CBPF-NF-042/94

# Algebraic characterization of anomalies in chiral $\mathcal{W}_3$−gravity

*Marcelo Carvalho, Luiz Claudio Queiroz Vilar* and *S.P. Sorella*

Centro Brasileiro de Pesquisas Físicas - CBPF
Rua Dr. Xavier Sigaud, 150
22290-180 Rio de Janeiro/RJ, Brasil

Abstract

The anomalies which occur in chiral $\mathcal{W}_3$−gravity are characterized by solving the $BRS$ consistency condition.

**Key-words:** Anomalies; BRS-cohomology.

**PACS:** 11.15.Bt

# 1 Introduction

$\mathcal{W}-$algebras [1] are an extension of the Virasoro algebra. They describe the commutation relations between the components of the stress-energy tensor $(T_{++}, T_{--})$ and currents $(W_{++++...}, W_{-----...})$ of higher spin (see ref. [2] for a general introduction).

Among the various $\mathcal{W}-$algebras considered in the recent literature, the so called $\mathcal{W}_3-$algebra plays a rather special role, due to the fact that it has a simple field theory realization. The corresponding field model, known as $\mathcal{W}_3-$gravity, yields a generalization of the usual bosonic string action. It is available in a chiral [3] as well as in a nonchiral version [4]. The starting field content of the model is given by a set of free scalar fields $\phi^i (i = 1, ..., D)$ which are used to study the current algebra of the spin-two and spin-three operators $T_{++}$ (resp. $T_{--}$) and $W_{+++}$ (resp. $W_{---}$)

$$T_{++} = \frac{1}{2}\partial_+\phi^i\partial_+\phi^i \ , \qquad W_{+++} = \frac{1}{3}d_{ijk}\partial_+\phi^i\partial_+\phi^j\partial_+\phi^k \ . \tag{1.1}$$

The quantities $d_{ijk}$ are totally symmetric and chosen to satisfy

$$d^m_{(ij}d_{k)lm} = \delta_{(ij}\delta_{k)l} \ . \tag{1.2}$$

The operators in (1.1) can be included in the initial free scalar action by coupling them to external fields $h_{--}, B_{---}$ (resp. $h_{++}, B_{+++}$). It is a remarkable fact then that the resulting action exhibits a set of local invariances which can be gauge fixed using the Batalin-Vilkovisky [5] procedure. The model is thus constrained by a classical Slavnov-Taylor identity which can be taken as the starting point for the analysis of the quantum properties of the $(T - \mathcal{W})-$current algebra. As it is well known, the existence of anomalies in the quantum extension of the Slavnov-Taylor identity turns out to be related to the presence of central charges in the corresponding current algebra.

This is the case of the $\mathcal{W}_3-$gravity, for which several types of anomalies have been found [3, 4]. Let us recall indeed that, besides the gravitational anomalies which depend on the fields $h_{--}, B_{---}$, the model possesses a pure matter field anomaly whose origin lies in the nonlinearity of the $\mathcal{W}_3-$algebra.

Let us also remark that these anomalies have been essentially detected by a direct computation of the relevant Feynman graphs which contribute to the one and two loop effective action. However, up to our knowledge, a purely algebraic characterization based on the cohomological analysis of the $BRS$ consistency condition is still lacking. The aim of the present work is to fill this gap.

In order to avoid too many technical details we shall limit here to present the analysis of the chiral $\mathcal{W}_3-$models for which the $d_{ijk}$'s in equations(1.1), (1.2) are traceless [1], i.e.

$$d^j_{ij} = 0 \ . \tag{1.3}$$

The generalization to the nontraceless case as well as to the nonchiral models can be done in the same way and does not present any additional complication.

The paper is organized as follows. In Sect. 2 the quantization of chiral $\mathcal{W}_3-$gravity is shortly recalled. In Sect. 3 we discuss the $BRS$ consistency condition for the anomalies and we derive the corresponding descent equations. The final part of this section is devoted to the introduction of an operator $\delta$ which allows to decompose the exterior space-time derivative as a $BRS$ commutator [7]. This important feature will provide a powerful method for solving the descent equations.

In Sect. 4 we present the computation of the cohomology of the linearized Slavnov-Taylor operator. In particular, we will be able to establish an useful formula, called here the *Russian-like* formula, which will greatly simplify the full algebraic analysis. This formula denotes a special combination of the ghost fields which allows to compute in a very simple and elegant way all the relevant cohomology classes. The *Russian-like* formula is one of the main results of the present paper. It will reveal a simple and so far unnoticed elementary structure of the $\mathcal{W}_3-$gravity.

In Sect. 5 the anomalies which affect the quantum extension of the Slavnov-Taylor identity are given. Sect. 6 deals with some interesting property of the $\mathcal{W}_3-$action. We will see indeed that, in complete analogy with the case of the bosonic string [8], the complete action of the $\mathcal{W}_3-$gravity turns out to be cohomologically trivial, suggesting then a topological interpretation of the model.

For the sake of clarity we have collected the most lengthy expressions in two Appendices, respectively *App. A* and *App. B*. Finally, in order to make contact with the two loop computation of ref [3], the *App. C* contains a detailed discussion of the $BRS$ consistency condition at the order $\hbar^2$.

# 2 The model and its quantization

In this section we briefly review the quantization of chiral $\mathcal{W}_3-$gravity. Following [3] the gauge-fixed action we start with is

$$S = S_{inv} + S_{gh} \ , \tag{2.1}$$

where

$$S_{inv} = \int d^2x \left( \frac{1}{2} \partial_+ \phi^i \partial_- \phi^i - h_{--} T_{++} - B_{---} W_{+++} \right) \ , \tag{2.2}$$

[1] Conditions (1.2) and (1.3) define a nonempty set of algebras, known as the magic Jordan algebras [6].

$T_{++}$ and $W_{+++}$ denoting respectively the operators

$$T_{++}=\frac{1}{2}\partial_+\phi^i\partial_+\phi^i\ , \qquad W_{+++}=\frac{1}{3}d_{ijk}\partial_+\phi^i\partial_+\phi^j\partial_+\phi^k\ . \tag{2.3}$$

They describe the coupling of a set of scalar fields $\phi^i(i=1,...,D)$ to external gauge fields $(h_{--},B_{---})$ [3]. The quantities $d_{ijk}$ in (2.3) are totally symmetric and satisfy the constraints

$$d^m_{(ij}d_{k)lm}=\delta_{(ij}\delta_{k)l}\ , \qquad d^j_{ij}=0\ . \tag{2.4}$$

The gauge-fixing term $S_{gh}$ of (2.1), using a Landau type background gauge [9], reads

$$\begin{aligned}
S_{gh}&=\int d^2x\, s\,(b_{++}h_{--}+\beta_{+++}B_{---})\\
&=-\int d^2x\, b_{++}\Big(\partial_-c_-+\partial_+h_{--}c_--h_{--}\partial_+c_-+2(\,\partial_+B_{---}\gamma_{--}-B_{---}\partial_+\gamma_{--})\,T_{++}\Big)\\
&\quad-\int d^2x\,\beta_{+++}\Big(\partial_-\gamma_{--}+2\partial_+h_{--}\gamma_{--}-h_{--}\partial_+\gamma_{--}-2B_{---}\partial_+c_-+\partial_+B_{---}c_-\Big)\ ,
\end{aligned} \tag{2.5}$$

where $(c_-,\gamma_{--})$ and $(b_{++},\beta_{+++})$ are respectively a pair of ghost and antighost fields and $s$ denotes the $BRS$ operator whose action on the fields is specified by

$$\begin{aligned}
sc_-&=c_-\partial_+c_-+2\gamma_{--}\partial_+\gamma_{--}T_{++}\ ,\\
s\gamma_{--}&=c_-\partial_+\gamma_{--}-2\partial_+c_-\gamma_{--}\ ,\\
s\phi^i&=c_-\partial_+\phi^i+\gamma_{--}d^i_{jk}\partial_+\phi^j\partial_+\phi^k+2b_{++}\gamma_{--}\partial_+\gamma_{--}\partial_+\phi^i\ ,\\
sh_{--}&=\partial_-c_-+\partial_+h_{--}c_--h_{--}\partial_+c_-+2(\partial_+B_{---}\gamma_{--}-B_{---}\partial_+\gamma_{--})T_{++}\ ,\\
sB_{---}&=\partial_-\gamma_{--}+2\partial_+h_{--}\gamma_{--}-h_{--}\partial_+\gamma_{--}-2B_{---}\partial_+c_-+\partial_+B_{---}c_-\ ,\\
sb_{++}&=s\beta_{+++}=0\ ,
\end{aligned} \tag{2.6}$$

and

$$sS=0\ . \tag{2.7}$$

Let us remark that the above transformations are nilpotent only *on-shell*, i.e.

$$\begin{aligned}
s^2h_{--}&=-2\gamma_{--}\partial_+\gamma_{--}\partial_+\phi^i\frac{\delta S}{\delta\phi^i}\ ,\\
s^2\phi^i&=2\gamma_{--}\partial_+\gamma_{--}\partial_+\phi^i\frac{\delta S}{\delta h_{--}}\ ,
\end{aligned} \tag{2.8}$$

and

$$s^2\,(c_-,\ \gamma_{--},\ B_{---},\ b_{++})=0\ . \tag{2.9}$$

However, coupling the nonlinear $BRS$ transformations of $(\phi^i,c_-,\gamma_{--})$ in (2.6) to external fields $(Y^i,\tau_+,\rho_{++})$

$$S_{ext}=\int d^2x\,\Big(Y^is\phi^i+\tau_+sc_-+\rho_{++}s\gamma_{--}\Big)\ , \tag{2.10}$$

and making use of the indentities

$$sh_{--} = -\frac{\delta S}{\delta b_{++}} \ , \qquad sB_{---} = -\frac{\delta S}{\delta \beta_{+++}} \ , \tag{2.11}$$

one easily verifies that the total action $\Sigma$

$$\Sigma = S_{inv} + S_{gh} + S_{ext} \ , \tag{2.12}$$

obeys the classical Slavnov-Taylor identity [3]

$$\mathcal{S}(\Sigma) = 0 \ , \tag{2.13}$$

with

$$\mathcal{S}(\Sigma) = \int d^2x \left( \frac{\delta\Sigma}{\delta Y^i}\frac{\delta\Sigma}{\delta\phi^i} + \frac{\delta\Sigma}{\delta\tau_+}\frac{\delta\Sigma}{\delta c_-} + \frac{\delta\Sigma}{\delta\rho_{++}}\frac{\delta\Sigma}{\delta\gamma_{--}} - \frac{\delta\Sigma}{\delta b_{++}}\frac{\delta\Sigma}{\delta h_{--}} - \frac{\delta\Sigma}{\delta\beta_{+++}}\frac{\delta\Sigma}{\delta B_{---}} \right) . \tag{2.14}$$

As in the case of the bosonic string [9], identity (2.13) is the starting point for studying the properties of the Green's functions of the model with insertion of the composite operators $(T_{++}, W_{+++})$, i.e. (2.13) yields an algebraic characterization of the $(T - W)$-current algebra.

Introducing now the linearized Slavnov-Taylor operator $\mathcal{B}_\Sigma$

$$\begin{aligned}\mathcal{B}_\Sigma = \int d^2x \Big( & \frac{\delta\Sigma}{\delta Y^i}\frac{\delta}{\delta\phi^i} + \frac{\delta\Sigma}{\delta\phi^i}\frac{\delta}{\delta Y^i} + \frac{\delta\Sigma}{\delta\tau_+}\frac{\delta}{\delta c_-} + \frac{\delta\Sigma}{\delta c_-}\frac{\delta}{\delta\tau_+} + \frac{\delta\Sigma}{\delta\rho_{++}}\frac{\delta}{\delta\gamma_{--}} + \frac{\delta\Sigma}{\delta\gamma_{--}}\frac{\delta}{\delta\rho_{++}} \\ & - \frac{\delta\Sigma}{\delta b_{++}}\frac{\delta}{\delta h_{--}} - \frac{\delta\Sigma}{\delta h_{--}}\frac{\delta}{\delta b_{++}} - \frac{\delta\Sigma}{\delta B_{---}}\frac{\delta}{\delta\beta_{+++}} - \frac{\delta\Sigma}{\delta\beta_{+++}}\frac{\delta}{\delta B_{---}} \Big) \ ,\end{aligned} \tag{2.15}$$

one gets the nonlinear algebraic relation

$$\mathcal{B}_\mathcal{F}\mathcal{S}(\mathcal{F}) = 0 \ , \tag{2.16}$$

valid for an arbitrary functional $\mathcal{F}$ with even ghost number. In particular if $\mathcal{F}$ is a solution of (2.13) then the corresponding linearized operator $\mathcal{B}_\mathcal{F}$ is nilpotent, i.e. $\mathcal{B}_\mathcal{F}\mathcal{B}_\mathcal{F} = 0$. Therefore

$$\mathcal{B}_\Sigma\mathcal{B}_\Sigma = 0 \ . \tag{2.17}$$

Let us conclude this section by displaying the ghost number $N_g$ and the dimension of all the fields and sources

| | $\phi^i$ | $h_{--}$ | $B_{---}$ | $c_-$ | $\gamma_{--}$ | $b_{++}$ | $\beta_{+++}$ | $Y^i$ | $\tau_+$ | $\rho_{++}$ | $\mathcal{B}_\Sigma$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $N_g$ | 0 | 0 | 0 | 1 | 1 | -1 | -1 | -1 | -2 | -2 | 1 |
| $dim$ | 0 | 0 | -1 | 0 | -1 | 1 | 2 | 1 | 1 | 2 | 1 |

Table 1: Ghost numbers and dimensions.

# 3 The consistency condition and the descent equations

At the quantum level the classical action (2.12) gives rise to an effective action

$$\Gamma = \Sigma + O(\hbar) \ , \tag{3.1}$$

which obeys the broken Slavnov-Taylor identity

$$\mathcal{S}(\Gamma) = \hbar \mathcal{A} \cdot \Gamma \ , \tag{3.2}$$

where the insertion $\mathcal{A} \cdot \Gamma$ represents the possible breaking induced by the radiative corrections. According to the Quantum Action Principle [10] the lowest order nonvanishing contribution to the breaking

$$\mathcal{A} \cdot \Gamma = \mathcal{A} + O(\hbar \mathcal{A}) \ , \tag{3.3}$$

is an integrated local functional of the fields and their derivatives with dimension three and ghost number one which obeys the consistency condition

$$\mathcal{B}_\Sigma \mathcal{A} = 0 \ . \tag{3.4}$$

The latter is the first-order expansion of the exact relation

$$\mathcal{B}_\Gamma \mathcal{A} \cdot \Gamma = 0 \ , \tag{3.5}$$

which easily follows from equation (2.16), i.e. from $\mathcal{B}_\Gamma \mathcal{S}(\Gamma) = 0$.

Setting $\mathcal{A} = \int \mathcal{A}^1_2$, condition (3.4) yields the nonintegrated equation [2],

$$\mathcal{B}_\Sigma \mathcal{A}^1_2 + d\mathcal{A}^2_1 = 0 \ , \tag{3.6}$$

where $\mathcal{A}^2_1$ is a local functional with ghost number two and form degree one and $d = dx^+ \partial_+ + dx^- \partial_-$ denotes the exterior space-time derivative which, together with the linearized operator $\mathcal{B}_\Sigma$, obeys [3]

$$\mathcal{B}_\Sigma d + d\mathcal{B}_\Sigma = d^2 = \mathcal{B}_\Sigma \mathcal{B}_\Sigma = 0 \ . \tag{3.7}$$

A solution $\mathcal{A}^1_2$ of eq. (3.6) is said nontrivial if

$$\mathcal{A}^1_2 \neq \mathcal{B}_\Sigma \hat{\mathcal{A}}^0_2 + d\hat{\mathcal{A}}^1_1 \ , \tag{3.8}$$

with $\hat{\mathcal{A}}^0_2$ and $\hat{\mathcal{A}}^1_1$ local functionals of the fields. In this case the integral of $\mathcal{A}^1_2$ on space-time, $\int \mathcal{A}^1_2$, identifies a cohomology class of the operator $\mathcal{B}_\Sigma$ in the sector

[2] We adopt here the usual convention of denoting with $\mathcal{A}^p_q$ a q-form with ghost number p.

[3] The operators $\mathcal{B}_\Sigma$ and $d$ increase respectively the ghost number and the form degree by one unit.

of the integrated local functionals with ghost number one. As it is well known this corresponds to the appearence of an anomaly in the quantum extension of the Slavnov-Taylor identity (2.13).

Equation (3.6), due to the relations (3.7) and to the algebraic Poincaré Lemma [11], generates a tower of descent equations

$$
\begin{aligned}
&\mathcal{B}_\Sigma \mathcal{A}^1_2 + d\mathcal{A}^2_1 = 0 \ , \\
&\mathcal{B}_\Sigma \mathcal{A}^2_1 + d\mathcal{A}^3_0 = 0 \ , \\
&\mathcal{B}_\Sigma \mathcal{A}^3_0 = 0 \ ,
\end{aligned}
\tag{3.9}
$$

with $\mathcal{A}^3_0$ a local functional with ghost number three and form degree zero. Let us also remark that, due to Table 1 and to the fact that the space-time derivatives $(\partial_-, \partial_+)$ are of dimension one, all the cocycles $\mathcal{A}^1_2, \mathcal{A}^2_1, \mathcal{A}^3_0$ in (3.9) have dimension three. The task of the next sections will be that of finding the most general local nontrivial solution of the descent equations (3.9), giving thus an algebraic characterization of the anomalies which can arise at the quantum level.

## 3.1 Decomposition of the exterior derivative

In order to solve the ladder (3.9) we follow the algebraic setup proposed by one of the authors [7] and successfully applied to the study of the Yang-Mills [12] and of the gravitational anomalies [13]. The method is based on the introduction of an operator $\delta$ which decomposes the exterior derivative $d$ as a $\mathcal{B}_\Sigma$ commutator, i.e.

$$
d = -\left[\mathcal{B}_\Sigma, \delta\right] \ . \tag{3.10}
$$

One easily verifies that, once the decomposition (3.10) has been found, successive applications of the operator $\delta$ on the cocycle $\mathcal{A}^3_0$ which solves the last equation of the tower (3.9) yield an explicit nontrivial solution for the higher cocycles $\mathcal{A}^1_2, \mathcal{A}^2_1$. It is interesting to observe that, actually, the decomposition (3.10) also occurs in the topological field theories [14] and in the string theory [8].

Let us emphasize that solving the last equation of (3.9) is a problem of local cohomology instead of a modulo $d$ one. One sees thus that, due to the operator $\delta$, the characterization of the cohomology of $\mathcal{B}_\Sigma$ modulo $d$ is essentially reduced to a problem of local cohomology. It is also worth to recall that the latter can be systematically studied by using several methods as, for instance, the spectral sequences technique [15, 16].

To find the decomposition (3.10) let us begin by specifying the local functional space $\mathcal{V}$ the operators $\mathcal{B}_\Sigma$ and $d$ act upon. Taking into account that the classical action (2.12) is invariant under the constant shift symmetry ($\phi^i \to \phi^i + const$) and that, due to the couplings with the fields ($h_{--}$, $B_{---}$) (see eq.(2.2)), the matter fields $\phi^i$ enter into the effective action $\Gamma$ with at least one derivative $\partial_+$, it follows that the space $\mathcal{V}$ can be identified as the space of the local formal power series in the variables ($h_{--}$, $B_{---}$, $c_-$, $\gamma_{--}$, $b_{--}$, $\beta_{+++}$ ,$Y^i$ ,$\tau_+$, $\rho_{++}$, $\partial_+\phi^i$) and their derivatives, i.e.

$$
\begin{aligned}
\mathcal{V} = \; & \textit{formal power series in the variables} \left(\partial_-^l\partial_+^m\xi,\ \partial_-^l\partial_+^{m+1}\phi^i\right) , \\
& \xi = \left(h_{--}, B_{---}, c_-, \gamma_{--}, b_{--}, \beta_{+++}, Y^i, \tau_+, \rho_{++}\right) , \qquad l, m = 0, 1, ....
\end{aligned} \tag{3.11}
$$

The solution of the descent equations (3.9) will be required to belong to $\mathcal{V}$, i.e. $(\mathcal{A}^1_2, \mathcal{A}^2_1, \mathcal{A}^3_0)$ will be space-time forms whose coefficients are elements of the local functional space (3.11). We also remark that when the integrated consistency condition (3.4) is translated at the nonintegrated level by means of the tower (3.9) one is no more allowed to make integration by parts. Therefore the fields and their derivatives have to be considered as independent variables.

On the local space $\mathcal{V}$ we introduce the two operators $\delta_+, \delta_-$ defined as

$$
\delta_+ = \frac{\partial}{\partial c_-} , \tag{3.12}
$$

and

$$
\begin{aligned}
\delta_- = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Bigg( \; & \partial_-^p\,\partial_+^q h_{--}\frac{\partial}{\partial\partial_-^p\,\partial_+^q c_-} + \partial_-^p\,\partial_+^q B_{---}\frac{\partial}{\partial\partial_-^p\,\partial_+^q \gamma_{--}} - \partial_-^p\,\partial_+^q \tau_+\frac{\partial}{\partial\partial_-^p\,\partial_+^q b_{++}} \\
& -\partial_-^p\,\partial_+^q \rho_{++}\frac{\partial}{\partial\partial_-^p\,\partial_+^q \beta_{+++}} - \partial_-^p\,\partial_+^q Y^i\frac{\partial}{\partial\partial_-^p\,\partial_+^{q+1} \phi^i}\Bigg) .
\end{aligned} \tag{3.13}
$$

They are easily seen to verify the algebraic relations

$$
\{\mathcal{B}_\Sigma, \delta_+\} = \partial_+ , \qquad \{\mathcal{B}_\Sigma, \delta_-\} = \partial_- , \tag{3.14}
$$

and

$$
\{\delta_-, \delta_+\} = 0 , \qquad \{\delta_\pm, d\} = 0 . \tag{3.15}
$$

The operator $\delta$,

$$
\delta = dx^+\delta_+ + dx^-\delta_- , \tag{3.16}
$$

obeys thus

$$
d = -\left[\mathcal{B}_\Sigma, \delta\right] , \qquad [\delta, d] = 0 , \tag{3.17}
$$

i.e., according to (3.10), the exterior space-time derivative $d$ has been decomposed as a $BRS$ commutator.

Suppose now that the solution $\mathcal{A}^3_0$ of the last equation of the tower (3.9) has been found, it is apparent then to check that the higher cocycles $\mathcal{A}^1_2, \mathcal{A}^2_1$ can be identified with the $\delta$-transform of $\mathcal{A}^3_0$ [7]:

$$
\mathcal{A}^2_1 = \delta\mathcal{A}^3_0 = dx^+\delta_+\mathcal{A}^3_0 + dx^-\delta_-\mathcal{A}^3_0 , \tag{3.18}
$$

and

$$\mathcal{A}^1_2 = \frac{1}{2}\delta^2\mathcal{A}^3_0 = -dx^+dx^-\delta_+\delta_-\mathcal{A}^3_0 \ . \tag{3.19}$$

Let us proceed thus with the characterization of $\mathcal{A}^3_0$, i.e. with the computation of the cohomology of $\mathcal{B}_\Sigma$ in the sector of the zero forms with ghost number and dimension three.

# 4 Cohomology of $\mathcal{B}_\Sigma$ in the sector of the zero forms with ghost number three and dimension three

In order to study the cohomology of $\mathcal{B}_\Sigma$ we introduce the filtering operator [4]

$$\widetilde{\mathcal{N}} = \int d^2x \left(\phi^i\frac{\delta}{\delta\phi^i} + Y^i\frac{\delta}{\delta Y^i}\right) \ , \tag{4.2}$$

according to which the linearized operator $\mathcal{B}_\Sigma$ decomposes as

$$\mathcal{B}_\Sigma = \mathcal{B}^{(0)} + \mathcal{B}^{(1)} + \mathcal{B}^{(2)} + \mathcal{B}^{(3)} \ , \tag{4.3}$$

with

$$\left[\widetilde{\mathcal{N}}, \mathcal{B}^{(n)}\right] = n\mathcal{B}^{(n)} \ , \qquad n = 0,1,2,3 \ . \tag{4.4}$$

The operators $\mathcal{B}^{(n)}$ in eq. (4.3) are given in *App.A*. In particular the nilpotency of $\mathcal{B}_\Sigma$ implies that

$$\sum_{j=0}^{n}\mathcal{B}^{(n-j)}\mathcal{B}^{(j)} = 0 \ , \qquad n = 0,1,2,3 \tag{4.5}$$

so that the operator $\mathcal{B}^{(0)}$ turns out to be nilpotent as well

$$\mathcal{B}^{(0)}\mathcal{B}^{(0)} = 0 \ . \tag{4.6}$$

The usefulness of the above decomposition relies on a very general theorem on the $BRS$ cohomology [15, 16]. The latter states that the cohomology of the full linearized Slavnov-Taylor operator $\mathcal{B}_\Sigma$ is isomorphic to a subspace of the cohomology of $\mathcal{B}^{(0)}$. Let us focus then on the study of the cohomology of the operator $\mathcal{B}^{(0)}$.

---

[4] We recall here that on the local space $\mathcal{V}$ the action of the generic functional differential operator $\int \mathcal{O}(\phi)\frac{\delta}{\delta\phi}$ is given by

$$\int d^2x\, \mathcal{O}(\phi)\frac{\delta}{\delta\phi} := \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\partial^p_-\partial^q_+\mathcal{O}(\phi)\frac{\partial}{\partial\partial^p_-\partial^q_+\phi^i} \ . \tag{4.1}$$

## 4.1 Cohomology of $\mathcal{B}^{(0)}$ in the sector of the zero forms with ghost number three and dimension three

In order to solve the equation

$$\mathcal{B}^{(0)}\Omega = 0 \ , \tag{4.7}$$

with $\Omega$ a zero form with ghost number and dimension three, we introduce a second filtering operator

$$\mathcal{N} = \int d^2x \left( c_- \frac{\delta}{\delta c_-} + h_{--} \frac{\delta}{\delta h_{--}} + B_{---} \frac{\delta}{\delta B_{---}} + \gamma_{--} \frac{\delta}{\delta \gamma_{--}} \right) . \tag{4.8}$$

It decomposes $\mathcal{B}^{(0)}$ and $\Omega$ as

$$\mathcal{B}^{(0)} = \delta^{(0)} + \delta^{(1)} + \delta^{(2)} \ , \tag{4.9}$$

and

$$\Omega = \sum_{\nu=0}^{\infty} \Omega^{(\nu)} \ , \tag{4.10}$$

with

$$\left[ \mathcal{N}, \delta^{(\nu)} \right] = \nu \delta^{(\nu)} \ , \qquad \mathcal{N}\Omega^{(\nu)} = \nu \Omega^{(\nu)} \ . \tag{4.11}$$

Condition (4.7) splits now into a system of equations

$$\sum_{p=0}^{\nu} \delta^{(p)} \Omega^{(\nu-p)} = 0 \ , \qquad \nu = 0, 1, 2, 3, ... \tag{4.12}$$

As before, the nilpotency of $\mathcal{B}^{(0)}$ implies

$$\sum_{k=0}^{q} \delta^{(q-k)} \delta^{(k)} = 0 \ , \qquad q = 0, 1, 2 \tag{4.13}$$

and

$$\delta^{(0)} \delta^{(0)} = 0 \ , \tag{4.14}$$

so that, according to the previous theorem, the cohomology of $\mathcal{B}^{(0)}$ is, in turn, isomorphic to a subspace of the cohomology of $\delta^{(0)}$. As we shall see the operator $\delta^{(0)}$ turns out to have a very elementary structure, making the computation of its cohomology rather easy. This will allow us to come back to the operators $\mathcal{B}^{(0)}$ and $\mathcal{B}_{\Sigma}$ and to identify their cohomologies in a simple way.

Using *App.A*, the operators $\delta^{(0)}, \delta^{(1)}, \delta^{(2)}$ are computed to be

$$\begin{aligned} \delta^{(0)} = \int d^2x \Big( & \partial_- c_- \frac{\delta}{\delta h_{--}} + \partial_- \gamma_{--} \frac{\delta}{\delta B_{---}} - \partial_+ \partial_- \phi^i \frac{\delta}{\delta Y^i} \\ & - \partial_- b_{++} \frac{\delta}{\delta \tau_+} - \partial_- \beta_{+++} \frac{\delta}{\delta \rho_{++}} \Big) \ , \end{aligned} \tag{4.15}$$

$$
\begin{aligned}
\delta^{(1)} = \int d^2x \Big( \quad & c_-\partial_+\phi^i \frac{\delta}{\delta\phi^i} \ + \ c_-\partial_+c_- \frac{\delta}{\delta c_-} + \Big(c_-\partial_+h_{--} - h_{--}\partial_+c_-\Big)\frac{\delta}{\delta h_{--}} \\
& + \Big(c_-\partial_+\gamma_{--} + 2\gamma_{--}\partial_+c_-\Big)\frac{\delta}{\delta\gamma_{--}} \\
& + \Big(2\gamma_{--}\partial_+h_{--} - h_{--}\partial_+\gamma_{--} - 2B_{---}\partial_+c_- + c_-\partial_+B_{---}\Big)\frac{\delta}{\delta B_{---}} \\
& + \Big(c_-\partial_+b_{++} - 2b_{++}\partial_+c_- + 2\gamma_{--}\partial_+\beta_{+++} - 3\beta_{+++}\partial_+\gamma_{--}\Big)\frac{\delta}{\delta b_{++}} \\
& + \Big(c_-\partial_+\beta_{+++} - 3\beta_{+++}\partial_+c_-\Big)\frac{\delta}{\delta\beta_{+++}} \\
& + \Big(\partial_+h_{--}\partial_+\phi^i + h_{--}{\partial_+}^2\phi^i - \partial_+(Y^ic_-)\Big)\frac{\delta}{\delta Y^i} \\
& + \Big(h_{--}\partial_+\beta_{+++} + 3\beta_{+++}\partial_+h_{--} + \partial_+(\rho_{++}c_-) + 2\rho_{++}\partial_+c_-\Big)\frac{\delta}{\delta\rho_{++}} \\
& + \Big(2b_{++}\partial_+h_{--} + h_{--}\partial_+b_{++} + 2B_{---}\partial_+\beta_{+++} + 3\beta_{+++}\partial_+B_{---}\Big)\frac{\delta}{\delta\tau_+} \\
& + \Big(2\tau_+\partial_+c_- + c_-\partial_+\tau_+ + 3\rho_{++}\partial_+\gamma_{--} + 2\gamma_{--}\partial_+\rho_{++}\Big)\frac{\delta}{\delta\tau_+} \quad \Big) ,
\end{aligned}
\tag{4.16}
$$

and

$$
\begin{aligned}
\delta^{(2)} = \int d^2x \Big( \quad & \Big(2b_{++}\gamma_{--}\partial_+\gamma_{--}\partial_+\phi^i\Big)\frac{\delta}{\delta\phi^i} \\
& + \Big(2\partial_+(b_{++}\partial_+B_{---}\gamma_{--}\partial_+\phi^i) - 2\partial_+(b_{++}B_{---}\partial_+\gamma_{--}\partial_+\phi^i)\Big)\frac{\delta}{\delta Y^i} \\
& - \Big(2\partial_+(Y^ib_{++}\gamma_{--}\partial_+\gamma_{--}) + 2\partial_+(\tau_+\gamma_{--}\partial_+\gamma_{--}\partial_+\phi^i)\Big)\frac{\delta}{\delta Y^i} \quad \Big)
\end{aligned}
\tag{4.17}
$$

Expression (4.15) shows that the variables $(h, \partial_-c)$, $(B, \partial_-\gamma)$, $(-Y^i, \partial_+\partial_-\phi^i)$, $(-\tau, \partial_-b)$, $(-\rho, \partial_-\beta)$ and their derivatives are grouped in $BRS$ doublets [11, 15, 16], i.e

$$
\delta^{(0)}\partial_-^l\partial_+^m u = \partial_-^l\partial_+^m v \ , \qquad \delta^{(0)}\partial_-^l\partial_+^m v = 0 \ , \tag{4.18}
$$

with

$$
\begin{aligned}
u &= \Big(h_{--}\ , B_{---}\ , -Y^i\ , -\tau_+\ , -\rho_{++}\Big) \ , \\
v &= \Big(\partial_-c_-\ , \partial_-\gamma_{--}\ , \partial_+\partial_-\phi^i\ , \partial_-b_{++}\ , \partial_-\beta_{+++}\Big) \ .
\end{aligned}
\tag{4.19}
$$

As it is well known, the cohomology does not depend on such variables [15, 16]. It follows thus that on the local space $\mathcal{V}$ the cohomology $\mathcal{H}(\delta^{(0)})$ of $\delta^{(0)}$ is spanned by

$$
\begin{aligned}
\mathcal{H}(\delta^{(0)}) = \ & \textit{formal power series in the variables} \ \Big(\partial_+^i\lambda,\ \partial_+^{i+1}\phi^i\Big) \ , \\
& \lambda = \Big(c_-\ , \gamma_{--}\ , b_{++}\ , \beta_{+++}\Big) \ , \qquad i = 0, 1, 2, ....
\end{aligned}
\tag{4.20}
$$

For the usefulness of the reader we display in *App.B* all the elements of $\mathcal{H}(\delta^{(0)})$ belonging to the sector of the zero forms with ghost number three and bounded by dimension three. The list is done according to the eigenvalues $\nu$ of the filtering operator $\mathcal{N}$ of (4.8). He will see, in particular, that the cohomology spaces of $\delta^{(0)}$ with $\nu \geq 5$ are empty. This is easily seen to imply that the system (4.12) reduces to a finite number of equations, i.e.

$$
\begin{aligned}
&\delta^{(0)}\,\Omega^{(6)} \,+\, \delta^{(1)}\,\Omega^{(5)} \,+\, \delta^{(2)}\,\Omega^{(4)} \,=0\;,\\
&\delta^{(0)}\,\Omega^{(5)} \,+\, \delta^{(1)}\,\Omega^{(4)} \,+\, \delta^{(2)}\,\Omega^{(3)} \,=0\;,\\
&\delta^{(0)}\,\Omega^{(4)} \,+\, \delta^{(1)}\,\Omega^{(3)} \,+\, \delta^{(2)}\,\Omega^{(2)} \,=0\;,\\
&\delta^{(0)}\,\Omega^{(3)} \,+\, \delta^{(1)}\,\Omega^{(2)} \,+\, \delta^{(2)}\,\Omega^{(1)} \,=0\;,\\
&\delta^{(0)}\,\Omega^{(2)} \,+\, \delta^{(1)}\,\Omega^{(1)} \,+\, \delta^{(2)}\,\Omega^{(0)} \,=0\;,\\
&\delta^{(0)}\,\Omega^{(1)} \,+\, \delta^{(1)}\,\Omega^{(0)} \,=0\;,\\
&\delta^{(0)}\,\Omega^{(0)} \,=0\;,
\end{aligned}
\tag{4.21}
$$

the conditions with higher eingenvalues ($\nu > 6$) corresponding to a trivial $\mathcal{B}^{(0)}$–cocycle.

It is not difficult now, using *App.B*, to work out the restrictions imposed by the system (4.21) on the cohomology $\mathcal{H}(\delta^{(0)})$ of $\delta^{(0)}$. We shall not enter here into the technical details of the computations, limiting ourselves to give the final result concerning the operator $\mathcal{B}^{(0)}$. It turns out that on the local space $\mathcal{V}$ the cohomology $\mathcal{H}(\mathcal{B}^{(0)})$ of $\mathcal{B}^{(0)}$ in the sector of zero forms with ghost number and dimension three contains five elements, a representative of which may be choosen as

$$
\begin{aligned}
\Bigg(&\Big(c_-\gamma_{--}\partial_+\gamma_{--}(\partial_+\phi^i\partial_+^3\phi^i - \partial_+^2\phi^i\partial_+^2\phi^i) + T_{++}\partial_+\gamma_{--}(\partial_+^2c_-\gamma_{--} + \frac{1}{2}c_-\partial_+^2\gamma_{--})\Big),\\
&c_-\partial_+c_-\partial_+^2c_-,\ c_-\partial_+c_-\gamma_{--}W_{+++},\ c_-\gamma_{--}\partial_+\gamma_{--}T_{++}T_{++},\ c_-\partial_+c_-\gamma_{--}\partial_+\gamma_{--}b_{++}T_{++}\Bigg)
\end{aligned}
\tag{4.22}
$$

$W_{+++}$ and $T_{++}$ denoting the operators of equation (2.3).

## 4.2 Completion of the cohomology of $\mathcal{B}_\Sigma$: a Russian formula

Having characterized the cohomology of $\mathcal{B}^{(0)}$ we turn to the study of the cohomology of the complete linearized Slavnov-Taylor operator $\mathcal{B}_\Sigma$ (2.15). Let us begin by observing that the cohomology space (4.22) decomposes according to the eigenvalues of the filtering operator $\widetilde{\mathcal{N}}$ of eq.(4.2) into the subspaces $\mathcal{Q}^{(0)}$, $\mathcal{Q}^{(2)}$, $\mathcal{Q}^{(3)}$, $\mathcal{Q}^{(4)}$

$$
\widetilde{\mathcal{N}}\mathcal{Q}^{(k)} = k\mathcal{Q}^{(k)}\;,
\tag{4.23}
$$

with

$$
\mathcal{Q}^{(0)} := c^3 = c_-\partial_+c_-\partial_+^2c_-\;,
\tag{4.24}
$$

and

$$\mathcal{Q}^{(2)} = m\mathcal{Q}_1^{(2)} + p\mathcal{Q}_2^{(2)} \ , \tag{4.25}$$

$\mathcal{Q}_1^{(2)}$, $\mathcal{Q}_2^{(2)}$ given respectively by

$$\mathcal{Q}_1^{(2)} = c_-\partial_+ c_- \gamma_{--}\partial_+\gamma_{--} b_{++} T_{++} \ , \tag{4.26}$$

$$\mathcal{Q}_2^{(2)} = c_-\gamma_{--}\partial_+\gamma_{--}\left(\partial_+\phi^i\partial_+^3\phi^i - \partial_+^2\phi^i\partial_+^2\phi^i\right) + T_{++}\partial_+\gamma_{--}\left(\partial_+^2 c_-\gamma_{--} + \frac{1}{2}c_-\partial_+^2\gamma_{--}\right) \ , \tag{4.27}$$

$(m, p)$ being arbitrary coefficients. Finally, for $\mathcal{Q}^{(3)}$, $\mathcal{Q}^{(4)}$ we get

$$\mathcal{Q}^{(3)} = c_-\partial_+ c_-\gamma_{--} W_{+++} \ , \tag{4.28}$$

$$\mathcal{Q}^{(4)} = c_-\gamma_{--}\partial_+\gamma_{--} T_{++} T_{++} \ . \tag{4.29}$$

In particular equations (4.24)-(4.29) show that the cohomology of $\mathcal{B}_\Sigma$ in the sector of zero forms with ghost number and dimension three can contain at most five elements. Let us try then to see if the above $\mathcal{B}^{(0)}-$cocycles can be promoted to nontrivial cocycles of $\mathcal{B}_\Sigma$, i.e. we look at the possibility of making a completion of $(\mathcal{Q}^{(0)}, \mathcal{Q}_1^{(2)}, \mathcal{Q}_2^{(2)}, \mathcal{Q}^{(3)}, \mathcal{Q}^{(4)})$ to obtain elements of the cohomology of $\mathcal{B}_\Sigma$. This means find, for each $\mathcal{Q}^{(i)}$, a term $\mathcal{R}_{\mathcal{Q}^{(i)}}$

$$\mathcal{Q}^{(i)} \rightarrow \hat{\mathcal{Q}}^{(i)} = \mathcal{Q}^{(i)} + \mathcal{R}_{\mathcal{Q}^{(i)}} \ , \tag{4.30}$$

such that

$$\begin{aligned} &i) \qquad \mathcal{B}_\Sigma\hat{\mathcal{Q}}^{(i)} = 0 \ , \\ &ii) \qquad \hat{\mathcal{Q}}^{(i)} \neq \mathcal{B}_\Sigma - variation \ . \end{aligned} \tag{4.31}$$

Of course the $\mathcal{R}_{\mathcal{Q}^{(i)}}$'s are always defined modulo trivial $\mathcal{B}_\Sigma-$cocycles.

The problem of the completion of the cohomology of $\mathcal{B}_\Sigma$ is in general rather complicated and, depending on the Lie algebra structure of the generators of the BRS symmetry, requires the knowledge of the corresponding Lie algebra cohomolgy (see for instance refs. [17] for the case of Yang-Mills). However we shall see that in the present case it is greatly simplified thanks to the existence of a *Russian-like* formula [18] which allows to map elements of the cohomology of $\mathcal{B}^{(0)}$ into nontrivial cocycles of $\mathcal{B}_\Sigma$ in a very simple and elegant way.

Let us proceed thus by discussing in detail the completion of the cocycle $c^3$ of equation (4.24). To this purpose we introduce the following combination, here called the *Russian-like* formula,

$$\hat{c}_- = c_- - 2b_{++}\gamma_{--}\partial_+\gamma_{--} \ . \tag{4.32}$$

Using *App.A* one easily verifies that $\hat{c}_-$ transforms under $\mathcal{B}_\Sigma$ in the same way as $c_-$ transforms under $\mathcal{B}^{(0)}$, i.e.

$$\begin{aligned} \mathcal{B}_\Sigma\hat{c}_- &= \hat{c}_-\partial_+\hat{c}_- \ , \\ \mathcal{B}^{(0)}c_- &= c_-\partial_+ c_- \ . \end{aligned} \tag{4.33}$$

It is apparent then from equation (4.22) that the cocycle

$$\hat{c}^3 = \hat{c}_- \partial_+ \hat{c}_- \partial_+^2 \hat{c}_- \tag{4.34}$$

is $\mathcal{B}_\Sigma$−invariant. It satisfies thus the first requirement of (4.31).

The expression (4.34) is computed to be

$$\hat{c}^3 = c^3 + \mathcal{R}_{c^3} \ , \tag{4.35}$$

with

$$\begin{aligned} \mathcal{R}_{c^3} = \ & -2c_- \partial_+ c_- (\partial_+^2 b_{++} \gamma_{--} \partial_+ \gamma_{--} + 2\partial_+ b_{++} \gamma_{--} \partial_+^2 \gamma_{--}) \\ & -2c_- \partial_+ c_- (b_{++} \partial_+ \gamma_{--} \partial^2 \gamma_{--} + b_{++} \gamma_{--} \partial_+^3 \gamma_{--}) \\ & +2c_- \partial_+^2 c_- (\partial_+ b_{++} \gamma_{--} \partial_+ \gamma_{--} + b_{++} \gamma_{--} \partial_+^2 \gamma_{--}) \\ & -2\partial_+ c_- \partial_+^2 c_- b_{++} \gamma_{--} \partial_+ \gamma_{--} \ . \end{aligned} \tag{4.36}$$

Notice that $\mathcal{R}_{c^3}$ belongs to the subspace of the filtering operator $\widetilde{\mathcal{N}}$ with zero eigenvalue. In addition it turns out to be an exact $\mathcal{B}^{(0)}$−cocycle

$$\begin{aligned} \mathcal{R}_{c^3} & = \mathcal{B}^{(0)} \mathcal{M}^{(0)} \\ & = \mathcal{B}^{(0)} \Big( 4\partial_+^2 c_- \gamma_{--} \partial_+ \gamma_{--} b_{++} - 2\partial_+ c_- \gamma_{--} \partial_+^2 \gamma_{--} b_{++} \\ & \qquad - 2\partial_+ c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+ b_{++} + \frac{3}{4} c_- \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} b_{++} \\ & \qquad + \frac{3}{2} \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} \beta_{+++} - \frac{1}{2} c_- \gamma_{--} \partial_+^3 \gamma_{--} b_{++} \Big) \ . \end{aligned} \tag{4.37}$$

Let us now prove that (4.35) also satisfies the second requirement of (4.31). As usual [16] we proceed by assuming the converse. Suppose then that

$$\hat{c}^3 = \mathcal{B}_\Sigma \Xi \ , \tag{4.38}$$

for some local formal power series $\Xi$. Decomposing (4.38) according to the eingenvalues of $\widetilde{\mathcal{N}}$ and taking into account that both $c^3$ and $\mathcal{R}_{c^3}$ belong to the subspace with zero eigenvalue, one gets

$$c^3 + \mathcal{R}_{c^3} = \mathcal{B}^{(0)} \Xi^{(0)} \ . \tag{4.39}$$

This condition, due to the $\mathcal{B}^{(0)}$−triviality of $\mathcal{R}_{c^3}$ (eq. (4.37)), yields

$$c^3 = \mathcal{B}^{(0)} (\Xi^{(0)} - \mathcal{M}^{(0)}) \ , \tag{4.40}$$

which would imply that $c^3$ is $\mathcal{B}^{(0)}$−trivial, in contradiction with (4.22). We have thus proven that expression (4.34) defines a cohomology class of the operator $\mathcal{B}_\Sigma$ in the sector of zero forms with ghost number and dimension three. In particular, equation (4.32) has been shown to give a simple and elegant way of obtaining cohomology classes of $\mathcal{B}_\Sigma$ from those corresponding to $\mathcal{B}^{(0)}$, justifying then the name of *Russian-like* formula.

Let us proceed to present the completion of the remaining cocycles (4.25)-(4.29). Concerning expression $\mathcal{Q}^{(4)}$ of eq. (4.29) one easily checks that it is $\mathcal{B}_\Sigma-$invariant

$$\mathcal{B}_\Sigma \mathcal{Q}^{(4)} = 0 \ . \tag{4.41}$$

However, due to

$$\mathcal{Q}^{(4)} = -\mathcal{B}_\Sigma (c_- \gamma_{--} \partial_+ \gamma_{--} b_{++} T_{++}) \ , \tag{4.42}$$

it is cohomologically trivial. Let us consider then the cocycle $\mathcal{Q}_2^{(2)}$ in (4.27). Using the *Russian-like* formula (4.32) it is not difficult to prove that a completion of $\mathcal{Q}_2^{(2)}$ is given by

$$\mathcal{Q}_2^{(2)} \rightarrow \hat{\mathcal{Q}}_2^{(2)} = \mathcal{Q}_2^{(2)} + \mathcal{R}_{\mathcal{Q}_2^{(2)}} \ , \tag{4.43}$$

with $\mathcal{R}_{\mathcal{Q}_2^{(2)}}$

$$\mathcal{R}_{\mathcal{Q}_2^{(2)}} = 3\hat{c}_- \gamma_{--} \partial_+ \gamma_{--} \left( \partial_+^3 \gamma_{--} \beta_{+++} + \partial_+^2 \gamma_{--} \partial_+ \beta_{+++} \right) \ . \tag{4.44}$$

As in the case of the cocycle $\hat{c}^3$, expression (4.43) can be proven to be $\mathcal{B}_\Sigma-$nontrivial. It identifies thus a cohomology class.

Finally, the two cocycles $\mathcal{Q}_1^{(2)}$ and $\mathcal{Q}^{(3)}$ yield a $\mathcal{B}_\Sigma-$invariant quantity only in the combination

$$\mathcal{Q}^{(3)} + 4\mathcal{Q}_1^{(2)} \ , \tag{4.45}$$

wich is seen to be trivial, i.e.

$$\left( \mathcal{Q}^{(3)} + 4\mathcal{Q}_1^{(2)} \right) = -\mathcal{B}_\Sigma (c_- \partial_+ c_- \gamma_{--} \beta_{+++}) \ . \tag{4.46}$$

This concludes the analysis of the completion of the cohomology of $\mathcal{B}_\Sigma$.

## 4.3 Summary

Let us summarize here, for the convenience of the reader, the final result on the cohomology of $\mathcal{B}_\Sigma$. On the local space $\mathcal{V}$ the cohomology of the linearized operator $\mathcal{B}_\Sigma$ in the sector of zero forms $\mathcal{A}_0^3$ (see eq. (3.9)) with ghost number and dimension three contains only two elements. They can be written as

$$\mathcal{A}_0^3 = \left( \hat{\mathcal{Q}}^{(0)}, \ \hat{\mathcal{Q}}_2^{(2)} \right) \tag{4.47}$$

with

$$\begin{aligned} \hat{\mathcal{Q}}^{(0)} = \hat{c}^3 = \ & c_- \partial_+ c_- \partial_+^2 c_- \\ & -2c_- \partial_+ c_- (\partial_+^2 b_{++} \gamma_{--} \partial_+ \gamma_{--} + 2\partial_+ b_{++} \gamma_{--} \partial_+^2 \gamma_{--}) \\ & -2c_- \partial_+ c_- (b_{++} \partial_+ \gamma_{--} \partial^2 \gamma_{--} + b_{++} \gamma_{--} \partial_+^3 \gamma_{--}) \\ & +2c_- \partial_+^2 c_- (\partial_+ b_{++} \gamma_{--} \partial_+ \gamma_{--} + b_{++} \gamma_{--} \partial_+^2 \gamma_{--}) \\ & -2\partial_+ c_- \partial_+^2 c_- b_{++} \gamma_{--} \partial_+ \gamma_{--} \ , \end{aligned} \tag{4.48}$$

and

$$
\begin{aligned}
\hat{\mathcal{Q}}_2^{(2)} = \; & \hat{c}_- \gamma_{--} \partial_+ \gamma_{--} \Big( \partial_+ \phi^i \partial_+^3 \phi^i - \partial_+^2 \phi^i \partial_+^2 \phi^i \Big) \\
& + T_{++} \partial_+ \gamma_{--} \Big( \partial_+^2 \hat{c}_- \gamma_{--} + \frac{1}{2} \hat{c}_- \partial_+^2 \gamma_{--} \Big) \\
& + 3 \hat{c}_- \gamma_{--} \partial_+ \gamma_{--} \Big( \partial_+^3 \gamma_{--} \beta_{+++} + \partial_+^2 \gamma_{--} \partial_+ \beta_{+++} \Big) \; .
\end{aligned} \tag{4.49}
$$

# 5 Anomalies

Having characterized the cohomology of $\mathcal{B}_\Sigma$ in the sector of the zero forms with ghost number and dimension three we are now ready to solve the ladder (3.9) and find the anomalies which occur in the quantum extension of the Slavnov-Taylor identity (2.13).

Using the decomposition (3.17) and the equations (3.18), (3.19) it is immediate to check that to each element of $\mathcal{A}_0^3$ in (4.47) corresponds a two form $\mathcal{A}_2^1$ with ghost number one which is a nontrivial solution of the consistency condition (3.6)

$$
\mathcal{A}_2^1 = \Big( \mathcal{U}_2^1 , \; \mathcal{T}_2^1 \Big) \; , \tag{5.1}
$$

with $\mathcal{U}_2^1$ and $\mathcal{T}_2^1$ given respectively by

$$
\mathcal{U}_2^1 = -dx^+ dx^- \delta_+ \delta_- \hat{\mathcal{Q}}^{(0)} , \qquad \mathcal{T}_2^1 = -dx^+ dx^- \delta_+ \delta_- \hat{\mathcal{Q}}_2^{(2)} \; . \tag{5.2}
$$

They read

$$
\begin{aligned}
\mathcal{U}_2^1 = dx^+ dx^- \Big( \; & \partial_+ h_{--} \partial_+^2 c_- - \partial_+^2 h_{--} \partial_+ c_- - 2 \partial_+ h_{--} \partial_+^2 b_{++} \gamma_{--} \partial_+ \gamma_{--} \\
& - 2 \partial_+^2 \tau_+ \partial_+ c_- \gamma_{--} \partial_+ \gamma_{--} - 2 B_{---} \partial_+ c_- \partial_+^2 b_{++} \partial_+ \gamma_{--} \\
& + 2 \partial_+ B_{---} \partial_+ c_- \partial_+^2 b_{++} \gamma_{--} \\
& - 4 \partial_+ h_{--} \partial_+ b_{++} \gamma_{--} \partial^2 \gamma_{--} - 4 \partial_+ \tau_+ \partial_+ c_- \gamma_{--} \partial_+^2 \gamma_{--} \\
& - 4 B_{---} \partial_+ c_- \partial_+ b_{++} \partial_+^2 \gamma_{--} + 4 \partial_+^2 B_{---} \partial_+ c_- \partial_+ b_{++} \gamma_{--} \\
& - 2 \partial_+ h_{--} b_{++} \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} - 2 \tau_+ \partial_+ c_- \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} \\
& - 2 \partial_+ B_{---} \partial_+ c_- b_{++} \partial_+^2 \gamma_{--} + 2 \partial_+^2 B_{---} \partial_+ c_- b_{++} \partial_+ \gamma_{--} \\
& - 2 \partial_+ h_{--} b_{++} \gamma_{--} \partial_+^3 \gamma_{--} - 2 \tau_+ \partial_+ c_- \gamma_{--} \partial_+^3 \gamma_{--} \\
& - 2 B_{---} \partial_+ c_- b_{++} \partial_+^3 \gamma_{--} + 2 \partial_+^3 B_{---} \partial_+ c_- b_{++} \gamma_{--} \\
& + 2 \partial_+^2 h_{--} \partial_+ b_{++} \gamma_{--} \partial_+ \gamma_{--} + 2 \partial_+ \tau_+ \partial_+^2 c_- \gamma_{--} \partial_+ \gamma_{--} \\
& + 2 B_{---} \partial_+^2 c_- \partial_+ b_{++} \partial_+ \gamma_{--} - 2 \partial_+ B_{---} \partial_+^2 c_- \partial_+ b_{++} \gamma_{--} \\
& + 2 \partial_+^2 h_{--} b_{++} \gamma_{--} \partial_+^2 \gamma_{--} + 2 \tau_+ \partial_+^2 c_- \gamma_{--} \partial_+^2 \gamma_{--} \\
& + 2 B_{---} \partial_+^2 c_- b_{++} \partial_+^2 \gamma_{--} - 2 \partial_+^2 B_{---} \partial_+^2 c_- b_{++} \gamma_{--} \Big) \; ,
\end{aligned} \tag{5.3}
$$

and

$$
\begin{aligned}
\mathcal{T}_2^1 = dx^+dx^-\Big( \; & B_{---}\partial_+\gamma_{--}\partial_+^3\phi^i\partial_+\phi^i \;-\; \partial_+B_{---}\gamma_{--}\partial_+^3\phi^i\partial_+\phi^i \\
& -\gamma_{--}\partial_+\gamma_{--}\partial_+^2Y^i\partial_+\phi^i \;-\; \gamma_{--}\partial_+\gamma_{--}Y^i\partial_+^3\phi^i \\
& -B_{---}\partial_+\gamma_{--}\partial_+^2\phi^i\partial_+^2\phi^i \;+\; \gamma_{--}\partial_+B_{---}\partial_+^2\phi^i\partial_+^2\phi^i \\
& +2\gamma_{--}\partial_+\gamma_{--}\partial_+Y^i\partial_+^2\phi^i \;-\; \frac{1}{2}\partial_+B_{---}\partial_+^2\gamma_{--}T_{++} \\
& +\frac{1}{2}\partial_+^2B_{---}\partial_+\gamma_{--}T_{++} \;+\; \frac{1}{2}\partial_+\gamma_{--}\partial_+^2\gamma_{--}Y^i\partial_+\phi^i \\
& +3B_{---}\partial_+\gamma_{--}\partial_+^3\gamma_{--}\beta_{+++} \;-\; 3\partial_+B_{---}\gamma_{--}\partial_+^3\gamma_{--}\beta_{+++} \\
& +3\partial_+^3B_{---}\gamma_{--}\partial_+\gamma_{--}\beta_{+++} \;+3\gamma_{--}\partial_+\gamma_{--}\partial_+^3\gamma_{--}\rho_{++} \\
& +3B_{---}\partial_+\gamma_{--}\partial_+^2\gamma_{--}\partial_+\beta_{+++} \;-3\partial_+B_{---}\gamma_{--}\partial_+^2\gamma_{--}\partial_+\beta_{+++} \\
& +3\partial_+^2B_{---}\gamma_{--}\partial_+\gamma_{--}\partial_+\beta_{+++} \;+3\gamma_{--}\partial_+\gamma_{--}\partial_+^2\gamma_{--}\partial_+\rho_{++}\Big) \; .
\end{aligned}
\tag{5.4}
$$

Let us remember that, in general [7], eqs. (3.18)-(3.19) give only a particular solution of the descent equations (3.9). However, in the present case, (5.3) and (5.4) yield the most general solution. This is due to the fact that the cohomology of $\mathcal{B}_\Sigma$ turns out to be vanishing in the sectors of the one forms with ghost number two and of the two forms with ghost number one [5].

Finally, for the integrated cocycles we get

$$
\begin{aligned}
\mathcal{A}_{\mathcal{U}} &= \int \mathcal{U}_2^1 \\
&= \int d^2x \Big( 2c_-\partial_+^3h_{--} - 4\partial_+^3h_{--}\gamma_{--}\partial_+\gamma_{--}b_{++} + 4\partial_+^3c_-B_{---}\partial_+\gamma_{--}b_{++} \\
&\qquad - 4\partial_+^3c_-\gamma_{--}\partial_+B_{---}b_{++} - 4\partial_+^3c_-\gamma_{--}\partial_+\gamma_{--}\tau_+\Big) \; ,
\end{aligned}
\tag{5.5}
$$

and

$$
\begin{aligned}
\mathcal{A}_{\mathcal{T}} &= \int \mathcal{T}_2^1 \\
&= \int d^2x \Big( \; 2\Big(B_{---}\partial_+\gamma_{--} - \partial_+B_{---}\gamma_{--}\Big)\partial_+^3\phi^i\partial_+\phi^i \\
&\qquad + \frac{3}{2}\Big(\partial_+^2B_{---}\partial_+\gamma_{--} - \partial_+B_{---}\partial_+^2\gamma_{--}\Big)T_{++} \\
&\qquad + \Big(\partial_+^3B_{---}\gamma_{--} - B_{---}\partial_+^3\gamma_{--}\Big)T_{++} \\
&\qquad - \gamma_{--}\partial_+\gamma_{--}\partial_+^2Y^i\partial_+\phi^i - \gamma_{--}\partial_+\gamma_{--}Y^i\partial_+^3\phi^i \\
&\qquad + 2\gamma_{--}\partial_+\gamma_{--}\partial_+Y^i\partial_+^2\phi^i + \frac{1}{2}\partial_+\gamma_{--}\partial_+^2\gamma_{--}Y^i\partial_+\phi^i\Big) \; .
\end{aligned}
\tag{5.6}
$$

[5]This property follows from the emptiness of the cohomology of $\mathcal{B}^{(0)}$ in the form sectors considered above, as it is easily proven by repeating the same analysis of the previous section.

Equations (5.5), (5.6) give the expressions of the anomalies which arise in chiral $\mathcal{W}_3-$gravity. They are in complete agreement with the one loop result of the Feynman graphs computation done in ref. [3]. In particular, the term $\mathcal{A}_{\mathcal{U}}$ of (5.5), also called the universal gravitational anomaly, is easily seen to be a generalization of the usual diffeomorphism anomaly of the bosonic string [9], while the second term $\mathcal{A}_{\mathcal{T}}$ in eq.(5.6) is a matter-dependent anomaly whose origin can be traced back to the nonlinearity of the $\mathcal{W}_3-$algebra [3, 4].

# 6 Triviality of the classical action

This section is devoted to display an interesting algebraic property of the classical $\mathcal{W}_3-$action of eq.(2.12).

As it happens in the case of the bosonic string [8] (see also ref. [19] for the generalization to superstring), the classical action (2.12) turns out to be cohomologically trivial, i.e. it is a pure $\mathcal{B}_\Sigma-$variation. It is easily checked indeed that

$$\Sigma = \mathcal{B}_\Sigma \int d^2x \left( \frac{1}{2} Y^i \phi^i - \tau_+ c_- - \frac{1}{2} \rho_{++} \gamma_{--} - \frac{1}{2} \beta_{+++} B_{---} \right) . \tag{6.1}$$

This property allows to interpret in a suggestive way the $\mathcal{W}_3-$gravity as a topological model of the Witten's type [20].

## Acknowledgements

The *Conselho Nacional de Desenvolvimento Cientifico e Tecnologico*, *CNPq*-Brazil is gratefully acknowledged for the financial support.

# A The linearized Slavnov-Taylor operator $\mathcal{B}_\Sigma$

As shown in Sect. 4, the linearized operator $\mathcal{B}_\Sigma$ decomposes according to the eigenvalues of the filtering operator $\widetilde{\mathcal{N}}$ of eq. (4.2) as

$$\mathcal{B}_\Sigma = \mathcal{B}^{(0)} + \mathcal{B}^{(1)} + \mathcal{B}^{(2)} + \mathcal{B}^{(3)} \ . \tag{A.1}$$

with

$$\left[\widetilde{\mathcal{N}}, \mathcal{B}^{(n)}\right] = n\mathcal{B}^{(n)} \ , \qquad n = 0,1,2,3 \ . \tag{A.2}$$

The operators $\mathcal{B}^{(n)}$ read explicitely

$$\begin{aligned}
\mathcal{B}^{(0)} = \int d^2x \quad & \Bigg( \Big( c_-\partial_+\phi^i + 2b_{++}\gamma_{--}\partial_+\gamma_{--}\partial_+\phi^i \Big)\frac{\delta}{\delta\phi^i} \\
& + c_-\partial_+c_-\frac{\delta}{\delta c_-} \ + \ \Big(\partial_-c_- + c_-\partial_+h_{--} - h_{--}\partial_+c_-\Big)\frac{\delta}{\delta h_{--}} \\
& + \Big(c_-\partial_+\gamma_{--} + 2\gamma_{--}\partial_+c_-\Big)\frac{\delta}{\delta\gamma_{--}} \\
& + \Big( -2B_{---}\partial_+c_- + c_-\partial_+B_{---}\Big)\frac{\delta}{\delta B_{---}} \\
& + \Big(\partial_-\gamma_{--} + 2\gamma_{--}\partial_+h_{--} - h_{--}\partial_+\gamma_{--}\Big)\frac{\delta}{\delta B_{---}} \\
& + \Big(c_-\partial_+b_{++} - 2b_{++}\partial_+c_- + 2\gamma_{--}\partial_+\beta_{+++} - 3\beta_{+++}\partial_+\gamma_{--}\Big)\frac{\delta}{\delta b_{++}} \\
& + \Big(c_-\partial_+\beta_{+++} - 3\beta_{+++}\partial_+c_-\Big)\frac{\delta}{\delta\beta_{+++}} \\
& + \Big( - \ \partial_+\partial_-\phi^i + \partial_+h_{--}\partial_+\phi^i + h_{--}{\partial_+}^2\phi^i + 2\partial_+(b_{++}\partial_+B_{---}\gamma_{--}\partial_+\phi^i) \\
& \quad - 2\partial_+(b_{++}B_{---}\partial_+\gamma_{--}\partial_+\phi^i) - \partial_+(Y^ic_-) - 2\partial_+(Y^ib_{++}\gamma_{--}\partial_+\gamma_{--}) \\
& \quad - 2\partial_+(\tau_+\gamma_{--}\partial_+\gamma_{--}\partial_+\phi^i)\Big)\frac{\delta}{\delta Y^i} \\
& + \Big( - \ \partial_-\beta_{+++} + h_{--}\partial_+\beta_{+++} + 3\beta_{+++}\partial_+h_{--} \\
& \quad + \partial_+(\rho_{++}c_-) + 2\rho_{++}\partial_+c_-\Big)\frac{\delta}{\delta\rho_{++}} \\
& + \Big( - \ \partial_-b_{++} + 2b_{++}\partial_+h_{--} + h_{--}\partial_+b_{++} + 2B_{---}\partial_+\beta_{+++} + c_-\partial_+\tau_+ \\
& \quad + 3\beta_{+++}\partial_+B_{---} + 2\tau_+\partial_+c_- + 3\rho_{++}\partial_+\gamma_{--} + 2\gamma_{--}\partial_+\rho_{++}\Big)\frac{\delta}{\delta\tau_+}\Bigg) \ ,
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\mathcal{B}^{(1)} = \int d^2x \quad & \Bigg( \Big(d_{ijk}\gamma_{--}\partial_+\phi^j\partial_+\phi^k\Big)\frac{\delta}{\delta\phi^i} \\
& + \Big(d_{ijk}\partial_+B_{---}\partial_+\phi^j\partial_+\phi^k + 2d_{ijk}B_{---}\partial_+\phi^j\partial_+^2\phi^k \\
& \quad + 2\partial_+(d_{ijk}\gamma_{--}Y^j\partial_+\phi^k)\Big)\frac{\delta}{\delta Y^i} \Bigg) \ ,
\end{aligned} \tag{A.4}$$

$$\begin{aligned}\mathcal{B}^{(2)} = \int d^2x \quad \Bigg( & \Big(2\gamma_{--}\partial_+\gamma_{--}T_{++}\Big)\frac{\delta}{\delta c_-} + T_{++}\frac{\delta}{\delta b_{++}} - Y^i\partial_+\phi^i\frac{\delta}{\delta\tau_+} \\ & + \Big(2(\partial_+B_{---}\gamma_{--} - B_{---}\partial_+\gamma_{--})T_{++} + 2Y^i\gamma_{--}\partial_+\gamma_{--}\partial_+\phi^i\Big)\frac{\delta}{\delta h_{--}} \\ & + \Big(-2\partial_+b_{++}\gamma_{--}T_{++} - 4b_{++}\partial_+\gamma_{--}T_{++} - 2b_{++}\gamma_{--}\partial_+T_{++}\Big)\frac{\delta}{\delta\beta_{+++}} \\ & + \Big(4b_{++}\partial_+B_{---}T_{++} + 2\partial_+b_{++}B_{---}T_{++} + 2b_{++}B_{---}\partial_+T_{++} \\ & \quad + 4Y^ib_{++}\partial_+\gamma_{--}\partial_+\phi^i + 2\partial_+Y^ib_{++}\gamma_{--}\partial_+\phi^i \\ & \quad + 2Y^i\partial_+b_{++}\gamma_{--}\partial_+\phi^i + 2Y^ib_{++}\gamma_{--}\partial_+^2\phi^i + 4\tau_+\partial_+\gamma_{--}T_{++} \\ & \quad + 2\partial_+\tau_+\gamma_{--}T_{++} + 2\tau_+\gamma_{--}\partial_+^2\phi^i\partial_+\phi^i\Big)\frac{\delta}{\delta\rho_{++}} \Bigg) \, ,\end{aligned} \tag{A.5}$$

and

$$\mathcal{B}^{(3)} = \int d^2x \left( W_{+++}\frac{\delta}{\delta\beta_{+++}} - d_{ijk}Y^i\partial_+\phi^j\partial_+\phi^k\frac{\delta}{\delta\rho_{++}} \right) . \tag{A.6}$$

# B Cohomology of $\delta^{(0)}$

We give here the list of all the elements belonging to the cohomology of $\mathcal{H}(\delta^{(0)})$ of the operator $\delta^{(0)}$ of eq. (4.15) in the sector $\Lambda$ of the zero forms with gost number and dimension three. The list is done according to the eigenvalues $\nu$ of the filtering operator $\mathcal{N}$ of (4.8). It turns out that the only nonvanishing subspaces are those corresponding to the eigenvalues 3 and 4 of $\mathcal{N}$, i.e.

$$\Lambda = \Lambda^{(3)} + \Lambda^{(4)} \ , \tag{B.1}$$

with $\Lambda^{(3)}$ and $\Lambda^{(4)}$ given respectively by

$$\begin{aligned}
\Lambda^{(3)} = \Big( \ & \partial_+^4 c_- \gamma_{--} \partial_+ \gamma_{--}, \quad \gamma_{--} \partial_+ \gamma_{--} \partial_+^5 \gamma_{--}, \quad \gamma_{--} \partial_+^2 \gamma_{--} \partial_+^4 \gamma_{--}, \\
& c_- \partial_+ c_- \partial_+^2 c_-, \quad c_- \partial_+^4 c_- \gamma_{--}, \quad \partial_+ c_- \partial_+^3 c_- \gamma_{--}, \\
& c_- \partial_+^3 c_- \partial_+ \gamma_{--}, \quad \partial_+ c_- \partial_+^2 c_- \partial_+ \gamma_{--}, \quad c_- \partial_+^2 c_- \partial_+^2 \gamma_{--}, \\
& c_- \partial_+ c_- \partial_+^3 \gamma_{--}, \quad c_- \gamma_{--} \partial_+^5 \gamma_{--}, \quad c_- \partial_+ \gamma_{--} \partial_+^4 \gamma_{--}, \\
& c_- \partial_+^2 \gamma_{--} \partial_+^3 \gamma_{--}, \quad \partial_+ c_- \gamma_{--} \partial_+^4 \gamma_{--}, \quad \partial_+ c_- \partial_+ \gamma_{--} \partial_+^3 \gamma_{--}, \\
& \partial_+^2 c_- \gamma_{--} \partial_+^3 \gamma_{--}, \quad \partial_+^2 c_- \partial_+ \gamma_{--} \partial_+^2 \gamma_{--}, \quad \partial_+^3 c_- \gamma_{--} \partial_+^2 \gamma_{--} \\
& \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} \partial_+^3 \gamma_{--}, \quad c_- \partial_+ c_- \gamma_{--} W_{+++}, \\
& c_- \partial_+ c_- \gamma_{--} \partial_+^2 \phi^i \partial_+ \phi^i, \quad c_- \partial_+^2 c_- \gamma_{--} T_{++}, \\
& c_- \partial_+ c_- \partial_+ \gamma_{--} T_{++}, \quad c_- \gamma_{--} \partial_+ \gamma_{--} T_{++} T_{++}, \\
& c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 \phi^i \partial_+ \phi^j \partial_+ \phi^k d_{ijk}, \quad \partial_+ c_- \gamma_{--} \partial_+ \gamma_{--} W_{+++}, \\
& c_- \gamma_{--} \partial_+^2 \gamma_{--} W_{+++}, \quad c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^3 \phi^i \partial_+ \phi^i, \\
& c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 \phi^i \partial_+^2 \phi^i, \quad \partial_+^2 c_- \gamma_{--} \partial_+ \gamma_{--} T_{++}, \\
& \partial_+ c_- \gamma_{--} \partial_+^2 \gamma_{--} T_{++}, \quad c_- \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} T_{++}, \\
& \partial_+ c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 \phi^i \partial_+ \phi^i, \quad c_- \gamma_{--} \partial_+^3 \gamma_{--} T_{++}, \\
& c_- \gamma_{--} \partial_+^2 \gamma_{--} \partial_+^2 \phi^i \partial_+ \phi^i, \quad \gamma_{--} \partial_+ \gamma_{--} \partial_+^3 \gamma_{--} T_{++}, \\
& \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} \partial_+^2 \phi^i \partial_+ \phi^i, \quad \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} W_{+++} \Big) \ ,
\end{aligned} \tag{B.2}$$

and

$$
\begin{aligned}
\Lambda^{(4)} = \Big( \; & c_- \partial_+ c_- \partial_+^2 c_- \gamma_{--} b_{++}, \quad c_- \partial_+ c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 b_{++}, \\
& c_- \partial_+^2 c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+ b_{++}, \quad c_- \partial_+ c_- \gamma_{--} \partial_+^2 \gamma_{--} \partial_+ b_{++}, \\
& \partial_+ c_- \partial_+^2 c_- \gamma_{--} \partial_+ \gamma_{--} b_{++}, \quad c_- \partial_+^3 c_- \gamma_{--} \partial_+ \gamma_{--} b_{++}, \\
& c_- \partial_+^2 c_- \gamma_{--} \partial_+^2 \gamma_{--} b_{++}, \quad c_- \partial_+ c_- \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} b_{++}, \\
& c_- \partial_+ c_- \gamma_{--} \partial_+^3 \gamma_{--} b_{++}, \\
& c_- \partial_+ c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+ \beta_{+++}, \quad c_- \partial_+^2 c_- \gamma_{--} \partial_+ \gamma_{--} \beta_{+++}, \\
& c_- \partial_+ c_- \gamma_{--} \partial_+^2 \gamma_{--} \beta_{+++}, \quad c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} \partial_+^2 b_{++}, \\
& \partial_+ c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} \partial_+ b_{++}, \quad c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^3 \gamma_{--} \partial_+ b_{++}, \\
& \partial_+^2 c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} b_{++}, \quad \partial_+ c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^3 \gamma_{--} b_{++}, \\
& c_- \gamma_{--} \partial_+^2 \gamma_{--} \partial_+^3 \gamma_{--} b_{++}, \quad c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^4 \gamma_{--} b_{++}, \\
& c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} \partial_+ \beta_{+++}, \quad \partial_+ c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} \beta_{+++}, \\
& c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^3 \gamma_{--} \beta_{+++}, \quad \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} \partial_+^3 \gamma_{--} b_{++}, \\
& c_- \partial_+ c_- \gamma_{--} \partial_+ \gamma_{--} b_{++} T_{++}, \\
& c_- \gamma_{--} \partial_+ \gamma_{--} \partial_+^2 \gamma_{--} b_{++} T_{++} \Big) \; .
\end{aligned}
\tag{B.3}
$$

# C The second order consistency condition

As we have seen in Sect. 5 the solution of the integrated consistency condition (3.4) contains two nontrivial elements, given respectively in eqs.(5.5), (5.6). Let us emphasize that, actually, their numerical coefficients turn out to be nonvanishing already at the one loop order [3]. Thus, according to the Quantum Action Principle [10], expressions (5.5), (5.6) yield the most general nontrivial breaking of the Slavnov-Taylor identity (2.13) at the order $\hbar$. This could seem to be in contradiction with the Feynman graphs computation of ref. [3], where a third term was found. However this will be not the case, as we shall see by discussing the consistency condition (3.5) at the second order (i.e. at the order $\hbar^2$). The reason relies on the fact that the third term found in [3] is of order $\hbar^2$, i.e. its numerical coefficient turns out to be related to two loops Feynman diagrams. Moreover, the presence of nonvanishing one loop anomalies implies that the consistency condition (3.5) at the order $\hbar^2$ is not simply given by the equation (3.4). This means that the second order breaking terms cannot be characterized as cohomology classes of the linearized Slavnov-Taylor operator $\mathcal{B}_\Sigma$, i.e. in other words they obey a more involved consistency condition.

For a better understanding of this point let us discuss in detail the broken Slavnov-Taylor identity (3.2) and the condition (3.5) at the order $\hbar^2$. Let us begin with the Slavnov-Taylor identity

$$\mathcal{S}(\Gamma) = \hbar\mathcal{A}\cdot\Gamma \ . \tag{C.1}$$

Expanding $\Gamma$ and $(\mathcal{A}\cdot\Gamma)$ [6] in powers of $\hbar$

$$\begin{aligned} \Gamma &= \Sigma + \hbar\Gamma^{(1)} + \hbar^2\Gamma^{(2)} + O(\hbar^3) \ , \\ \hbar\mathcal{A}\cdot\Gamma &= \hbar\mathcal{A} + \hbar^2(\mathcal{A}\cdot\Gamma)^{(2)} + O(\hbar^3) \ , \end{aligned} \tag{C.2}$$

and using the identities

$$\mathcal{S}(\Gamma) = \frac{1}{2}\mathcal{B}_\Gamma\Gamma \ , \qquad \mathcal{B}_{\mathcal{F}_1}\mathcal{F}_2 = \mathcal{B}_{\mathcal{F}_2}\mathcal{F}_1 \ , \tag{C.3}$$

with $\mathcal{F}_1$, $\mathcal{F}_2$ arbitrary functionals with even ghost number, one gets

$$\mathcal{B}_\Sigma\Gamma^{(1)} = \mathcal{A} \ , \qquad order\ \hbar \ , \tag{C.4}$$

and

$$\mathcal{B}_\Sigma\Gamma^{(2)} + \frac{1}{2}\mathcal{B}_{\Gamma^{(1)}}\Gamma^{(1)} = (\mathcal{A}\cdot\Gamma)^{(2)} \ , \qquad order\ \hbar^2 \ . \tag{C.5}$$

In particular, equations (C.4), (C.5) show that, while the lowest order term $\mathcal{A}$ of the breaking $(\mathcal{A}\cdot\Gamma)$ is obtained by simply computing the $\mathcal{B}_\Sigma-$variation of the one loop effective action $\Gamma^{(1)}$, the second order term $(\mathcal{A}\cdot\Gamma)^{(2)}$ requires contributions from both $\Gamma^{(1)}$ and $\Gamma^{(2)}$.

Expanding now the consistency condition

$$\mathcal{B}_\Gamma(\mathcal{A}\cdot\Gamma) = 0 \ , \tag{C.6}$$

one gets

$$\mathcal{B}_\Sigma\mathcal{A} = 0 \ , \qquad order\ \hbar \ , \tag{C.7}$$

and

$$\mathcal{B}_\Sigma(\mathcal{A}\cdot\Gamma)^{(2)} + \mathcal{B}_{\Gamma^{(1)}}\mathcal{A} = 0 \ , \qquad order\ \hbar^2 \ . \tag{C.8}$$

As it is well known, equation (C.7) implies that the lowest nontrivial order of $(\mathcal{A}\cdot\Gamma)$ belongs to the cohomology of $\mathcal{B}_\Sigma$ in the class of the integrated local functionals with ghost number one. Writing then the most general solution of (C.7) as

$$\mathcal{A} = \mathcal{A}^{(anom)} + \mathcal{B}_\Sigma\Delta \ , \tag{C.9}$$

$\Delta$ and $\mathcal{A}^{(anom)} = (\mathcal{A}_\mathcal{U}, \mathcal{A}_\mathcal{T})$ (see eqs. (5.5), (5.6)) denoting respectively the trivial and the nontrivial part of $\mathcal{A}$, equation (C.8) becomes now

$$\mathcal{B}_\Sigma(\mathcal{A}\cdot\Gamma)^{(2)} + \mathcal{B}_{\mathcal{A}^{(anom)}}\Gamma^{(1)} + \mathcal{B}_{(\mathcal{B}_\Sigma\Delta)}\Gamma^{(1)} = 0 \ , \tag{C.10}$$

[6] We recall that in the present case the lowest order of $\mathcal{A}\cdot\Gamma$ is the order $\hbar$

where use has been made of (C.3). Eq.(C.8) is the consistency condition for the second order breaking term $(\mathcal{A}\cdot\Gamma)^{(2)}$. One sees thus that, in the presence of non-vanishing one loop anomalies, $(\mathcal{A}\cdot\Gamma)^{(2)}$ cannot be simply detected as a cohomology class of the operator $\mathcal{B}_{\Sigma}$. Let us conclude by recalling that the Quantum Action Principle [10] ensures that the right-hand side of (C.1) is the insertion of an integrated local functional

of the fields and their derivatives. This means that only the lowest order of $(\mathcal{A}\cdot\Gamma)$ is local. Therefore the analysis of the second order consistency condition (C.10) will, in general, require the knowledge of the local as well as of the nonlocal part of $(\mathcal{A}\cdot\Gamma)^{(2)}$.

# References


[1] A. B. Zamolodchikov, *Teor. Mat. Fiz* 65 (1985) 205;
V. A. Fateev and A. B. Zamolodchikov, *Nucl. Phys.* B280 (1987) 644;
J. Thierry-Mieg, *Phys. Lett.* B197 (1987) 368;
V. A. Fateev and S. Lykanov, *Int. J. Mod. Phys.* A3 (1988) 507;

[2] C. M. Hull, Classical and Quantum $\mathcal{W}-$gravity, *Summer School in High Energy Physics and Cosmology, Trieste 1992, I.C.T.P* ;

[3] C.M. Hull, *Phys. Lett.* 240B (1990) 110; *Nucl. Phys.* 353B (1991) 707;
*Phys. Lett.* 259B (1991) 68; *Nucl. Phys.* 364B (1991) 621;
*Phys. Lett.* 269B (1991) 257; *Nucl. Phys.* 367B (1991) 731;
*Comm. Math. Phys.* 156 (1993) 245; *Int. J. Mod. Phys.* 8 (1993) 2419;

[4] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, *Comm. Math. Phys.* 124 (1989) 87;
E. Bergshoeff, C. N. Pope, L. J. Romans, E. Sezgin, X. Shen and K. S. Stelle, *Phys. Lett.* B243 (1990) 350;
K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, *Phys. Lett.* B243 (1990) 245;
K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, *Nucl. Phys.* B364 (1991) 584;
A. T. Ceresole, M. Frau, J. McCarthy and A. Lerda, *Phys. Lett.* B265 (1991) 72;
E. Bergshoeff, C. N. Pope, L. J. Romans, E. Sezgin, X. Shen and K. S. Stelle, *Nucl. Phys.* B363 (1991) 163;

[5] I. A. Batalin and G. A. Vilkovisky, *Phys. Lett.* B102 (1981) 27;
I. A. Batalin and G. A. Vilkovisky, *Phys. Rev.* D28 (1983) 2567;

[6] L. J. Romans, *Nucl. Phys.* B352 (1991) 829;

[7] S. P. Sorella, *Comm. Math. Phys.* 157 (1993) 231;

[8] M. Werneck de Oliveira, M. Schweda and S. P. Sorella, *Phys. Lett.* B315 (1993) 93;

[9] L. Baulieu, C. Becchi and R. Stora, *Phys. Lett.* B180 (1986) 55;
C. Becchi, *Nucl. Phys.* B304 (1988) 513;

[10] Y. M. P. Lam, *Phys. Rev.* D6 (1972) 2145;
Y. M. P. Lam, *Phys. Rev.* D6 (1972) 2161;
T. E. Clark and J. H. Lowenstein, *Nucl. Phys.* B113 (1976) 109;

[11] F. Brandt, N. Dragon and M. Kreuzer, *Nucl. Phys.* B340 (1990) 187;

[12] S.P. Sorella and L. Tataru, *Phys. Lett.* B324 (1994) 351;

[13] M. Werneck de Oliviera and S. P. Sorella, *Int. J. Mod. Phys.* 9 (1994) 2979;
O. Moritsch, M. Schweda and S.P. Sorella, *Class. Quantum Grav.* 11 (1994) 1225;

[14] E. Guadagnini, N. Maggiore and S. P. Sorella, *Phys. Lett.* B255 (1991) 65;
C. Lucchesi, O. Piguet and S. P. Sorella, *Nucl. Phys.* B395 (1993) 325;
D. Birmingham and M. Rakowski, *Phys. Lett.* B269 (1991) 103;
D. Birmingham and M. Rakowski, *Phys. Lett.* B272 (1991) 217;
D. Birmingham and M. Rakowski, *Phys. Lett.* B275 (1992) 289;
D. Birmingham and M. Rakowski, *Phys. Lett.* B289 (1992) 271;

[15] J. A. Dixon, *Comm. Math. Phys.* 139 (1991) 495;

[16] C. Becchi, A. Blasi, G. Bonneau, R. Collina and F. Delduc, *Comm. Math. Phys.* 120 (1988) 121;

[17] M. Dubois-Violette, M. Talon and C.M. Viallet, *Comm. Math. Phys.* 102 (1985) 105;
F. Brandt, N. Dragon and M. Kreuzer, *Phys. Lett.* B231 (1989) 263; *Nucl. Phys.* B332 (1990) 224, 250;

[18] J. Thierry-Mieg, *J. Math. Phys.* 21 (1980) 2834; *Nuovo Cim.* 56A (1980) 396;
J. Manes, R. Stora and B. Zumino, *Comm. Math. Phys.* 102 (1985) 157;

[19] A. Boresch, M. Schweda and S. P. Sorella, *Phys. Lett.* B328 (1994) 36;

[20] E. Witten, *Comm. Math. Phys.* 117 (1988) 353;
E. Witten, *Comm. Math. Phys.* 121 (1989) 351;