ISSN 0029 - 3865

CBPF-NF-043/88

QUANTUM FIELD THEORY IN NON-STATIONARY COORDINATE SYSTEMS AND GREEN FUNCTIONS

by

B.F. SVAITER, and N.F. SVAITER

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

ABSTRACT

In this paper we studied a neutral massive scalar field in a bi-dimensional Milne space time. The quantization is made on hyperboles which are Lorentz invariant surfaces. The expansion for the field operator was carried on using a complete set of orthonormal modes which have definite positive and negative dilatation frequence.

We have calculated the advanced and retarded Green function and proved that the Feynman propagator diverges in the usual sense.

Key-words: Quantum-field theory; Curvilinear coordinates; Green functions.

1 - INTRODUCTION

Interesting possibilities were revealed when attempts were made to quantize the gravitational field. Although up to now these attempts have all failed, there have been other important results in quantum field theory in curved space and in curvelinear coordinate systems.

According to Fulling[1] a uniformly accelerating detector in a Minkovski space time observers a thermal spectrum, while an inertial observer measure the field in its vacuum state.

We will not go over the detector problem since this subject has been widelly discussed in the literature[2]. We will deal only with the formal part of the quantization of a neutral scalar field.

This will be done in a two dimensional flat space time using a particular curvilienar coordinate system.

In 1975 Kalnins[3] proved that in a two dimensional flat space time there are only ten coordinate systems in which the Klein-Gordon equation has separable variables.

In one of these systems, the Lorentz invariant surfaces (x^2 = cte) arise naturally.

Fubini, Hansen and Jackiw[4] quantized a massless neutral scalar field using this type of surface.

di Sessa[5], Sommerfield[6] and Rothe[7] et al. did the same with a massive neutral scalar field, but only di Sena delas with the problem of the associated Green function.

In this paper we use the same coordinate system and quantization as Sommerfield (massive neutral scalar field). The

Pauli Jordan and the advanced and retarded Green functions will be calculated and the divergence of the Feynman propagator will be demonstrated.

In section 2 after a brief exposition of the two dimensional Milne and Rindler space times we display the Klein Gordon equation in the Milne system. Two complete set of modes solution are presented.

In section 3 two criteria of choosing positive and negative frequence mode are discussed and the Sommerfield criterium is adopted.

In section 4 we calculate the Pauli Jordan function, the advanced and retarded Green function and we demonstrate that the Feynman propagator diverge.

The convergence and evaluation of certain integrals in the complex plane is discussed in Appendix A.

The additional theorem for the cylinder functions will be generalized in Appendix B.

In this paper we use the convention $\hbar = c = k_B = 1$.

2 - MASSIVE SCALAR FIELD IN MILNE'S UNIVERSE

Let us consider a two-dimensional Minkovski space time with line element

$$ds^2 = (dy^0)^2 - (dy^1)^2 \quad . \tag{2.1}$$

We shall use the following coordinate transformation

$$
\begin{aligned}
y^0 &= \eta \sinh \xi \qquad & 0 < \eta < \infty \\
y^1 &= \eta \cosh \xi \ . \qquad & -\infty < \xi < \infty
\end{aligned} \tag{2.2}
$$

In this case the line element (2.1) becomes

$$
ds^2 = \eta^2 d\xi^2 - d\eta^2 \quad . \tag{2.3}
$$

This transformation covers only the region $y^1 > |y^0|$. The (ξ,η) coordinate system is called Rindler's coordinates. It can be shown that this system is one naturally suited to an observer with constant proper acceleration(3).

As this system does not cover the whole Minkovski space time, we shall selected the following additional coordinate transformation (see Fig. 1)

$$
\begin{cases} y^0 = -\eta \sinh \xi \\ y^1 = -\eta \cosh \xi \end{cases} \qquad \text{Region II (Rindler)} \tag{2.4a}
$$

$$
\begin{cases} y^0 = \eta \cosh \xi \\ y^1 = \eta \sinh \xi \end{cases} \qquad \text{Region F (Milne)} \tag{2.4b}
$$

$$
\begin{cases} y^0 = -\eta \cosh \xi \\ y^1 = -\eta \sinh \xi \ . \end{cases} \qquad \text{Region P (Milne)} \tag{2.4c}
$$

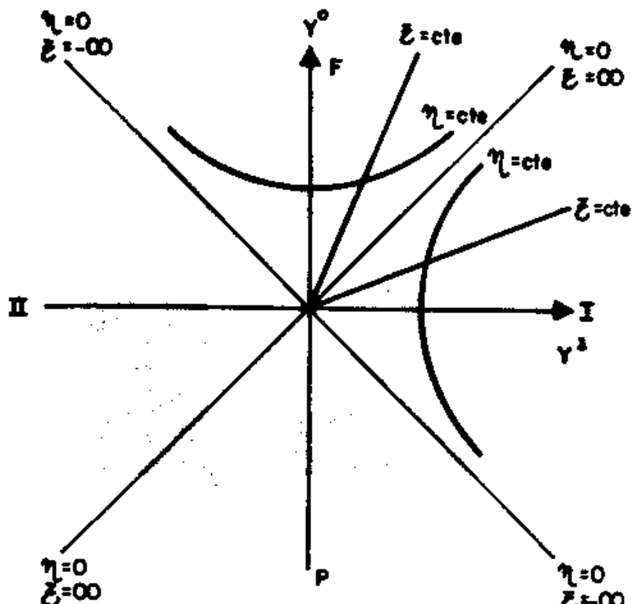


FIG. 1

The four coordinate transformations (2.2), (2.4a), (2.4b) and (2.4c) together cover all Minkovski space time.

The coordinate systems that cover the region inside the light cone are a two-dimensional Milne Universe.

Using the transformation (2.4b) the line element (2.1) becomes

$$ds^2 = d\eta^2 - \eta^2 d\xi^2 \quad . \tag{2.5}$$

Observers who perceive the universe expanding from $y^0 = 0$ have world lines ξ = cte. The surfaces η = cte are hyperboles where we postulate the commutation relation between the fields.

It is useful to define new variables γ, τ in the region (F)

$$\xi = a\gamma \qquad a > 0 \tag{2.6a}$$

$$\eta = \frac{1}{a} e^{a\tau} \qquad . \qquad \infty < \gamma,\tau < \infty \tag{2.6b}$$

In order to quantize a neutral massive scalar field it is necessary to solve the Klein Gordon equation in the Milne Universe.

It becomes

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\gamma^2} + e^{2a\tau}m^2\right)\Phi(\tau,\gamma) = 0 \quad . \tag{2.7}$$

The time dependent part of the Klein Gordon equation is a Bessel equation

$$\left(\frac{d^2}{d\eta^2} + \frac{1}{\eta}\frac{d}{d\eta} + m^2 + \frac{\lambda^2}{\eta^2}\right) \chi_\lambda(m\eta) = 0 \quad . \tag{2.8}$$

The set of solutions $\phi_\lambda \propto e^{i\lambda\xi}\chi_\lambda(m\eta)$ and $\phi_\lambda^* \propto e^{-i\lambda\xi}\chi_\lambda^*(m\eta)$ is complete and the Φ field can be expanded in the form

$$\Phi(\eta,\xi) = \int_{-\infty}^{\infty} d\lambda \left[a(\lambda)\phi_\lambda + a^\dagger(\lambda)\phi_\lambda^*\right] \quad . \tag{2.9}$$

We shall take the scalar product

$$(\phi_1,\phi_2) = -i \int_\Sigma \sqrt{-g_\Sigma}\, d\Sigma^\mu\, \phi_1(x)\overleftrightarrow{\partial}_\mu\phi_2^*(x) \tag{2.10}$$

where $d\Sigma^\mu = \eta^\mu d\Sigma$, with η^μ a future directed unit vector orthogonal to the space-like hypersurface Σ , and $d\Sigma^\mu$ is the volume element in Σ.

The Klein Gordon equation (2.7) posesses two distinct complete sets of orthornormal mode solutions (orthonormal under the scalar product (2.10)).

$\{u_\lambda, u^*_\lambda\}$ and $\{v_\lambda, v^*_\lambda\}$ namely

$$\begin{cases} u_\lambda(\eta,\xi) = -\dfrac{i}{2\sqrt{2}}\, e^{\pi\lambda/2}\, e^{i\lambda\xi}\, H^{(2)}_{i\lambda}(m\eta) & (2.11a) \\ u^*_\lambda(\eta,\xi) = \dfrac{i}{2\sqrt{2}}\, e^{\pi\lambda/2}\, e^{-i\lambda\xi}\, H^{(1)}_{-i\lambda}(m\eta) & (2.11b) \end{cases}$$

and

$$\begin{cases} v_\lambda(\eta,\xi) = -\dfrac{i}{2}\, (\sinh\pi|\lambda|)^{-1/2}\, e^{i\lambda\xi}\, J_{-i|\lambda|}(m\eta) & (2.12a) \\ v^*_\lambda(\eta,\xi) = \dfrac{i}{2}\, (\sinh\pi|\lambda|)^{-1/2}\, e^{-i\lambda\xi}\, J_{i|\lambda|}(m\eta) & (2.12b) \end{cases}$$

$H^{(1)}_{i\lambda}$ and $H^{(2)}_{i\lambda}$ are the Bessel functions of the third kind or Hankel function of imaginary order. $J_{i\lambda}$ is the Bessel function of first kind with imaginary order[9].

Positive and negative frequence modes must be distinguished in the quantization in order to identify $a(\lambda)$ and $a^\dagger(\lambda)$ as annihilation an creation operators of quanta of the field. If the space time has a stationary geometry there exist a time-like Killing vector K. This vector generates a one parameter Lie group of isometries, and the orthonormal modes satisfy

$$L_K\, u = -i\omega u \qquad . \tag{2.13}$$

where L_K is the Lie derivative with respecto to K. In this case there is a natural way of defining positive and negative frequency modes.

The vacuum associated with these modes is called trivial or Killing vacuum[10]. However the line element (2.5) is time (η) dependent, and there is no simple way of defining positive and negative frequency modes. Different solutions for this problem were presented by Sommerfield and di Sessa. For each way of defining positive and negative modes we have different quantizations.

3 - THE DI SESSA AND SOMMERFIELD QUANTIZATION

(a) di Sessa Criterion

This authors claims that the concept of positive frequency requires for its definition a complexification of the real Lorentzian manifold. In this situation the positive frequency modes are those which vanish when $t \to -i\infty$. It is easy to see that

$$\lim_{\eta \to -i\infty} H^{(2)}_{i\lambda}(m\eta) = 0 \quad . \tag{3.1}$$

Then (2.11a) and (2.11b) are positive and negative frequency modes respectively.

$J_{i\lambda}$ and $J_{-i\lambda}$ do not vanish when $\eta \to -i\infty$, so (2.12a) and (2.12b) do not have definite positive or negative frequency in the di Sessa criterion.

The vacuum associated with (2.11a) and (2.11b) will be represented by $|0\rangle$.

(b) Sommerfield Criterion

The operator

$$D = \frac{1}{2}\int_{-\infty}^{\infty} d\gamma \left[\left(\frac{\partial}{\partial\tau}\Phi\right)^2 + \left(\frac{\partial}{\partial\gamma}\Phi\right)^2 + e^{2a\tau}m^2\Phi^2 \right] \qquad (3.2)$$

generates translation in τ, and is called dilatation generator. It satisfies the Heisenberg equation

$$\left[\Phi(\tau,\gamma),D\right] = i\frac{\partial}{\partial\tau}\Phi \quad . \qquad (3.3)$$

Sommerfield use this fact and the additional fact that in the light cone ($\eta \to 0$ or $\tau \to -\infty$) we have

$$\lim_{\substack{\tau\to 0 \\ \text{or} \quad \eta\to-\infty}} J_{i\lambda}(m\eta) \propto \frac{e^{ia\lambda\tau}}{2^{i\lambda}\Gamma(1+i\lambda)} \qquad (3.4)$$

to choose (2.12a) and (2.12b) as positive and negative dilatation frequency modes respectively.

Using (2.9), (2.12a), (2.12b), (3.2) and (3.4) we obtain

$$\lim_{\tau\to-\infty} D(\tau) \propto \frac{1}{2}\int_{-\infty}^{\infty} d\lambda|\lambda| \left[a(\lambda)a^{\dagger}(\lambda) + a^{\dagger}(\lambda)a\ (\lambda)\right] \ . \qquad (3.5)$$

So the Fock space can be constructed and the associated vacuum will be represented by $|\overline{0}>$. The problem is to find the Green functions associated with the (2.12a), (2.12b) modes.

The Feynman propagator of the modes (2.11a) and (2.11b) has already been calculated[5].

It will be shown that the Feynman propagator associated with the modes (2.12a) and (2.12b) diverges. The others propagators, the retarded and advanced Green functions G_R and G_A are defined respectively by

$$G_R(x,x') = -\Theta(x^0-x^{0'})G(x,x') \tag{3.6}$$

$$G_A(x,x') = \Theta(x^{0'}-x^0)G(x,x') \tag{3.7}$$

where G(x,x') is known as the Pauli Jordan function which is defined as the expected value of the commutator of the field in the vacuum state.

$$iG(x,x') = \langle\bar{0}|[\Phi(x),\Phi(x')]|\bar{0}\rangle \tag{3.8}$$

The Feynman propagator G_F is defined as the time ordered product of fields

$$iG_F(x,x') = \langle\bar{0}|T\Phi(x)\Phi(x')|\bar{0}\rangle =$$

$$= \Theta(x^0-x^{0'})G^+(x,x') + \Theta(x^{0'}-x^0)G^-(x,x') \tag{3.9}$$

where

$$\Theta(x^0) = \begin{cases} 1 & x^0 > 0 \\ 0 & x^0 < 0 \end{cases} \tag{3.10}$$

and $G^+(x,x')$ and $G^-(x,x')$ are the Wightman functions.

4 - THE GREEN FUNCTION OF THE FIELDS

The Pauli Jordan function of the fields will be calculated in this section using the complete set (2.12a), (2.12b) and the result is the same as that obtained using the set (2.11a), (2.11b) (di Sessa modes).

The coincidence of the Pauli Jordan function when calculated using the modes (2.12a) and (2.12b) or (2.11a) and (2.11b) can also be demonstrated using the fact that both modes are related by an Bogoliubov transformation ($\alpha_{\mu\nu} \neq 0$, $\beta_{\mu\nu} \neq 0$). It seems useful to calculate directly the Pauli Jordan function using the modes (2.12a) and (2.12b). If the Feynman's propagator divergence can be eliminated, the calculi shall be done using those modes.

The Pauli Jordan function can be split into its positive and negative frequence parts as

$$iG(x,x') = G^{+}(x,x') - G^{-}(x,x') \tag{4.1}$$

where

$$G^{+}(x,x') = \int_{-\infty}^{\infty} d\lambda \; v^{+}(x)\,(v^{+}(x'))^{*} \tag{4.2a}$$

and

$$G^{-}(x,x') = \int d\lambda \; v^{-}(x)\,(v^{-}(x'))^{*} \tag{4.2b}$$

Substituting (2.12a), (2.12b) in (4.2a) and (4.2b) we have

$$G^{+}(\eta,\xi;\eta',\xi') = \frac{1}{4}\int_{-\infty}^{\infty} \frac{d\lambda}{\mathrm{senh}\pi|\lambda|}\, e^{i\lambda(\xi-\xi')}\, J_{-i|\lambda|}(m\eta)J_{i|\lambda|}(m\eta') \tag{4.3a}$$

$$G^{-}(\eta,\xi;\eta'\xi') = \frac{1}{4}\int_{-\infty}^{\infty} \frac{d\lambda}{\mathrm{senh}\pi|\lambda|}\, e^{i\lambda(\xi-\xi')}\, J_{i|\lambda|}(m\eta)J_{-i|\lambda|}(m\eta') \tag{4.3b}$$

The Feynman propagator (3.9) diverge because the integrals (4.3a) and (4.3b) calculated individually are divergent since when $\lambda \to 0^{+},0^{-}$ the integrand behaves like $\frac{1}{\pi|\lambda|} J_0(m\eta)J_0(m\eta')$ (zero order) + r, $|r| < \infty$ near the origin ($\lambda = 0$), but this divergence can be eliminated if we calculate $G^{+}-G^{-}$ toghether. If is straightforward to conclude

$$G^{+}(\eta,\xi;\eta',\xi') - G^{-}(\eta,\xi;\eta'\xi') =$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} \frac{d\lambda}{\sinh\pi\lambda}\, e^{i\lambda(\xi-\xi')} J_{-i\lambda}(m\eta)J_{i\lambda}(m\eta') +$$

$$- \frac{1}{4}\int_{-\infty}^{\infty} \frac{d\lambda}{\sinh\pi\lambda}\, e^{i\lambda(\xi-\xi')}\, J_{i\lambda}(m\eta)J_{-i\lambda}(m\eta') \quad . \tag{4.4}$$

Defining

$$f_{\lambda}(z_1,z_2,z_3) = \frac{1}{\sinh\pi\lambda}\, e^{i\lambda z_1}\, J_{-i\lambda}(mz_2)J_{i\lambda}(mz_3) \tag{4.5}$$

The expression (4.4) can be written as

$$G^{+} - G^{-} = I_1 - I_2 \tag{4.6}$$

where

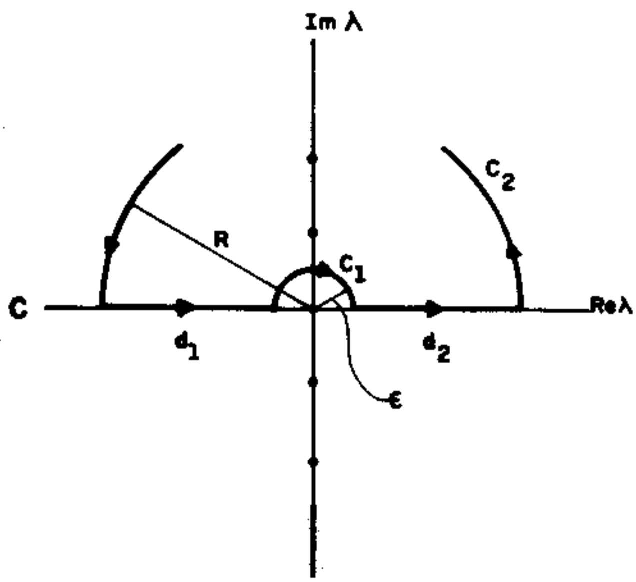
$$I_1 = \frac{1}{4}\int_{-\infty}^{\infty} d\lambda\, f_\lambda(\xi-\xi',\eta,\eta') \tag{4.7a}$$

$$I_2 = \frac{1}{4}\int_{-\infty}^{\infty} d\lambda\, f_\lambda(\xi-\xi',\eta',\eta) \quad . \tag{4.7b}$$

When $m\eta$ and $m\eta'$ are not roots of J_0, $|f_\lambda|$ tends to infinity in the order λ^{-1}, when $\lambda \to 0$. However, the integrals are finite if we adopt the principal value in the origin ($\lambda = 0$). The function f_λ is analytic with respect to λ in the whole complex plane except in the points $\lambda = ni$ ($n \in Z$). We have and infinite number of first order poles, and the residue of f_λ in these points is

$$\mathrm{Res}(f_\lambda;ni) = \frac{1}{\pi}\, e^{-nz_1}\, J_n(mz_2)J_n(mz_3) \quad . \tag{4.8}$$

Two distinct contour C and C' will be used to calculate I_1 and I_2 (see Fig. 2).



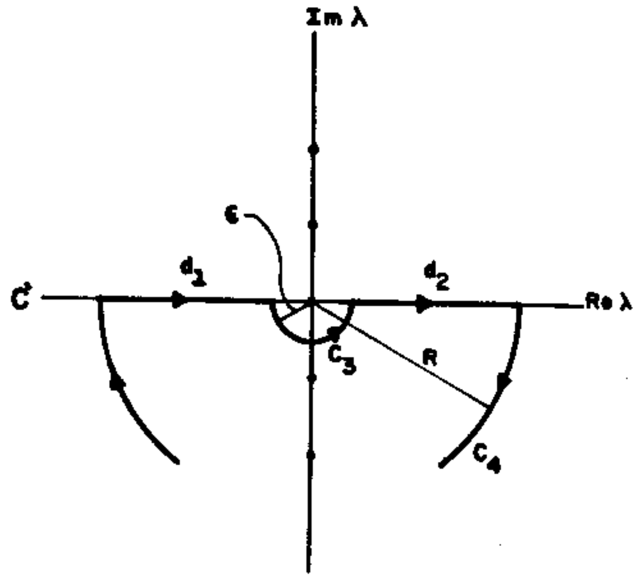


FIG.2

C_2 and C_4 cross the imaginary axis at the middle point of the adjacent poles i.e.

$$R = q + \frac{1}{2} \qquad q \in N$$

then

$$\lim_{R=q+\frac{1}{2}\to\infty} \int_{C_2} d\lambda f_\lambda(z_1,z_2,z_3) = 0 \quad \text{if} \quad e^{z_1}\frac{z_3}{z_2} > 1 \qquad (4.9a)$$

and

$$\lim_{R=q+\frac{1}{2}\to\infty} \int_{C_4} d\lambda f_\lambda(z_1,z_2,z_3) = 0 \quad \text{if} \quad e^{z_1}\frac{z_3}{z_2} < 1 \qquad (4.9b)$$

In the Appendix A we demonstrate (4.9a) and (4.9b).

By the Cauchy theorem

$$\int_C f_\lambda(z_1,z_2,z_3)d\lambda = 2\pi i \sum_{n=1}^{q} \text{Res}(f_\lambda;ni) \qquad (4.10)$$

If $e^{z_1} \frac{z_3}{z_2} > 1$, and we take the limit of the equation (4.10) when $\varepsilon \to 0$, $q \to \infty$ we get

$$\int_{-\infty}^{\infty} f_\lambda(z_1,z_2,z_3)d\lambda - \pi i \ \text{Res}(f_\lambda;0) =$$

$$= 2\pi i \sum_{n=1}^{\infty} \text{Res}(f_\lambda;ni) \quad . \qquad (4.11)$$

Therefore using (4.8)

$$\int_{-\infty}^{\infty} d\lambda f_\lambda(z_1,z_2,z_3) = i\ J_0(mz_2)J_0(mz_3) +$$

$$+ 2i \sum_{n=1}^{\infty} e^{-nz_1} J_n(mz_2)J_n(mz_3) \quad \text{if} \quad e^{z_1}\frac{z_3}{z_2} > 1 \qquad (4.12)$$

Similarly, when $e^{z_1}\frac{z_3}{z_2} < 1$ starting from

$$\int_{C'} f_\lambda(z_1,z_2,z_3)d\lambda = -2\pi i \sum_{n=-1}^{-q} \text{Res}(f_\lambda;ni) \qquad (4.13)$$

we get taking the limit

$$\int_{-\infty}^{\infty} f_\lambda(z_1,z_2,z_3)d\lambda + \pi i \ \text{Res}(f_\lambda;0) =$$

$$= -2\pi i \sum_{n=1}^{-\infty} \text{Res}(f_\lambda;ni) \quad . \qquad (4.14)$$

Therefore using (4.8)

$$\int_{-\infty}^{\infty} f_\lambda(z_1,z_2,z_3)\,d\lambda = -i\, J_0(mz_2)J_0(mz_3) +$$

$$- 2i \sum_{n=-1}^{-\infty} e^{-nz_1} J_n(mz_2)J_n(mz_3) \qquad \text{if} \quad e^{z_1}\frac{z_3}{z_2} < 1. \tag{4.15}$$

Using (4.7a) and (4.7b) we have

$$I_1 = \begin{cases} \frac{i}{4}\left[J_0(m\eta)J_0(m\eta') + 2\sum_{n=1}^{\infty} e^{-n(\xi-\xi')}J_n(m\eta)J_n(m\eta')\right] & \text{if } e^{(\xi-\xi')}\frac{\eta'}{\eta} > 1 \quad (4.16a) \\ -\frac{i}{4}\left[J_0(m\eta)J_0(m\eta') + 2\sum_{n=-1}^{-\infty} e^{-n(\xi-\xi')}J_n(m\eta)J_n(m\eta')\right] & \text{if } e^{(\xi-\xi')}\frac{\eta'}{\eta} < 1 \quad (4.16b) \end{cases}$$

$$I_2 = \begin{cases} \frac{i}{4}\left[J_0(m\eta)J_0(m\eta') + 2\sum_{n=1}^{\infty} e^{-n(\xi-\xi')}J_n(m\eta)J_n(m\eta')\right] & \text{if } e^{(\xi-\xi')}\frac{\eta}{\eta'} > 1 \quad (4.17a) \\ -\frac{i}{4}\left[J_0(m\eta)J_0(m\eta') + 2\sum_{n=-1}^{-\infty} e^{-n(\xi-\xi')}J_n(m\eta)J_n(m\eta')\right] & \text{if } e^{(\xi-\xi')}\frac{\eta}{\eta'} < 1 \quad (4.17b) \end{cases}$$

The space time interval $\sigma = (y^0-y^{0'})^2-(y^1-y^{1'})^2$ in the coordinates (η,ξ) can be written as

$$\sigma = \eta^2+\eta'^2 - 2\eta\eta'\cosh(\xi-\xi') =$$

$$= -\eta\eta' \, e^{-(\xi-\xi')} \left(e^{(\xi-\xi')} \frac{\eta'}{\eta} - 1 \right) \left(e^{(\xi-\xi')} \frac{\eta}{\eta'} - 1 \right) . \tag{4.18}$$

If $\sigma < 0$ (which correspond to space-like separated events) there are two possibilities

$$e^{(\xi-\xi')} \frac{\eta'}{\eta} > 1 \quad \text{and} \quad e^{(\xi-\xi')} \frac{\eta}{\eta'} > 1 \tag{4.19a}$$

or

$$e^{(\xi-\xi')} \frac{\eta'}{\eta} < 1 \quad \text{and} \quad e^{(\xi-\xi')} \frac{\eta}{\eta'} < 1 . \tag{4.19b}$$

In the cases (4.19a) and (4.19b) $I_1 = I_2$ so $G(x,x') = 0$.

If $\sigma > 0$ (which correspond to time-like separated events) there are again two possibilities

$$e^{(\xi-\xi')} \frac{\eta'}{\eta} > 1 \quad \text{and} \quad e^{(\xi-\xi')} \frac{\eta}{\eta'} < 1 \tag{4.20a}$$

or

$$e^{(\xi-\xi')} \frac{\eta'}{\eta} < 1 \quad \text{and} \quad e^{(\xi-\xi')} \frac{\eta}{\eta'} > 1 \tag{4.20b}$$

It should be noted that in (4.20a) $\eta' > \eta$ and in (4.20b) $\eta > \eta'$.

In the case (4.20a) ($\sigma > 0$, $\eta' > \eta$)

$$I_1 - I_2 = \frac{i}{2} \sum_{n=-\infty}^{\infty} e^{(\xi-\xi')} J_n(m\eta) J_n(m\eta') \quad . \tag{4.21a}$$

In the case (4.20b) ($\sigma > 0$, $\eta > \eta'$)

$$I_1-I_2 = -\frac{i}{2}\sum_{n=-\infty}^{\infty} e^{n(\xi-\xi')}\, J_n(m\eta)J_n(m\eta') \quad . \qquad (4.21b)$$

Thus using (4.6), (4.21a) and (4.21b)

$$G^+-G^- = -\frac{i}{2}\left[\theta(\sigma)\varepsilon(\eta-\eta')\sum_{n=-\infty}^{\infty} e^{n(\xi-\xi')}J_n(m\eta)J_n(m\eta')\right] \qquad (4.22)$$

where $\varepsilon(\eta) = \begin{cases} 1 & \eta > 0 \\ -1 & \eta < 0 \end{cases}$.

The addition theorem of the Bessel functions states that

$$J_0(ms) = \sum_{n=-\infty}^{\infty} J_n(m\eta)J_n(m\eta')\, e^{in\Theta} \qquad (4.23)$$

where

$$s = (\eta^2+\eta'^2 - 2\eta\eta' \cos\Theta)^{1/2}$$

Taking the analitic extension of $\Theta = i(\xi-\xi')$ we get, using (4.18)

$$s = \sigma^{1/2}$$

$$J_0(m\sigma^{1/2}) = \sum_{n=-\infty}^{\infty} J_n(m\eta)J_n(m\eta')e^{n(\xi-\xi')} \qquad (4.24)$$

(see Appendix B for a more detailed demonstration).

Finally substituting (4.24) in (4.22) and using (4.1) we get

$$iG = G^+-G^- = -\frac{i}{2}\left[\theta(\sigma)\varepsilon(\eta-\eta')J_0(m^2\sigma)^{1/2}\right] \quad . \qquad (4.25)$$

Acknowledgements

One of us (N.F. Svaiter) would like to thank Prof. Mario Novello and Drs. Isaias Costa and Adolfo Malbouisson of Centro Brasileiro de Pesquisas Físicas for many helpful suggestions and clarifying discussions.

We are grateful to Prof. Samuel Mac Dowell from Pontifícia Universidade Católica-PUC/RJ for stimulating comments.

This paper was partially suported by CNPq.

APPENDIX A

It is known that

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(z/2)^{2k+\nu}}{\Gamma(\nu+k+1)} \quad .$$

Defining

$$(a)_0 = 1$$

$$(a)_k = a(a+1)\ldots(a+k-1) \tag{A.1}$$

we get

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!\,(\nu+1)_k} \tag{A.2}$$

The same formula can also be expressed using the hipergeometric functions[11]

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \, {}_0F_1 \left(\nu+1; -\frac{1}{4} z^2\right) \quad .$$

Suposing (and that is most important) $\exists \; n_0 > 0$ such that

$$\forall n \in \mathbb{Z} \; |\nu+n| \geq n_0 \quad .$$

Using the definition (A.2)

$$|(\nu+1)_k| \geq (n_0)^k \quad \forall \, k \in \mathbb{N} \tag{A.3}$$

Now using (A.2) and (A.3)

$$|J_\nu(z)| \le \frac{|(z/2)^\nu|}{|\Gamma(\nu+1)|} \sum_{k=0}^{\infty} \frac{|(-\frac{z^2}{4})^2|}{k!\,|(\nu+1)_k|} \le$$

$$\frac{|(z/2)^\nu|}{|\Gamma(\nu+1)|} \sum_{k=0}^{\infty} \frac{|\frac{z^2}{4}|^k}{k!\,n_0^{\,k}} \quad .$$

This inequality can be simplified

$$|J_\nu(z) \le \frac{|(z/2)^\nu|}{|\Gamma(\nu+1)|} \exp\left(\frac{z^2}{4n_0}\right) \tag{A.4}$$

if $|\nu+n| \ge n_0 \quad \forall\, n \in \mathbb{Z}$.

Let us suppose

$$|\nu| = q + \frac{1}{2} \qquad q \in \mathbb{N}$$

then

$$|\nu+k| \ge \frac{1}{2} \qquad k \in \mathbb{Z}$$

and using (A.4) we get

$$|J_\nu(z)| \le \frac{|(z/2)^\nu|}{|\Gamma(\nu+1)|} \exp\left(\frac{z^2}{2}\right)$$

(A.5)

if

$$|\nu| = q + \frac{1}{2} \text{ for some } q \in \mathbb{N}$$

In the article (4.5) is

$$f_\lambda(z_1,z_2,z_3) = \frac{1}{\operatorname{senh}\pi\lambda}\, e^{i\lambda z_1}\, J_{-i\lambda}(mz_2) J_{i\lambda}(mz_3) \quad .$$

In the contour C_2 and C_4

$$|\pm i\lambda| = q + \frac{1}{2} \qquad q \in \mathbb{N} \tag{A.6}$$

Using (A.5) and (A.6), we get

$$|f_\lambda(z_1,z_2,z_3)| \le \left|\frac{1}{\sinh\pi\lambda}\, e^{i\lambda z_1}\right| \left|\frac{(z_3/z_2)^{i\lambda}}{\Gamma(i\lambda+1)\Gamma(-i\lambda+1)}\right| \, e^{\frac{m^2}{2}(z_2^2+z_3^2)}$$

in C_2 and C_4.

Because

$$\Gamma(i\lambda+1)\Gamma(-i\lambda+1) = \frac{\pi\lambda}{\sinh\pi\lambda}$$

the inequality becomes

$$|f_\lambda(z_1,z_2,z_3)| \le \left|\frac{1}{\lambda}\,(e^{z_1}\,\frac{z_3}{z_2})^{i\lambda}\right| \,\frac{1}{\pi}\, e^{\frac{m^2}{2}\,(z_2^2+z_3^2)}$$

in C_2 and C_4.

The expression

$$\frac{1}{\pi}\; e^{\frac{m^2}{2}\,(z_2^2+z_3^2)}$$

do not change in the contour C_2 and C_4 so we call it M, and we get

$$|f_\lambda(z_1,z_2,z_3)| \le \left|\,(e^{z_1}\,\frac{z_3}{z_2})^{i\lambda}\right| \,\frac{M}{|\lambda|} \tag{A.7}$$

in C_2 and C_4.

In our problem $z_1 = \xi-\xi'$ is a real number and z_2, z_3 (η or η') are positive .

Than there exist $K \in \mathbb{R}$ such that

$$e^{z_1} \frac{z_3}{z_2} = e^K \qquad . \qquad \text{(A.8)}$$

Let studie the case $e^{z_1} \frac{z_3}{z_2} > 1$ $(K > 0)$.

Using (A.7) and (A.8)

$$\left| \int_{C_2} f_\lambda(z_1,z_2,z_3)d\lambda \right| \le \int_{C_2} \left| f_\lambda(z_1,z_2,z_3) \right| |d\lambda| \le \int_{C_2} M|e^{i\lambda k}| \frac{|d\lambda|}{|\lambda|}$$

We can chose the parametrization

$$C_2 : \lambda(\Theta) = (q + \tfrac{1}{2})e^{i\Theta} \qquad 0 \le \Theta \le \pi$$

The inequality above becomes

$$\left| \int_{C_2} f_\lambda d\lambda \right| \le M \int_0^\pi |e^{i\lambda K}| d\Theta = M \int_0^\pi e^{-k(q + \frac{1}{2})\sin\Theta} d\Theta =$$

$$= 2M \int_0^{\pi/2} e^{-(q + \frac{1}{2})\sin\Theta} d\Theta \qquad \text{(A.9)}$$

If $0 \le \Theta \le \pi/2$ then $2\Theta/\pi \le \sin\Theta$.

We are studing the case $K > 0$. So

$$e^{-K(q + \frac{1}{2}) \text{ sen}\Theta} \le e^{-K(q + \frac{1}{2}) \frac{2\Theta}{\pi}} \qquad 0 \le \Theta \le \pi$$

and

$$\left|\int_{C_2} f_\lambda d\lambda\right| \leq 2M \int_0^{\pi/2} e^{-\frac{2K}{\pi}(q+\frac{1}{2})\Theta} d\Theta =$$

$$= -\frac{M\pi}{K(q+\frac{1}{2})}\left(e^{-K(q+\frac{1}{2})} - 1\right) \leq \frac{M\pi}{K(q+\frac{1}{2})} \qquad \text{(A.10)}$$

Thus using (A.10)

$$\lim_{\substack{q\to\infty \\ q\in\mathbb{N}}} \left|\int_{C_2} f_\lambda d\lambda\right| = 0 \quad \text{if} \quad e^{z_1}\frac{z_3}{z_2} > 1 \quad .$$

For the case $e^{z_1}\frac{z_3}{z_2} < 1$ $(K < 0)$ the contour C_4 is the adequated one.

Similar calculations give us

$$\lim_{\substack{q\to\infty \\ q\in\mathbb{N}}} \left|\int_{C_4} f_\lambda d\lambda\right| = 0 \quad \text{if} \quad e^{z_1}\frac{z_3}{z_2} < 1 \quad .$$

APPENDIX B

We will define

$$g(z) = J_0\left[m(\eta^2+\eta'^2-\eta\eta'(z + \frac{1}{z}))^{1/2}\right]$$

$$= J_0\left[m((\eta+\eta' z)(\eta + \frac{\eta'}{z}))^{1/2}\right] . \qquad (B.1)$$

J_0 is analitic in the whole complex plane and its expantion in power serie centered at zero contains only even powers. Then the square rot above can be naturally eliminated and g(z) is analitic in whole complex plane except the origin. We will define "another" function

$$h(z) = \sum_{n=-\infty}^{\infty} J_n(m\eta)J_n(m\eta')z^n \qquad (B.2)$$

The serie (B.2) is convergent if $z \neq 0$. Than h(z) is analitic in the whole complex plane except at $z = 0$.

If we take $|z| = 1$,

$$z = e^{i\Theta} \qquad \Theta \in \mathbb{R}$$

$$h(z) = \sum_{n=-\infty}^{\infty} J_n(m\eta)J_n(m\eta')e^{in\Theta} =$$

$$= J_0(m\eta)J_0(m\eta') + 2\sum_{n=1}^{\infty} J_n(m\eta)J_n(m\eta')\cos n\Theta .$$

Using the addition theorem for cilinder function we get

$$h(z) = J_0\left[m(\eta^2+\eta'^2-2\eta\eta'\cos\Theta)^{1/2}\right]$$

Now

$$g(z) = g(e^{i\Theta}) = J_0\left[m(\eta^2+\eta'^2-2\eta\eta'\cos\Theta)^{1/2}\right]$$

we obtain $g(z) = h(z)$ if $|z| = 1$.

$g(z)-h(z)$ is analitic in $\mathbb{C} - \{0\}$ and vanish in $|z| = 1$, than it must be equal zero in $\mathbb{C} - \{0\}$.

This occurs because the zeros of any analityc function are isolated inside its domain (open and conected) or the function vanish in all the domain. What we obtained is that

$$J_0\left[m(\eta^2+\eta'^2-\eta\eta'(z+\tfrac{1}{z}))^{1/2}\right] =$$

$$= \sum_{n=-\infty}^{\infty} J_n(m\eta)J_n(m\eta')z^n \qquad z \neq 0 \quad . \qquad (B.3)$$

If

$$z = e^{i\Theta} \quad \Theta \in \mathbb{C}$$

than

$z(\Theta) \neq 0 \quad \forall\, \Theta \in \mathbb{C}$ and using (B.3) we get

$$J_0\left[n(\eta^2+\eta'^2-2\eta\eta'\cos\Theta)^{1/2}\right] = \sum_{n=-\infty}^{\infty} J_n(m\eta)J_n(m\eta')e^{in\Theta} \qquad \Theta \in \mathbb{C}$$

If $\Theta = i(\xi-\xi')$ we find

$$J_0[m(\sigma)^{1/2}] = \sum_{n=-\infty}^{\infty} J_n(n\eta)J_n(m\eta')e^{n(\xi-\xi')}$$

where $\sigma = (\eta^2+\eta'^2-2\eta\eta'\cosh(\xi-\xi'))$.

REFERENCES

1 - Fulling S., Ph.D. Thesis, Princeton (1972).

2 - Sciama D.W., Candelas P. and Deutsh D. Adv. in Phys. 30 (1981) 327.

3 - Kalnins E.G., Siam J. Math. Annal. 6 (1975) 340.

4 - Fubini S., Hanson A.J. and Jackiw R., Phys. Rev. D 7 (1973) 1732.

5 - di Sessa A., Journ. of Math. Phys. 15 (1974) 1892.

6 - Sommerfield C.M., Annals of Phys. 84 (1974).

7 - Gromes D., Rothe H. and Stech B., Nuclear Phys. B 75 (1974) 313.

8 - Birrel N.D. and Davies P.C.W., Quantum Fields in Curved Space. Cambridge University Press.

9 - Lebedev N.N., Spatial Functions and Their Applications - Dover.

10 - Castagnino M. Lectures on Semiclassical Quantum Gravity-IV School of Cosmology and gravitation. CBPF, Brasil.