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ON THE CHARACTERS AND IRREPS OF SOLVABLE FINITE GROUPS

by

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ABSTRACT

We propose a recurrent method to simultaneously calculate the characters and the irreducible representations of finite solvable groups. The representations of a normal subgroup are used as well as in the traditional induction procedure. However, our method is applicable even when the stabilizer of the representations of the subgroup is the group itself and it has the advantage that the representations obtained are already symmetry adapted to the composition series of the group.

Key-words: Characters; IRREPS; Solvable groups.

I. INTRODUCTION

A group structure is completely determined if we know the multiplication table from which it is possible to obtain the properties of the finite group and in particular its character table and the correspondent unitary irreducible representations (irreps).

Burnside (1955), Dixon (1967), Chen et al (1985) and Lee and Chen (1986) have proposed some methods to obtain the character tables of finite groups. Basically those methods involve the simultaneous diagonalisation of the structure constants of the centre of the algebra of the finite group.

Similarly, in order to obtain the symmetry adapted irreps to a canonical sequence, Chen et al (1985), Lee and Chen (1986) and Nogueira et al (1988) have proposed the diagonalisation of different linear combinations of the elements of the algebra of the centre of the nested groups in the series.

These approaches are more convenient than the traditional induction method when we are interested in calculate the symmetry adapted irreps to a chain of subgroups as is the case of spontaneous symmetry break. On the other hand, their application implies a diagonalisation of at least one matrix of dimension equal to the number of elements of the group. This fact limits the application of these methods to groups of small order.

The purpose of this work is to give a recurrent method applicable to solvable groups that will enable us to simultaneously calculate the character table and the correspondent irreps of the group. Moreover, the irreps obtained by this method will be already adapted to the composition series of the solvable group.

In section II we give the method for a fundamental sequence $\mathbb{G} \triangleright \mathbb{H}$ such that $\mathbb{G}/\mathbb{H} \simeq \mathbb{C}_p$, where p is a prime number. Since the factors of a composition series of a solvable group are isomorphic to cyclic groups of prime order, the method can be applied recurrently from the trivial subgroup of order one.

As an example, we consider in section III the octahedral group. We construct its irreps and compare them with those obtained by the traditional induction method using the irreps of normal subgroups.

Finally, in section IV we discuss the advantages in using symmetry adapted irreps to composition series of solvable groups and the extension of our method to groups which are not solvable.

II. CHARACTERS AND REPRESENTATIONS

Let $\mathbb{G}$ be a finite group with a normal subgroup $\mathbb{H}$ such that $\mathbb{G}/\mathbb{H} = \langle t\mathbb{H} \mid (t\mathbb{H})^p = \mathbb{H} \rangle$ and p is a prime number.

Let $\gamma \in \text{irrep}(\mathbb{H})$; the stabilizer (*little group*) of γ in $\mathbb{G}$ is defined by $\mathbb{S}_{\mathbb{G}}(\gamma) = \{ g{\in}\mathbb{G} \mid \gamma^g(h) = \gamma(ghg^{-1}) \cong \gamma(h) \} \ \forall\, h \in \mathbb{H}$. First we want to express in a way applicable to our calculus two results already known on the irreps of the group $\mathbb{G}$(*).

1) The induced representation $\gamma \uparrow \mathbb{G}$ is irreducible in $\mathbb{G}$ if the stabilizer of γ in $\mathbb{G}$ is $\mathbb{H}$.

The matrix elements of the induced representation of $\mathbb{G}$ are given by

$$\gamma \uparrow \mathbb{G}\,(g)_{i,j} = \begin{cases} \gamma(t^i g\, t^{-j}) \text{ if } t^i g\, t^{-j} \in \mathbb{H} \\ 0 \quad \text{otherwise} \end{cases} \tag{1}$$

From this equation, it is easy to find the characters $\chi^{\gamma\uparrow\mathbb{G}}$ of $\gamma\uparrow\mathbb{G}$,

$$\chi^{\gamma\uparrow\mathbb{G}}(h) = \sum_{k=1}^{p} \chi^{\gamma}(t^k h t^{-k}) \ , \quad \chi^{\gamma\uparrow\mathbb{G}}(t^n h) = 0 \ , \ h \in \mathbb{H} \ , \ 0<n<p \ .$$

Using this result, we can calculate

$$(1/|\mathbb{G}|)\sum_{g\in\mathbb{G}} |\chi^{\gamma\uparrow\mathbb{G}}(g)|^2 = (1/|\mathbb{G}|)\sum_{k=1}^{p}\ \sum_{h\in\mathbb{H}} |\chi^{\gamma}(t^k h t^{-k})|^2 = p\ |\mathbb{H}|/|\mathbb{G}| = 1,$$

which allows us to conclude that the induced matrix $\gamma\uparrow\mathbb{G}$ is

(*) See for example Janssen and Boon, "Theory of finite groups. Applications in Physics", North Holland (1967),Ch. II. 6.

irreducible in $\mathbb{G}$, Q E D.

Now, since p is a prime number, the other alternative for $S_{\mathbb{G}}(\gamma)$ is $\mathbb{G}$. Then,

2) When $S_{\mathbb{G}}(\gamma) = \mathbb{G}$, the induced representation $\gamma \uparrow \mathbb{G}$ is reducible in p irreps Γ_k of $\mathbb{G}$ related by

$$\Gamma_j(t^l h) = \omega^{(j-k)l}\, \Gamma_k(t^l h).$$

In this case we have that $\gamma(ghg^{-1}) = U(g)\gamma(h)U(g^{-1})$, $\forall\, g \in \mathbb{G}$ with $U(g)$ the unitary matrix representing g. Then, $g \to U(g)$, is a projective irreducible representation of $\mathbb{G}$, since $U(h) = \gamma(h)$. On the other hand, we can write $U(t^l h) = U(t)^l \gamma(h)$, and the phase factor of $U(t^l)$ can be expressed as a function of the phase factor $e^{i\varphi}$ of $U(t)$. If we write $U(t) = V(t)e^{i\varphi}$, we have $U(t)^p = U(h_o) = \gamma(h_o) = V(h_o)e^{ip\varphi}$ $\therefore$ $e^{ip\varphi} = 1$. This defines $e^{i\varphi}$ up to a power m of ω, with $\omega^p = 1$, m integer. Further, since $\mathbb{G}/\mathbb{H} \simeq \mathbb{C}_p$, $\mathbb{G}$ has at least p one-dimensional irreps given by $\lambda_k(t^l h) = \eta_k(t^l \mathbb{H}) = \omega^{kl}$, where $1 \le k \le p$ and $\eta_k(t^l \mathbb{H})$ are the irreps of $\mathbb{G}/\mathbb{H}$. Therefore, a particular choice of the exponent m will only produce a rearrangement of the irreps Γ_k of $\mathbb{G}$, which are of the form $\Gamma_k(t^l h) = U(t^l h)\omega^{kl}$ QED.

This result encouraged the search of a procedure to obtain the representations of a group $\mathbb{G}$ in an irreducible form when the stabilizer of the irreps of its normal subgroup is $\mathbb{G}$ itself.

Making use of the results above, it is immediate to obtain the following relations for the characters of the irreps Γ_k within a given coset $t^l\mathbb{H}$, for $0 < l \le p$ and $\Gamma(h) = \gamma(h)$:

$$\sum_{h \in \mathbb{H}} |\chi^{\Gamma_k}(t^l h)|^2 = |\mathbb{H}| \quad \text{if} \quad S_{\mathbb{G}}(\gamma) = \mathbb{G} \quad . \tag{2}$$

From this equation we see that there is at least one character different from zero in each coset $t^l\mathbb{H}$. On the other hand, if $A_1 \in \text{irrep}(\mathbb{H})$ is not contained in the restriction of an irrep Γ_k of $\mathbb{G}$ on $\mathbb{H}$, the following equality must hole

$$\sum_{h \in \mathbb{H}} \chi^{\Gamma_k}(t^l h) = 0 \ . \tag{3}$$

Equation (3) shows that if there are two or more classes of $\mathbb{G}$ within the coset $t^l\mathbb{H}$, at least two characters $\chi^\Gamma(t^lh)$ must be different from zero. On the other hand, if the coset $t^l\mathbb{H}$ only contains one complete class of $\mathbb{G}$, the equality $\chi^\Gamma(t^lh) = 0$ must hold, in contradiction with equation (2). Compatibility between these two results, implies that if $\langle t^l\mathbb{H}\rangle = \mathscr{C}(t^l)$, there are no irreps Γ_k of $\mathbb{G}$ such that $\Gamma_k(h) = \gamma(h) \neq A_1(h)$. This result, can be shown using the following arguments :

As a first step, let us prove that the transversal t can always be chosen such that its order $o(t)$ is p^α , α a positive integer. Suppose that $v\mathbb{H}$ is the generator of $\mathbb{G}/\mathbb{H}$. Then we have $v^p = h_o$, $h_o \in \mathbb{H}$. If we take $o(h_o) = kp^{\alpha-1}$, with $(k,p) = 1$, and since $(v\mathbb{H})^k \to (v\mathbb{H})$ is an automorphism of $\mathbb{G}/\mathbb{H}$, we can always write $t = v^k$ and therefore $o(t) = p^\alpha$ QED.

Now we show that the cosets $t^k\mathbb{H}$ contain elements in complete classes $\mathscr{C}(t)$ of $\mathbb{G}$ and that either all of them contain only one class or all of them contain two or more classes of $\mathbb{G}$.

In order to prove this, it is convenient to write a generic element of $\mathbb{G}$ as t^kh, $h \in \mathbb{H}$. Since $\mathbb{H}$ is normal in $\mathbb{G}$, we have

$$(t^kh_j)(t^lh)(t^kh_j)^{-1} = t^l(t^{k-l}h_jt^{l-k})(t^khh_j^{-1}t^{-k}) = t^lh', \ h'\in \mathbb{H}.$$

Suppose now that $t^l\mathbb{H}$ contains only one class $\mathscr{C}(t^l)$ of $\mathbb{G}$. In this case, the centralizer of t^l in $\mathbb{G}$, $\mathbb{C}_{\mathbb{G}}(t^l)$ is $\langle t \mid t^p = 1 \rangle$ and since $(l,p) = 1$, it follows that $\mathbb{C}_{\mathbb{G}}(t^l) = \mathbb{C}_{\mathbb{G}}(t^k)$ $\forall$ k,l QED. Then, equation (3) shows that if $\langle t^l\mathbb{H}\rangle = \mathscr{C}(t^l)$, and if $A_1 \in$ irrep($\mathbb{H}$) is not contained in $\Gamma \in$ irrep($\mathbb{G}$), $\chi^\Gamma(t^lh) = 0$ must hold $\forall$ $0 < l < p$, $h \in \mathbb{H}$.

We have recently shown (Nogueira et al 1988) that if $\chi^\Gamma(t^lh) = 0$, the restriction of Γ of $\mathbb{G}$ on $\mathbb{H}$ is the direct sum of p conjugate irreps γ_i of $\mathbb{H}$. Therefore, all the irreps Γ of $\mathbb{G}$ which do not contain A_1 of $\mathbb{H}$ can be calculated by the induction method and those which contain A_1 are given by $\lambda_k(t^lh) = \omega^{kl}$.
The dihedral groups $\mathbb{D}_{2n+1}$ and the tetrahedral group $\mathbb{T}$ are examples of groups with these characteristics.

We are now ready to describe the procedure allowing the calculation of the characters and the irreps of the finite group $\mathbb{G}$ as a function of the irreps of its subgroup $\mathbb{H}$, if $\mathbb{G}/\mathbb{H} \cong \mathbb{C}_p$ and $\mathbb{S}_{\mathbb{G}}(\gamma) = \mathbb{G}$. Our method is based on three fundamental equations which are obtained from the orthogonality relationships for the matrix elements of the irreps. The first fundamental relation is(†),

$$\chi^{\Gamma}(th_1)\chi^{\Gamma}(th_2)^* = [|\gamma|/|\mathscr{C}(th_2)|]\sum \chi^{\gamma}(h_1h_i^{-1}), \tag{4}$$

where the sum is over every h_i such that $th_i \in \mathscr{C}(th_2)$ and Γ runs on the p irreps of $\mathbb{G}$.

As we have shown, there always exists an element $th_a \in (t\mathbb{H})$ such that $\chi^{\Gamma}(th_a) \neq 0$. This allows us to define

$$\xi^{\Gamma}(th)^* = [|\gamma|/|\mathscr{C}(th_a)||\chi^{\Gamma}(th_a)|] \sum_{h_i} \chi^{\gamma}(hh_i^{-1}), \tag{5}$$

where

$$\xi^{\Gamma}(th) = \chi^{\Gamma}(th)\ e^{-i\varphi}, \tag{6}$$

i.e. $\xi^{\Gamma}(th)$ is related to the characters of the elements of the coset $t\mathbb{H}$ by the phase factor

$$e^{i\varphi} = \chi^{\Gamma}(th_a)/|\chi^{\Gamma}(th_a)|. \tag{6'}$$

The second relation is(†)

$$\chi^{\Gamma}(t^{l+1}h_1) = (|\gamma|/|\mathbb{H}|)\sum_{h\in\mathbb{H}} \chi^{\Gamma}(t^{l}h_1h)\ \chi^{\Gamma}(th^{-1}). \tag{7}$$

If we use this equation recurrently starting by $l = 1$, we can define

$$\xi^{\Gamma}(t^{l}h) = \chi^{\Gamma}(t^{l}h)e^{-il\varphi}. \tag{8}$$

Putting this in equation (7), we obtain

$$\xi^{\Gamma}(t^{l+1}h) = (|\gamma|/|\mathbb{H}|) \sum_{h_1\in\mathbb{H}} \xi^{\Gamma}(t^{l}hh_1)\ \xi^{\Gamma}(th_1^{-1}). \tag{9}$$

From equations (5) and (9) we can calculate $\xi^{\Gamma}(t^{l}h)\ \forall\ 0<l<p$. In order to calculate $e^{i\varphi}$ we put $l = p - 1$, $h = (h_oh_a)^{-1}$ in equation (8):

$$e^{ip\varphi} = \xi^{\Gamma}[t^{p-1}(h_oh_a)^{-1}]/\xi^{\Gamma}(th_a)^*,\ h_o = t^{p}. \tag{10}$$

(†) See Appendix A.

This determines $e^{i\varphi}$ up to a factor ω^{m} ($\omega^{p}=1$).

Note that the choice of a particular value of m will only interchange the irreps Γ_k and, therefore, we can calculate the characters of any irrep Γ_k from relations (4)-(8). Then, if χ^{Γ_k} is the character of a particular irrep Γ_k, we have our third relation[†]

$$\Gamma_k(t) = (|\gamma|\omega^{k}/|\mathbb{H}|)\sum_{h} \chi^{\Gamma}(th^{-1})\ \gamma(h) \ . \tag{11}$$

$\mathbb{G}$	$\chi^{\Gamma_{i,\alpha}}$	$\chi^{\Gamma_{k,\beta}}$
$\mathbb{H}$	$\sum_{j=1}^{p} \chi^{\gamma_\alpha}(t^{-j}\ h\ t^{j})$	$\chi^{\gamma_\beta}(h)$
...	0	...
$t^{l}\mathbb{H}$	0	$\omega^{kl}\ \chi^{\Gamma_{k',\beta}}(t^{l}h)$
...	0	...
LITTLE GROUP	$S_{\mathbb{G}}(\gamma_\alpha) = \mathbb{H}$	$S_{\mathbb{G}}(\gamma_\beta) = \mathbb{G}$
IRREPS	$\Gamma_{i,\alpha}(t)_{m,m+1} = \gamma_\alpha(1)$, $0<m<p$; $\Gamma_{i,\alpha}(t)_{p,1} = \gamma_\alpha(t^{p})$	$\Gamma_{k,\beta}(t) = \frac{\lvert\gamma_\beta\rvert}{\lvert\mathbb{H}\rvert}\sum_{h\in\mathbb{H}} \chi^{\Gamma_{k,\beta}}(th^{-1})\gamma_\beta(h)$

Table I: Character table and irreps of $\mathbb{G}$.

α and β label the irreps of $\mathbb{H}$, k' is a fixed value of k and p is a prime number such that $\omega^{p}=1$.

Table I shows the characters of the irreps of the group $\mathbb{G}$

calculated by this method, together with the correspondent stabilizers.

III. THE IRREPS OF THE OCTAHEDRAL GROUP

As an illustration we calculate the irreps of $\mathbb{O}$. This group is of interest by several reasons :

i) It is one of the better known groups among the spectroscopists and as it has only 24 elements its subgroups structure is very simple.

ii) Since the stabilizer of the irreps $B_i(i=1,2,3)$ of $\mathbb{D}_2$ in $\mathbb{O}$ is $S_{\mathbb{O}}(B_i) = \mathbb{D}_4$, in order to obtain the irreps of $\mathbb{O}$, we must search for the allowable representation of $\mathbb{D}_4$. Therefore, this is a typical example where the stabilizer is such that $\mathbb{H} \subsetneq S_{\mathbb{G}} \subsetneq \mathbb{G}$.

iii) The group $\mathbb{O}$ is a monomial group (Robinson 1982), i.e. all the representations can be induced from the one-dimensional irreps of the subgroups. This makes easy the calculation using alternative methods.

$\mathbb{D}_2$	1	C_2^z	C_2^y	C_2^x	$S_{\mathbb{O}}(\gamma)$	$S_{\mathbb{T}}(\gamma)$
A_1	1	1	1	1	$\mathbb{O}$	$\mathbb{T}$
B_1	1	1	-1	-1	$\mathbb{D}_4\langle C_4^z, C_2^y\rangle$	$\mathbb{D}_2$
B_2	1	-1	1	-1	$\mathbb{D}_4\langle C_4^y, C_2^z\rangle$	$\mathbb{D}_2$
B_3	1	-1	-1	1	$\mathbb{D}_4\langle C_4^x, C_2^y\rangle$	$\mathbb{D}_2$

Table II: Characters of the irreps of $\mathbb{D}_2$ and their stabilizers.

First we obtain the irreps from the traditional induction method. Table II shows the character table of the group $\mathbb{D}_2$ and the stabilizers of the irreps. Using this table, we calculate the characters of the induced representation $A_1\uparrow \mathbb{O}$:

$$\chi^{A_1\uparrow\mathbb{O}}(1) = \chi^{A_1\uparrow\mathbb{O}}(3C_2) = 6 \;,\; \chi^{A_1\uparrow\mathbb{O}}(g) = 0 \quad \forall \quad g \in \{\mathbb{O} - \mathbb{D}_2\}.$$

From the characters of $\mathbb{O}$, it can be deduced that $A_1\uparrow \mathbb{O} = A_1 \oplus A_2 \oplus 2E$. It should be noted that this illustrates the case in which the induction procedure breaks down, i.e. for those irreps of $\mathbb{G}$ induced by allowable representations of a stabilizer which is the same as $\mathbb{G}$ itself. In order to calculate T_1 and T_2 from B_1 of $\mathbb{D}_2$ we need to know the allowable irreps Γ_i of $\mathbb{D}_4$ which subduce B_1, i.e. $\Gamma_i\downarrow \mathbb{D}_2 = B_1$. From a character table of $\mathbb{D}_4$ (see Table III) it is easy to see that the allowable representations are A_2 and B_2, and they give the induced representations T_1 and T_2 of $\mathbb{O}$ in an irreducible form.

$\mathbb{D}_4$	$\mathbb{D}_2$			$\alpha\,\mathbb{D}_2$	
	1	C_2^z	C_2^x, C_2^y	$2C_2'$	$2C_4$
A_1	1	1	1	1	1
A_2	1	1	-1	-1	1
B_1	1	1	1	-1	-1
B_2	1	1	-1	1	-1
E	2	-2	0	0	0

Table III: Characters and irreps of $\mathbb{D}_4$
$(\alpha = C_2^{x(-y)})$

Now, using that

$$\mathbb{O} = \mathbb{D}_4 \oplus \beta\,\mathbb{D}_4 \oplus \beta^2\,\mathbb{D}_4 \;, \quad \mathbb{D}_4 = \mathbb{D}_2 \oplus \alpha\,\mathbb{D}_2 \;,$$

where $\alpha = C_2^{x(-y)}$, $\beta = C_3^{xyz}$ with $\beta\alpha\beta = \alpha$,we obtain for T_1 of $\mathbb{O}$,

$$T_1(\alpha) = -\begin{Bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{Bmatrix} \qquad T_1(\beta) = \begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{Bmatrix} \;,$$

$$T_1(h) = \begin{Bmatrix} B_2(h) & 0 & 0 \\ 0 & B_3(h) & 0 \\ 0 & 0 & B_1(h) \end{Bmatrix} \ , \ B_i(h) \in \text{irreps}(\mathbb{D}_2) \ .$$

Finally, if A_2 is the one-dimensional irrep of $\mathbb{O}$ with Kernel $\mathbb{T}$, it must be $T_2 = A_2\, T_1$. Then,

$$T_2(\alpha) = -T_1(\alpha) \ , \ T_2(\beta) = T_1(\beta) \ , \ T_2(h) = T_1(h) \quad \forall\, h \in \mathbb{D}_2 \ .$$

In order to calculate the character table and the irreps of $\mathbb{O}$ by our method, we first make the calculations for the group $\mathbb{T}$, starting with the irreps of $\mathbb{D}_2$ and after that we obtain the irreps and characters of $\mathbb{O}$ from those of $\mathbb{T}$. Since the tetrahedral group is $\mathbb{T} = \mathbb{D}_2 \oplus \beta\, \mathbb{D}_2 \oplus \beta^2\, \mathbb{D}_2$, where $\beta\, \mathbb{D}_2 = \mathscr{C}(\beta)$, the character table for $\mathbb{T}$ can be calculated in the following form. From equation (4) we obtain $\chi^{\lambda}(th) = e^{i\varphi}$, where λ is one of the one-dimensional irreps λ_k of $\mathbb{T}$. Using equation (7) we have $\chi^{\lambda}(t^{\iota}h) = e^{i\iota\varphi}$. But if we set $\iota = 3$, we see that $e^{3i\varphi} = 1$. Then, since the stabilizer of the irreps of $\mathbb{D}_2$ in $\mathbb{T}$ is $\mathbb{T}$ itself only for $\gamma = A_1$, there are only three one-dimensional irreps for $\mathbb{T}$. Finally, since the irreps B_i of $\mathbb{D}_2$ are conjugate representations, the other irrep of $\mathbb{T}$ is three-dimensional, and its characters can be calculated from

$$\chi^{T}(h) = \sum_{i=1}^{3} \chi^{B_i}(h) \ , \ \chi^{T}(g) = 0 \quad \forall\, g \in \{\mathbb{T} - \mathbb{D}_2\}.$$

$\mathbb{T}$	$\mathbb{D}_2$ 1	$3C_2^z$	$\beta\mathbb{D}_2$ $4C_3$	$\beta^2\mathbb{D}_2$ $4C_3^2$	$S_{\mathbb{O}}(\gamma)$
A_1	1	1	1	1	$\mathbb{O}$
E_1	1	1	ω	ω^2	$\mathbb{T}$
E_2	1	1	ω^2	ω	$\mathbb{T}$
T	3	−1	0	0	$\mathbb{O}$

Table IV : Characters and irreps of $\mathbb{T}$ and their stabilizers

($\beta = C_3^{xyz}$, $\omega = \exp(2\pi i/3)$)

The result is given in Table IV, which also shows the stabilizers of the irreps of $\mathbb{T}$ in $\mathbb{O}$. The representation T of $\mathbb{T}$ is given using equation (1) by

$$T(\beta) = \begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{Bmatrix},\ T(h) = \begin{Bmatrix} B_2(h) & 0 & 0 \\ 0 & B_3(h) & 0 \\ 0 & 0 & B_1(h) \end{Bmatrix},\ B_i(h) \in \text{irreps}(\mathbb{D}_2).$$

$\mathbb{O}$	$\mathbb{T}$			$\alpha\mathbb{T}$	
	1	$3C_2$	$8C_3$	$6C_2'$	$6C_4$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
E	2	2	−1	0	0
T_1	3	−1	0	−1	1
T_2	3	−1	0	1	−1

Table V: Characters and irreps of $\mathbb{O}$
$(\alpha = C_2^{x(-y)})$

We must now calculate the table of characters of $\mathbb{O}$ and its irreps E, T_1 and T_2. Using table I, equations (4) and (7), and knowing that $\mathbb{O} = \mathbb{T} \oplus \alpha\,\mathbb{T}$, where the coset $\alpha\,\mathbb{T}$ contains the ambivalent classes $6C_2'$ and $6C_4$, we obtain Table V. This table shows the characters of the irreps of $\mathbb{O}$ and, in order to coincide with Mulliken's nomenclature, we have taken $\chi^{T_1}(6C_4) = 1$.

Now, using equation (1), it immediately follows

$$E(\alpha) = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix},\quad E(h) = \begin{Bmatrix} E_2(h) & 0 \\ 0 & E_1(h) \end{Bmatrix},\ E_i(h) \in \text{irreps}(\mathbb{T}).$$

Finally, we calculate T_1 and T_2 using equation (11). The elements of $\mathbb{O}$ for the correspondent classes are

$$6C_2' = \alpha \{ 1, C_2^z, \beta(1 , C_2^x), \beta(1 ,C_2^y) \}$$
$$6C_4 = \alpha \{ 1, C_2^z, \beta(C_2^y , C_2^z), \beta(1 ,C_2^y) \} C_2^x .$$

If we put the expression of $T \in \text{irrep}(\mathbb{T})$ obtained from $\mathbb{D}_2$ in equation (11), we have the following result for the element α, which is in accordance with that obtained by the induction method,

$$T_1(\alpha) = - \begin{Bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{Bmatrix} \quad \text{and} \quad T_2 = A_2 T_1.$$

IV. CONCLUSIONS

As we have mentioned for $\mathbb{O}$, there is a subclass of solvable groups, the monomial $\mathbb{M}$-groups, for which the traditional method of induction gives all the irreps of order greater than one. These groups have subgroups $\mathbb{H}_i$ such that, every $\Gamma \in \text{irrep}(\mathbb{M})$ can be induced from a one-dimensional irrep of $\mathbb{H}_i$ and its dimension is given by $|\Gamma| = |\mathbb{M}:\mathbb{H}_i|$. However, in problems where we need the symmetry adapted irreps to a chain of maximal subgroups, the irreps obtained by that method are not convenient since they are adapted to a different sequence given by $\mathbb{G} \triangleright \mathbb{H}_i^{Core}$, with $\mathbb{H}^{Core} = \bigcap g \, \mathbb{H} \, g^{-1} \; \forall \; g \in \mathbb{G}$. The octahedral group $\mathbb{O}$, which we took as example, is monomial and if we calculate the irreps using that property we will find the irrep E adapted to the sequence $\mathbb{O} \triangleright \mathbb{T}$ and T_1 and T_2 adapted to $\mathbb{O} \triangleright \mathbb{D}_4^{Core} = \mathbb{D}_2$. A further adaptation of T_1 and T_2 to the sequence $\mathbb{O} \triangleright \mathbb{T} \triangleright \mathbb{D}_2$, would destroy the monomial form.

It is important to bear in mind that if we have a solvable group $\mathbb{G} = \mathbb{G}_n$, the symmetry adapted irreps to the composition series are completely determined, given only the matrices

$$\delta_{\mu_n,\mu_n'} \Gamma_{\mathbb{G}_n}(t_i)_{\mu,\mu'} = \left(\prod_{j=n}^{i} \delta_{\mu_j,\mu_j'} \right) \Gamma_{\mathbb{G}_k}(t_i)_{\nu,\nu'} \; , \; i \le k \qquad (12)$$

where $\mu = \mu_n \cdots \mu_i \cdots \mu_1$, $\nu = \mu_k \cdots \mu_i \cdots \mu_1$ and μ_k numbers the irreps of the subgroup $\mathbb{G}_k$ in the composition series of $\mathbb{G}_n$.

This simplification is due to the fact that, since the cosets $t_i^k\, \mathbb{G}_{i-1}$ are disjoint, every element $g \in \mathbb{G}_n$ can be written in the form of a product

$$g = t_n^{k_n} \cdots\ t_i^{k_i} \cdots\ t_1^{k_1},$$

of powers of transversals t_n , where $0 \le k_i < p_i$. Then, equation (12) not only allows a drastic simplification in the expression of the irreps of $\mathbb{G}_n$, but at the same time shows the descent in symmetry of the irreps of the subgroups that occur in the composition series of $\mathbb{G}_n$.

Finally, our method is also applicable to every finite group for which the irreps of the tail group in its derived series are known. This is so because every finite group has a derived series that can be refined to another in a way that the factor groups in the chain are cyclic groups of prime order. Examples of this might be, for instance, those groups with derived series ending in simple groups as is the case of $\mathbb{A}_n$ for $n \ge 5$.

APPENDIX A

Let $\mathbb{G}$ be a finite group with an invariant subgroup $\mathbb{H}$ such that $\mathbb{G}/\mathbb{H} \cong \mathbb{C}_p = \langle t\mathbb{H} \mid (t\mathbb{H})^p = \mathbb{H} \rangle$. An element $g \in \mathbb{G}$ can always be written as $t^k h_i$ with $0 \leq k < p$, $t^p = h_o \in \mathbb{H}$ and $h_i \in \mathbb{H}$. Since for different values of k, the cosets $t^k\mathbb{H}$ are disjoint, $t^k h_i$ covers all the elements of $\mathbb{G}$ ($t^k h_i \neq t^l h_j$ if $k \neq l$ or $h_i \neq h_j$).

If $\Gamma \in$ irreps($\mathbb{G}$) and $\Gamma(h) = \gamma(h)$, $h \in \mathbb{H}$ and $\gamma \in$ irreps($\mathbb{H}$), from the orthogonality relationships for $\gamma(h)_{\alpha\beta}$ matrix elements, it is quite easy to see that

$$\Gamma(t^l h_1)_{\alpha\tau}\ \Gamma(h_3 t^m)_{\sigma\beta} = (|\Gamma|/|\mathbb{H}|) \sum_{h\in\mathbb{H}} \Gamma(t^l h_1 h)_{\alpha\beta}\ \Gamma(h_3 t^m h^{-1})_{\sigma\tau} \qquad .(A)$$

Making some substitutions in this relation we can obtain equations (4), (7) and (11) of section II.

Setting $\tau = \alpha$, $\sigma = \beta$, $h_3 = h_2^{-1}$, $l = 1$, $m = -1$ and summing over α and β, equation (A) becomes,

$$\chi^\Gamma(th_1)\ \chi^\Gamma(th_2)^* = (|\Gamma|/|\mathbb{H}|) \sum_{h\in\mathbb{H}} \chi^\Gamma[th_1 h(th_2)^{-1}h^{-1}]$$

But, since $hth_2h^{-1} = th_i$, and remembering that $\chi^\Gamma(h) = \chi^\gamma(h)$, we obtain equation (4),

$$\chi^\Gamma(th_1)\ \chi^\Gamma(th_2)^* = [\,|\Gamma|/|\mathbb{C}(th_2)|\,] \sum_i \chi^\Gamma(h_1 h_i^{-1}) \ ,\ th_i \in \mathbb{C}(th_2) \ .$$

Now, if we set $\beta = \alpha$, $\tau = \sigma$, $h_3 = 1$, $m = 1$ and sum over α and β in (A), we directly obtain equation (7)

$$\chi^\Gamma(t^{l+1}h_1) = (|\Gamma|/|\mathbb{H}|) \sum_{h\in\mathbb{H}} \chi^\Gamma(t^l h_1 h)\ \chi^\Gamma(th^{-1}) \ .$$

Finally, setting $m = p$, $h_3 = t^{-p}$, $\beta = \alpha$ and summing over α in (A), we obtain equation (11)

$$\Gamma(t^l h_1)_{\sigma\tau} = (|\Gamma|/|\mathbb{H}|) \sum_{h\in\mathbb{H}} \chi^{\Gamma_k}(t^l h_1 h)\ \gamma(h^{-1})_{\sigma\tau} \ .$$

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