CBPF-NF-042/99

Quantum Geometry of Bosonic Strings – Revisited

Luiz C.L. Botelho and Raimundo C.L. Botelho[*]

Centro Brasileiro de Pesquisas Físicas - CBPF
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brasil

Departamento de Física
Universidade Federal Rural do Rio de Janeiro
23851–180 – Itaguaí, RJ, Brasil

We review the original paper by A.M. Polyakov (Quantum Geometry of Bosonic Strings, Phys. Lett. 103B, 207 (19981)) with corrections and improvements the concepts exposed there and following as closely as possible to the original A.M. Polyakov's paper.

*Ianomami Center for Advanced Research, Travessa Curuzu, 556, Cep 66000-240, Belém, PA, Brazil

In opinion of the author of ref. [1] “there are methods and formulae in science, which serve as master-key to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion (A.M. Polyakov) at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned sum over random paths. The replacement is necessary, because today gauge invariance plays the central role in physics”.

The general picture has been envisaged as follows ([2], [3], [4]): one should try to solve loop-space or generalized Schrödinger functional wave equations by the appropriate flux lines functionals represented by transition amplitudes given by the sums over all possible surfaces with fixed boundary

$$G(C)=\sum_{(S_C)} exp\left\{-\frac{1}{2\pi\alpha'}\ A(S_c)\right\} \tag{1}$$

here C is some loop (smooth or a random closed path), S_C is a surface bounded by the loop C and $A(S_C)$ is the area of this surface and α' an extrinsinc (lenght square) constant (the Regge slope parameter).

The main point on Polyakov's propose is to introduce besides the surface parametrization $X_\mu(\xi_1,\xi_2)$, an intrinsic metric tensor $g_{ab}(\xi_1,\xi_2)$ and a quadratic functional on the random surface $X_\mu(\xi_1,\xi_2)$ field substituting the area functional in eq. (1) (with $2\pi\alpha'=1$)

$$A(S_C)=\frac{1}{2}\int_D\ d^2\xi(\sqrt{g}g^{ab}\partial_a X_\mu\partial_b X^\mu)(\xi) \tag{2}$$

It is very important to remark that the above 2D-gravity induced surface functional has the geometrical meaning of the area spanned by the surface $X_\mu(\xi_1,\xi_2)$ *only at the classical level* $\alpha'\to 0$ (see ref. [5] for a study for the pure geometrical Nambu-Goto action in the framework of these reparametrization invariant functional integrals).

In order to proceed to the quantum theory, A.M. Polyakov has proposed that the quantum surface average of any extended reparametrization invariant functional $\Phi[X_\mu(\xi_1,\xi_2);g_{ab}(\xi_1,\xi_2)]$ should be given by the following expression without given the derivation

$$\begin{aligned}\int d\mu[S]\phi(S_C) &\overset{def}{\equiv} \int [Dg_{ab}(\xi)]exp(-\mu_{\rm bare}\int\sqrt{g}d^2\xi)\int[D[X_\mu(\xi)]\\ &\left[exp\left(-\frac{1}{2}\int_D(\sqrt{g}g^{ab}\partial_a X_\mu\partial_b X_\mu)(\xi)d^2\xi\right)\right]\Phi[X_\mu(\xi),g_{ab}(\xi)]\end{aligned} \tag{3}$$

The reparametrization invariant functional measures on eq. (3) are associated to the following functional measures (the well-known de Witt functional metrics ([6])

$$\|\delta X^\mu\|^2 = \int d^2\xi (g(\xi))^{1/2}\ \delta X_\mu(\xi)\delta X_\mu(\xi) \tag{4}$$

and

$$\|\delta g_{ab}\|^2 = \int d^2\xi [g(\xi)]^{1/2}\ (g^{aa'}g^{bb'} + C g^{ab}g^{a'b'})\delta g_{ab}\delta g_{a'b'} \tag{5}$$

where $C \neq -\frac{1}{2}$ is an arbitrary constant.

The reparametrization invariant gaussian functional integral $X_\mu(\xi_1,\xi_2)$ is easily evaluated with the result in the conformal gauge $g_{ab} = \rho^2\delta_{ab}$ (for closed boundary less 2D-compact Riemannian manifolds)

$$det^{-D/2}(-\Delta_{g_{ab}=\rho^2\delta_{ab}}) = exp\left\{\frac{D}{48\pi}\int d^2\xi \left[\frac{(\partial_a\rho)^2}{\rho^2} + \left(\lim_{\varepsilon\to 0}\ \frac{D}{4\pi\varepsilon}\right)\rho^2\right]\right\} \tag{6}$$

The functional integration on the intrinsic metric field is well-known ([1]) with infinitesimal coordinate transformations $\{\in_a(\xi_1,\xi_2)\}$ around the conformal orbit (i.e., $\nabla^c_{g_{ab}=\rho^2\delta_{ab}} \cdot \in_c = 0$)

$$\|\delta g_{ab}\|^2 = (1+2c)\int d^2\xi\ \delta\rho(\xi)\delta\rho(\xi) + \int d^2\xi\ \sqrt{g}\phi^b_a\phi^a_b \tag{7}$$

Here

$$\phi_{ab} = (\nabla_a \in_b + \nabla_b \in_a)_{g_{ab}=\rho^2\delta_{ab}} \tag{8}$$

From eq. (7) we derive the correct integration measure in terms of the Feynman measures, denoted by the symbol $D^F(\cdot) = \prod_\xi d(\cdot)$

$$[Dg_{ab}(\xi)] = D^F[\rho(\xi)]D^F[\in_a(\xi)](det^{1/2}\mathcal{L}) \tag{9}$$

Here the Polyakov's operator $\mathcal{L}$ is obtained from eq. (7) and given by

$$(\mathcal{L}\in)_a = \nabla^b(\nabla_a \in_b + \nabla_b \in_a)|_{g_{ab}=\rho^2\delta_{ab}} \tag{10}$$

and its functional determinant was exactly evaluated (acting on smooth C^∞ compact support vector-sections on S)

$$-\frac{1}{2}log\ det\mathcal{L} = \frac{13}{6\pi}\int_\xi \left[\frac{1}{2}\ \frac{(\partial_a\rho)^2}{\rho^2}\right] + \int_\xi \left(\lim_{\varepsilon\to 0}\ \frac{2}{4\pi\varepsilon}\right)\rho^2\ (\xi)\|d^2\xi) \tag{11}$$

By combining eq. (6) with eq. (11) and eq. (3), we obtain the partition function for the closed surfaces *defined in terms of the correct conformal quantum degrees of freedom* $\rho(\xi_1, \xi_2)$

$$Z = \int D^F[\rho(\xi)] exp\left(-\frac{(26-D)}{12\pi}\int_\xi\left[\frac{1}{2}\ \frac{(\partial_a\rho)^2}{\rho^2}\right] + \int_\xi \mu_R^2\rho^2\right) \tag{12}$$

This expression shows the origin of the commonly known critical dimension 26 in the string theory: at this value of the dimension one does not have dynamics for the metric field $g_{ab}(\xi) = \rho^2(\xi)\delta_{ab}$. However for $D < 26$ one must examine the "σ-model like" in eq. (12) which is not the Liouville field theory wrongly stated by A.M. Polyakov [(1)] because the correct theory's dynamical variable in this framework is the scalar field $\rho(\xi)$ instead of that mistakenly used by Polyakov $2lg\ \rho(\xi) = \varphi(\xi)$. These above cited 2D-theories coincides only for very weak fluctuations around the 2D-flat metric $\rho(\xi) = 1 + \varepsilon\bar{\rho}$ $(\varepsilon \to 0)$.

Note that the quantum field equation associated to the obtained effective partition functional is given by (the *the two-dimensional* effective Einstein equations for this induced 2D-gravitation!)

$$(\partial^a\partial_a)\rho(\xi) = \frac{12\pi\mu_R^2}{(26-D)}\ (\rho(\xi))^3 + \frac{12\pi}{(26-D)}\ \frac{(\partial_a\rho)^2}{\rho} \tag{13}$$

Note that our σ-model like (Euclidean) lagrangian (with $\mu_R^2\ = \mu_{\text{bare}}^2 + \lim_{\in\to 0^+} \frac{(2-D)}{4\pi\varepsilon}$) describing the closed random surface sum

$$\mathcal{L}(\rho, \partial_a\rho) = \frac{26-D}{12\pi}\int_\xi\left[\frac{1}{2}\partial_a\left(\frac{1}{\rho}\right)\partial_a(\rho)\right](\xi)d^2\xi + \mu_R^2\int_\xi \rho^2(\xi)d^2\xi \tag{14}$$

does not possesses in principle any conformal symmetry as it was imposed in ref. [1] as a consequence of the incorrect variable to be quantized. It is worth remark that even in the original Polyakov's work the symmetry which remains after specification of the conformal gauge are the ? conformal transformations of the ξ-domain $|\frac{dw}{dz}|^2 = 1$ (see eq. (31)-[1]) for $\phi(z, \bar{z})$ defined as a scalar field. *We conjecture* that the only phase in which the 2D-quantum field theory makes sense is its perturbative phase around the "flat" configuration $\rho^2(\xi) = 1 + \frac{1}{D}\tilde{\rho}_q^2(\xi)$ in a $\frac{1}{D}$-expansion or other suitable classical $\rho_{cl}(\xi)$ solution of eq. (13) $\rho^2(\xi) = \rho_{cl}^2(\xi) + \frac{1}{D}\ \tilde{\rho}_q^2(\xi)$.

The intercept point probabilities (the scalar N-scattering amplitude) in this Randosurface theory is straightforwardly reduced to the average

$$A^{(\delta)}(p^1;\cdots,p^N) = (\delta)^{(\sum_{i=1}^{N} p_i^2)} \int_\xi \Pi_{i=1}^N d^2\xi_j \left(\Pi_{i<j}^N |\xi_i - \xi_j|^{p^i\cdot p^j}\right)$$
$$\times \int D^F[\rho] e^{-\mathcal{L}(\rho,\partial_a\rho)} \left(\Pi_{i=1}^N [\rho(\xi_j)]^{+2(1-p_i^2)}\right) \quad (15)$$

It is possible to show that only for (Euclidean) values of external momenta $1 - p_i^2 = -1, -2, \cdots$ or $p_i^2 = 0, -1, -2, \cdots$, the quantum field average eq. (14) makes sense and leading, thus, to a spectrum without the usual lowest state being a tachyon (see [7] for the same physical result in a approximate context).

So, our main conclusion is that the summation of Bosonic random surfaces understood as 2D-induced quantum gravitation as originally proposed by A.M. Polyakov in ref. [1] is reduced to a massive σ-model scalar field lagrangean obtained in eq. (12), and not to the Liouville ill-defined model as originaly put forward in ref. [1] (see Appendix 1 for some related comments). Note that the simplest supersymmetric version of the Bosonic Quantum Field eq. (12) describes the sum of fermionic random surfaces with critical dimension $D = 10$ ([7]).

Acknowledgements: Luiz C.L. Botelho is grateful to CBPF for the warm hospitality of DCP-CBPF-CNPq, Rio de Janeiro. Raimundo C.L. Botelho wants to state here that his view on the theory of space, time and gravitation was formed under the influence of the philosophy of capitalism-materialism, in particular under the influence of Regan-Gate's, Capitalism and Empiriocriticism ideas. The study of the new american capitalism materialism help him to approach critically Polyakov's work and to think it out anew. It helped him to understand correctly and to interpret, the new results obtained. He wishes to state this here, although this paper does not explicitly touch philosophical questions (see ref. [8], page 8 for similar ideas).

Appendix – The Liouville model

Let us point out that there is a formal propose to describe the closed random surface partitional functional eq. (21) by means of Liouville-Polyakov degree of freedom $\phi(\xi) = 2lg\ \rho(\xi)$ which has the advantages of taking into account directly in the path integral *the positivity* of the quantum field $\rho(\xi)$. The important formal step in this study is the variable functional change

$$D^F[\rho(\xi)] = \Pi_\xi\ d\left[e^{\frac{\phi}{2}(\xi)}\right] = \Pi_\xi\left(det\left(e^{\frac{\phi}{2}}\right)(\xi)\right)d(\phi(\xi)) \tag{16}$$

Unfortunately the *functional Jacobian* $det\left(e^{\frac{\phi}{2}}\right)$ does not makes sense as a functional change of functional measures. However, one can propose a definition for the above cited Jacobian as in the original Fujikawa's "hand-wave" prescription to handle the axial anomaly as follows:

$$\begin{aligned}
&det_F\left[\left(e^{\frac{\phi}{2}}\right)(\xi)\right] = \lim_{\varepsilon\to 0^+}\ exp\ Tr_{(\xi)}\left[lg\left(e^{\frac{\phi}{2}}\right)(\xi)e^{-\varepsilon\Delta_{g_{ab}=e^\phi\delta_{ab}}}\right] = \\
&\lim_{\varepsilon\to 0^+}\ exp\left\{\int d^2\xi\ e^{\phi(\xi)}\frac{\phi}{2}(\xi)\left[\frac{1}{4\pi\varepsilon}-\frac{1}{12\pi}\left(e^{-\phi}\Delta\phi\right)\right](\xi)\right\} = \\
&exp\left\{\frac{1}{48\pi}\int_\xi\left[\frac{1}{2}(\partial_a\phi)^2\right]\right\}\ exp\left\{\frac{1}{8\pi\varepsilon}\int_\xi\ e^{\phi(\xi)}\phi(\xi)\right\}
\end{aligned} \tag{17}$$

By analyzing eq. (17) we feel that eq. (17) is not sound as it stands since 1) one could use other regularizing operator as that one of eq. (10); 2) the term in front of kinetic term for the Liouville weight *decreases* and leading to a new (incorrect) critical dimension for string theory, etc... Anyway eq. (16) deserves further studies.

REFERENCES

[1] A.M. Polyakov, Phys. Lett. 103B, 207 (1981).

[2] Luiz C.L. Botelho, J. Math. Phys. 30 (9), 2160 (1989).
Phys. Rev. 41D, 3283 (1990).
Phys. Rev. 56D, 1338 (1997).

[3] Luiz C.L. Botelho, Phys. Rev. 52D, 6941 (1995).

[4] Luiz C.L. Botelho, Phys. Lett. 415B, 231 (1997).

[5] Luiz C.L. Botelho, Phys. Rev. 490, 1975 (1994).

[6] Luiz C.L. Botelho, Phys. Rev. 38D, 2464 (1988).

[7] A.M. Polyakov, Phys. Lett. 103B, 211 (1981).

[8] V. Fock, "The Theory of Space, Time and Gravitation, 2^{nd} Revised Edition, Pergamon Press, 1964.