g \right)_{\varepsilon=0}$ belongs to the integrated cohomology of $b_{\widetilde{\mathcal{S}}}$. Equation (5.162) cannot thus be satisfied, meaning that $\partial \widetilde{\mathcal{S}} / \partial g$ identifies indeed a nontrivial element of the cohomology of $\mathcal{B}_{\widetilde{\mathcal{S}}}$. We can conclude therefore that the symmetry content of the topological Yang-Mills theory allows for a unique invariant nontrivial counterterm whose most general expression can be written as

$$\Delta^{count} = \varsigma_g \frac{\partial \widetilde{\mathcal{S}}}{\partial g} + \mathcal{B}_{\widetilde{\mathcal{S}}} \Delta^{-1} \,, \tag{5.166}$$

ς_g being an arbitrary free parameter corresponding to a possible renormalization of the gauge coupling constant g. The result (5.166) is in complete agreement with that found in ref. *[13]* in the case of untwisted $N = 2$ YM. This concludes the algebraic renormalization analysis of Witten's TYM.

Remark 19 *Observe that eqs.*(5.164), (5.165) *imply that the expression of the original TYM action* (3.52) *can be rewritten as*

$$\mathcal{S}_{TYM} \;=\; -\left(\frac{1}{3g^3} \varepsilon^{\mu\nu\rho\tau} \mathcal{W}_\mu \mathcal{W}_\nu \mathcal{W}_\rho \mathcal{W}_\tau \int d^4x \; tr \left(\frac{\phi^2}{2} \right) \right)_{\varepsilon=\omega=0} \;+\; b_{\widetilde{\mathcal{S}}} (...) \,, \tag{5.167}$$

This suggestive formula shows that the origin of the Witten's action can be in fact traced back, modulo an irrelevant exact cocycle, to the invariant polynomial $tr\phi^2$.

Remark 20 *Let us also recall here that the explicit Feynman diagrams computation yields a nonvanishing value for the renormalization constant ς_g, meaning that TYM (understood here as the twisted version of $N = 2$ YM) possesses a nonvanishing β function for the gauge coupling g. The latter agrees with that of the pure $N = 2$ untwisted Yang-Mills [22]. Moreover, it is well known that the β function of $N = 2$ Yang-Mills theory receives only one loop order contributions [23]. This important result is commonly understood in terms of the nonrenormalization theorem for the $U(1)$ axial anomaly which, due to supersymmetry, belongs to the same supercurrent multiplet of the energy-momentum tensor. This implies that there is a relationship between the coefficient of the axial anomaly and the β function, providing then a useful argument in order to understand the absence of higher order corrections for the $N = 2$ gauge Yang-Mills theory. On the other hand, the formula* (5.167) *shows that the TYM action $\mathcal{S}_{TYM}$ is directly related to the invariant polynomial $tr\,(\phi^2)$. It is known since several years that in the $N = 2$ susy gauge theories the Green's functions with the insertion of composite operators of the kind of the invariant polynomials $\mathcal{P}_n(\phi)$ of eq.*(5.151) *display remarkable finiteness properties and can be computed exactely, even when nonperturbative effects are taken into account [27]. It is natural therefore to expect that the finiteness properties of $tr\,(\phi^2)$ are at the origin of the absence of higher order corrections for the gauge β function of both twisted and untwisted $N = 2$ gauge theories. In other words the relation* (5.165) *could give us an alternative understanding of the nonrenormalization theorem for the $N = 2$ gauge β function. We shall hope to report soon on this aspect in a more formal and detailed work.*

.

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A Appendix

A.1 Susy conventions in euclidean space-time

The supersymmetric conventions adopted here are those which can be found in ref. *[28]*. For the matrices $(\sigma_\mu,\ \overline{\sigma}_\mu)$ we have

$$(\sigma^\mu)_{\alpha\dot{\alpha}} = (1, i\vec{\sigma}) \ , \quad (\overline{\sigma}^\mu)_{\dot{\alpha}\alpha} = (1, -i\vec{\sigma}) \ , \tag{A.168}$$

$\vec{\sigma}$ being the Pauli matrices. As usual, the spinor indices $(\alpha, \dot{\alpha})$ are rised and lowered by means of the antisymmetric tensors $\varepsilon_{\alpha\beta},\ \varepsilon_{\dot{\alpha}\dot{\beta}}$. The matrices $(\sigma_\mu,\ \overline{\sigma}_\mu)$ obey the following algebra

$$\begin{aligned} \sigma_\mu\overline{\sigma}_\nu + \sigma_\nu\overline{\sigma}_\mu &= 2g_{\mu\nu} \ , \\ \overline{\sigma}_\mu\sigma_\nu + \overline{\sigma}_\nu\sigma_\mu &= 2g_{\mu\nu} \ , \end{aligned} \tag{A.169}$$

as well as the completeness relations

$$(\sigma^\mu)_{\alpha\dot{\alpha}} (\overline{\sigma}_\mu)^{\dot{\beta}\beta} = 2\delta^\beta_\alpha \delta^{\dot{\beta}}_{\dot{\alpha}} \ , \tag{A.170}$$

with $g_{\mu\nu}$ the flat euclidean metric $g_{\mu\nu} = diag(+,+,+,+)$.

For the antisymmetric matrices $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ we have respectively

$$\begin{aligned} \sigma^{\mu\nu} &= \frac{1}{2}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu) \ , \qquad \sigma^{\mu\nu} = \widetilde{\sigma}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} \ , \\ \overline{\sigma}^{\mu\nu} &= \frac{1}{2}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu) \ , \qquad \overline{\sigma}^{\mu\nu} = -\widetilde{\overline{\sigma}}^{\mu\nu} = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\overline{\sigma}_{\rho\sigma} \ , \end{aligned} \tag{A.171}$$

and

$$\begin{aligned} \sigma^\mu\overline{\sigma}^\nu &= g^{\mu\nu} + \sigma^{\mu\nu} \ , \\ \overline{\sigma}^\mu\sigma^\nu &= g^{\mu\nu} + \overline{\sigma}^{\mu\nu} \ . \end{aligned} \tag{A.172}$$

The following useful relations hold

$$\begin{aligned} \sigma^{\mu\nu}\sigma^\lambda &= g^{\nu\lambda}\sigma^\mu - g^{\mu\lambda}\sigma^\nu - \varepsilon^{\mu\nu\lambda\rho}\sigma_\rho \ , \\ \overline{\sigma}^{\mu\nu}\overline{\sigma}^\lambda &= g^{\nu\lambda}\overline{\sigma}^\mu - g^{\mu\lambda}\overline{\sigma}^\nu + \varepsilon^{\mu\nu\lambda\rho}\overline{\sigma}_\rho \ , \\ \sigma^\lambda\overline{\sigma}^{\mu\nu} &= g^{\mu\lambda}\sigma^\nu - g^{\nu\lambda}\sigma^\mu - \varepsilon^{\mu\nu\lambda\rho}\sigma_\rho \ , \\ \overline{\sigma}^\lambda\sigma^{\mu\nu} &= g^{\mu\lambda}\overline{\sigma}^\nu - g^{\nu\lambda}\overline{\sigma}^\mu + \varepsilon^{\mu\nu\lambda\rho}\overline{\sigma}_\rho \ . \end{aligned} \tag{A.173}$$

B Appendix

B.1 Tensorial self-dual transformations

As expected, the TYM action of eq.(3.52) is left invariant by a set of nonlinear transformations whose generators $\delta_{\mu\nu}$ are self-dual, *i.e.* $\delta_{\mu\nu} = \widetilde{\delta}_{\mu\nu}$. They read

$$
\begin{aligned}
\delta_{\mu\nu}A_\sigma &= -(\varepsilon_{\mu\nu\sigma\tau}\psi^\tau + g_{\mu\sigma}\psi_\nu - g_{\nu\sigma}\psi_\mu)\ , \\
\delta_{\mu\nu}\psi_\sigma &= -(\varepsilon_{\mu\nu\sigma\tau}D^\tau\phi + g_{\mu\sigma}D_\nu\phi - g_{\nu\sigma}D_\mu\phi)\ , \\
\delta_{\mu\nu}\phi &= 0\ , \\
\delta_{\mu\nu}\overline{\phi} &= 8\chi_{\mu\nu} \\
\delta_{\mu\nu}\eta &= -4F^+_{\mu\nu} \\
\delta_{\mu\nu}\chi_{\sigma\tau} &= \frac{1}{8}(\varepsilon_{\mu\nu\sigma\tau} + g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma})\left[\phi,\overline{\phi}\right] \\
&\quad +(F^+_{\mu\sigma}g_{\nu\tau} - F^+_{\nu\sigma}g_{\mu\tau} - F^+_{\mu\tau}g_{\nu\sigma} + F^+_{\nu\tau}g_{\mu\sigma}) \\
&\quad +(\varepsilon_{\mu\nu\sigma}{}^{\alpha}F^+_{\tau\alpha} - \varepsilon_{\mu\nu\tau}{}^{\alpha}F^+_{\sigma\alpha} + \varepsilon_{\sigma\tau\mu}{}^{\alpha}F^+_{\nu\alpha} - \varepsilon_{\sigma\tau\nu}{}^{\alpha}F^+_{\mu\alpha})\ .
\end{aligned} \tag{B.174}
$$

The above transformations are checked to give rise, together with transformations (3.59) and (3.79), to the complete twisted $N = 2$ supersymmetric algebra of eqs.(3.72) and (3.73).

Remark 21 *According to the algebraic set up of Sect.4, the tensor self-dual transformations* $(B.174)$ *can be easily encoded in the Slavnov-Taylor identity* (4.119) *by means of the introduction of a suitable constant tensor self-dual ghost. However, the inclusion of the self-dual symmetry does not modify the previous results on the renormalizability of TYM. The theory will remain anomaly free and will admit a unique invariant nontrivial counterterm. As one can easily understand, this is essentially due to the fact that the tensor transformations* $(B.174)$ *do not actually act on the scalar field* ϕ*, so that they cannot modify the cohomology result of eq.*(5.151)*.*

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