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Dynamical system approach to generalized $su(2)$ algebras

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Abstract

We present an iterative algebraic formalism that contains the $su(2)$ algebra as a particular case. This algebra is written in terms of a functional of one generator of the algebra, $f(J_0)$, that can be any analytical function. We construct the representation of the algebra for $f(J_0) = rJ_0 - s$ and build a map connecting this generalization with $su_q(2)$ algebra. The algebra generated by the non-linear (quadratic) function f is investigated as well. We study the representations of this algebra through the dependence of these representations on the fixed points of the function f, and their stability properties.

Key-words: $su(2)$ algebra; dynamical systems; attractors; quantum algebras; $su_q(2)$ algebra.

1 Introduction

Recently, there has been intense activity in the study of deformed algebras. These algebras recover known algebras when a parameter goes to a specific value [1]. Deformed theories have been largely studied in both physics and mathematics with success [1, 2]. In physics one could interpret, for instance, special relativity and quantum mechanics as very successful deformations ($c \to \infty$ and $\hbar \to 0$ recovers classical physics). In the last years there has been an increasing interest in the generalized statistical mechanics, which is a deformation of the Boltzmann-Gibbs statistics [3].

Last year, a class of generalized Heisenberg algebras was introduced where, within this class, we find deformed and non-deformed Heisenberg-type algebras [4, 5]. Associated with this class of algebras there is a function, f, of one of the generators called the characteristic function of the algebra. When f is linear this algebra becomes the well-known q-oscillators [6]. Moreover, it was shown that this class describes the Heisenberg-type algebraic structure of a family of one-dimensional quantum systems having any two successive energy levels related by $\epsilon_{n+1} = f(\epsilon_n)$, where f is the above mentioned characteristic function of the algebra [7]. Among the physical possible consequences of this family of Heisenberg-type algebras, it was shown that it is possible to construct a non-standard free quantum field theory based on this algebra [8]. Also, the representations of the algebra are constructed using concepts of dynamical systems as attractors and their stabilities [5].

Stimulated by the physical connection of the generalized Heisenberg algebra and its algebraic richness, it was very recently introduced an analogous iterative generalization of $su(2)$ [9]. In this case there is also a characteristic function of one of the generators of the algebra such that when $f(x) = x - 1$ the well-known $su(2)$ algebra is recovered.

In this paper we apply this new approach to $su(2)$, i.e., we construct an iterative generalized $su(2)$ algebra and discuss their representations. These representations are constructed through their dependence on the fixed points of the function f, and their stability properties. We construct a map connecting the linear case of this generalized $su(2)$ algebra with $su_q(2)$ and discuss some aspects of the generalized algebra when f is non-linear.

In section II we introduce the iterative generalized $su(2)$ algebra and study its representation theory. In section III we establish the connection of this algebra with the $su(2)$ and $su_q(2)$ algebraic structures. In section IV we study the algebra for non-linear functions f through their dependence on the fixed points of the function f, and their stability properties. We construct the finite dimensional representations associated with the cycles and show the regions in the parameter space of the algebra where possibly these representations occur. Section V is devoted to the conclusions.

2 Iterative generalized $su(2)$ algebra

Let us propose the following algebraic relations among the operators J_0, J_+ and J_-:

$$J_0\, J_- \;=\; J_-\, f(J_0)\;, \tag{1}$$
$$J_+\, J_0 \;=\; f(J_0)\, J_+\;, \tag{2}$$
$$[J_+, J_-] \;=\; J_0(J_0+1) - f(J_0)(f(J_0)+1)\quad , \tag{3}$$

where we are assuming $J_- = J_+^\dagger$, $J_0^\dagger = J_0$ and that $f(J_0)$ is an analytical function in J_0.

This algebra satisfies, for all function f, the Jacobi identity

$$[J_0, [J_+, J_-]] + [J_-, [J_0, J_+]] + [J_+, [J_-, J_0]] = 0\,. \tag{4}$$

The first term of the L.H.S., due to eq. (3), is identically null. To show that the sum of the other two terms is equal to zero it is enough to expand them and use the property, derived from eqs. 1 and 2 , that $[J_0, J_+\, J_-] = 0$.

Using the algebraic relations in eqs. (1-3) it can be shown that the operator

$$C = 1/2\ \{J_+\, J_- + J_-\, J_+ + J_0(J_0+1) + f(J_0)(f(J_0)+1)\} \tag{5}$$

satisfies the commutation relations

$$[C, J_0] = [C, J_\pm] = 0, \tag{6}$$

i.e., C is one Casimir operator of the algebra.

If we substitute the specific function

$$f(J_0) = J_0 - 1\,, \tag{7}$$

in eqs. (1-3), these relations can be written as,

$$[J_0, J_\pm] \;=\; \pm J_\pm \tag{8}$$
$$[J_+, J_-] \;=\; 2J_0\,, \tag{9}$$

reobtaining the well-known $su(2)$ algebra. Thus, the algebraic relations proposed in eqs. (1-3) contain, as a particular case, the $su(2)$ algebra when we choose a specific linear function of J_0. To discuss this algebra and the role of the function f in it, we analyse its respective representation theory.

Let us assume that we have an irreducible representation of the algebra given in eqs. (1-3) such that the Hermitian operator J_0 is diagonal. As C comutes with J_0, the eigenvectors of J_0 are also eigenvectors of C. Let us label these eigenvectors as $|a, \alpha\rangle$, where

$$J_0|a, \alpha\rangle \;=\; \alpha|a, \alpha\rangle \tag{10}$$
$$C|a, \alpha\rangle \;=\; a|a, \alpha\rangle\,, \tag{11}$$

being a and α real numbers.

To examine the action of $J_\pm$ on these states, let us consider $|a, \alpha_i\rangle$ a normalized eigenstate of J_0 with respective eigenvalue α_i, i.e.,

$$J_0\,|a, \alpha_i\rangle = \alpha_i\,|a, \alpha_i\rangle\,, \tag{12}$$

where i is an integer.

Applying eq. (1) to $|a, \alpha_i\rangle$ we have

$$J_0\,(J_-\,|a, \alpha_i\rangle) = J_- f(J_0)|a, \alpha_i\rangle = f(\alpha_i)(J_-|a, \alpha_i\rangle)\,. \tag{13}$$

Thus, we see that $J_-\,|a, \alpha_i\rangle$ is also a J_0 eigenvector with eigenvalue $f(\alpha_i)$ that we call from now on α_{i-1}. Starting from $|a, \alpha_i\rangle$ and applying successively J_- to $|a, b_i\rangle$ we create different states with J_0 eigenvalues given by

$$J_0\left(J_-^m|a, \alpha_i\rangle\right) = f^m(\alpha_i)\left(J_-^m|a, \alpha_i\rangle\right) \propto \alpha_{i-m}|a, \alpha_{i-m}\rangle\,, \tag{14}$$

where $f^m(\alpha_i)$ denotes the m-th iterate of α_i through f. We will consider from now on only functions and initial values such that future iterations of these initial values are always lower than the previous values. This means that the function f and initial value we are assuming here satisfy $\alpha_j > f(\alpha_j) > f(f(\alpha_j)) > \ldots > f^m(\alpha_j) > \ldots$, where f^m means the m-th iterate of α_j through f and m is a positive integer. In a general case, however, the only necessary condition on the function f is that any future iteration of the highest weight, let us call it α_j, is lower than itself.

Since the application of J_- on $|\alpha_i\rangle$ creates a new vector, whose respective J_0 eigenvalue has iterations of α_i through f increased by one unit, but lower in absolute value, it is convenient to define the new vectors $J_-^m|a, \alpha_i\rangle$ as proportional to $|a, \alpha_{i-m}\rangle$, as was done in eq. (14), and we then call J_- a lowering operator. Note that

$$\alpha_{i-m} = f^m(\alpha_i) = f(\alpha_{i-m+1})\,, \tag{15}$$

where m denotes the number of iterations of α_i through f. Notice that the structure of the algebraic relations in eqs. (1, 2) yields the iterative eq. (15) among the eigenvalues of J_0; that is why we call iterative this algebraic structure.

Following the same procedure for J_+, applying eq. (2) to $|a, \alpha_i\rangle$, we have

$$\begin{aligned} J_+ J_0|a, \alpha_i\rangle &= \alpha_i\,(J_+|a, \alpha_i\rangle) \\ &= f(\alpha_{i+1})\,(J_+|a, \alpha_i\rangle) \\ &= f(J_0)\,(J_+|a, \alpha_i\rangle)\,, \end{aligned} \tag{16}$$

implying that $J_+|a, \alpha_i\rangle$ is also a J_0 eigenvector with eigenvalue α_{i+1}. Then, $J_+|a, \alpha_i\rangle$ is proportional to $|a, \alpha_{i+1}\rangle$ showing that J_+ is a raising operator.

Applying J_+ to $|a, \alpha_i\rangle$ we will get states with increasing J_0 eigenvalues. For a finite dimensional representation of the algebra given by the eqs. (1-3), there will be a highest weight α_j such that

$$J_+|a,\alpha_j\rangle = 0\,. \tag{17}$$

Clearly then, $J_-J_+|a,\alpha_j\rangle = 0$. From eq. (3) we have that $J_+J_- = J_-J_+ + J_0(J_0+1) - f(J_0)(f(J_0)+1)$. Substituting this expression in eq. (5), we see that C can be rewritten as

$$C = J_-J_+ + J_0(J_0+1)\,. \tag{18}$$

Using eq. (18) we have

$$J_-J_+|a,\alpha_j\rangle = (C - J_0(J_0+1))|a,\alpha_j\rangle = 0\,. \tag{19}$$

As $C|a,\alpha_j\rangle = a|a,\alpha_j\rangle$, it follows that

$$a = \alpha_j(\alpha_j+1)\,, \tag{20}$$

showing that if we write the eigenvalue of C as equal to $\alpha_j(\alpha_j+1)$ then α_j will be the highest possible value for J_0. Note that the Casimir of this generalized algebra presents the same formal expression as the Casimir of the angular momentum algebra.

By an analogous reasoning, for a finite dimension representation there is also a lowest J_0 eigenvalue, let us call it α_b, such that it satisfies

$$J_-|a,\alpha_b\rangle = 0\,, \tag{21}$$

implying that $J_+J_-|a,\alpha_b\rangle = 0$. Using a procedure similar to that developed just above giving eq. (19), we get

$$J_+J_-|a,\alpha_b\rangle = (C - f(J_0)(f(J_0)+1)|a,\alpha_b\rangle = 0\,, \tag{22}$$

leading to the relation

$$a = f(\alpha_b)(f(\alpha_b)+1)\,. \tag{23}$$

Comparing eq. (20) with eq. (23) we get that

$$f(\alpha_b) = -\alpha_j - 1\,, \tag{24}$$

and the allowed values of α_i satisfy

$$\alpha_j \geq \alpha_i \geq \alpha_b\,(= f^{(-1)}(-\alpha_j - 1))\,. \tag{25}$$

If we start from $|a,\alpha_b\rangle$ and apply J_+ to it we will reach $|a,\alpha_j\rangle$ after $d-1$ steps, where d is the dimension of the representation, a positive integer number. We then have

$$\alpha_b = f^{d-1}(\alpha_j)\,. \tag{26}$$

As the eigenvalue a is related to α_j, we will denote in the following the orthonormal eigenvectors $|a,\alpha_{j-m}\rangle \equiv |\alpha_j, j-m\rangle$ for simplicity and for easier comparison with standard

$su(2)$ notation. We also call, without loss of generality, the dimension $d = 2j + 1$. Note that $2j$ is an integer, associated with the number of iterations ($2j$) of α_j through f until to reach α_b and α_j is, in this general case, a real number. Therefore, there is a hidden symmetry in the sequence of eigenvalues of this generalized $su(2)$ algebra.

In general, we obtain the following expressions for a general m lying between 0 and $2j$:

$$J_0|\alpha_j, j-m\rangle = \alpha_{j-m}|\alpha_j, j-m\rangle \tag{27}$$

$$J_+|\alpha_j, j-m\rangle = N_{m-1}|\alpha_j, j-m+1\rangle \tag{28}$$

$$J_-|\alpha_j, j-m\rangle = N_m|\alpha_j, j-m-1\rangle \tag{29}$$

$$C|\alpha_j, j-m\rangle = \alpha_j(\alpha_j+1)|\alpha_j, j-m\rangle\,, \tag{30}$$

where $N_m^2 = (\alpha_j - \alpha_{j-m-1})(\alpha_j + \alpha_{j-m-1} + 1) = \alpha_j(\alpha_j+1) - \alpha_{j-m-1}(\alpha_{j-m-1}+1)$ and that can be proved in a similar way to the proof made for the generalized Heisenberg algebra in [5]. We observe that if we put $m = 0$ in eq. (28) then $N_{-1} \equiv 0$, which is consistent with eq. (17). Also, if we put $m = d-1$ in eq. (29) we should have $N_{d-1} = 0$, giving us a cut condition on the eigenvalues, i.e,

$$\alpha_j + \alpha_{j-d} + 1 = 0\,. \tag{31}$$

The eq. (31) also gives us the condition the eigenvalues of C should satisfy. The only α_j values that allow us to have finite dimension representations are those that satisfy eq. (31), i.e., after some finite number of iterations (say d), the d-th iteration of α_j through f gives exactly $-\alpha_j - 1$. Otherwise, the dimension of the representation will be infinite. This condition gives us the set of allowed values of α_j; this set consists in general of an infinitely enumerated number of values. Also, as $\alpha_j \geq \alpha_{j-d}$, from eq. (31) we have

$$\alpha_j \geq -\alpha_j - 1 \Rightarrow \alpha_j \geq -1/2\,, \tag{32}$$

a result that is valid for any function f and α_j satisfying eq. (25). Clearly, once the value of α_j is chosen, this value will be the highest value of the operator J_0 in the associated finite dimension representation.

Let us now analyse the algebraic relations given by eqs. (1-3) when the function f is a linear one.

3 Linear functions

3.1 $su(2)$

Let us first study the $su(2)$ case where $f(J_0) = J_0 - 1$ implying that $f(\alpha) = \alpha - 1$. As we have seen, with this function we reproduce the commutation relations of the $su(2)$ algebra. It is straightfoward to verify that a general eigenvalue α_{j-m} can be written as:

$$\alpha_{j-m} = f^m(\alpha_j) = \alpha_j - m\,. \tag{33}$$

Table 1: (C, J_0) and (J^2, J_z) similarities

C	J^2 ($su(2)$ algebra)
$[C, J_0] = 0$	$[J^2, J_z] = 0$
$C\|\alpha_j, j-m\rangle = \alpha_j(\alpha_j+1)\|\alpha_j, j-m\rangle$	$J^2\|j,m\rangle = j(j+1)\|j,m\rangle$
$J_0\|\alpha_j, j-m\rangle = \alpha_{j-m}\|\alpha_j, j-m\rangle$	$J_z\|j,m\rangle = m\|j,m\rangle$
α_j (real) is the highest J_0 eigenvalue	j (semi-integer) is the highest J_z eigenvalue
$C =$	J^2 (if $f(J_0) = J_0 - 1$)

Using eq. (33) in eq. (31) and remembering that $d = 2j+1$, i.e., $\alpha_{j-d} = \alpha_j - (2j+1)$ we have

$$\alpha_j + (\alpha_j - 2j - 1) + 1 = 2\alpha_j - 2j = 0\,, \tag{34}$$

implying that $\alpha_j \equiv j$. As $2j+1$ is an integer, this means that j and consequently α_j can be an integer or a semi-integer, as it is well-known. Also, from eq. (26), we see that the lowest eigenvalue is $\alpha_b = f^{2j}(\alpha_j) = \alpha_j - 2j = -j$, and the eigenstates can be written as $|j, j-m\rangle$, where m goes from zero to $2j$.

The operator C turns out to be

$$C_{su(2)} = (1/2)\{J_+J_- + J_-J_+\} + J_0^2\,, \tag{35}$$

which is exactly the expression for the squared total angular momentum operator $J^2 = J_x^2 + J_y^2 + J_z^2$ if we write $J_+ = J_x + iJ_y$, $J_- = J_x - iJ_y$ and $J_z^2 = J_0^2$. It can be imediately seen that the eq. (20) can be written now in the usual way $j(j+1)$. It is interesting to note that this form is preserved even for the general case where α_j is real. There are interesting analogies between the operator J^2 of the angular momentum algebra and our C operator that is shown in table I. This suggest that this generalized algebra could describe, for at least some f, a generalized angular momemtum algebra.

In general, in order to see the iterations through a graphical analysis of the function f we graph $y = f(x)$ together with $y = x$. Where the lines intersect we have $x = y = f(x)$, so that the intersections are precisely the fixed points. Now, for a point x_0, different from the fixed point, in order to follow its path through iterations with the function f we perform the following steps

1. move vertically to the graph of $f(x)$,

2. move horizontally to the graph of $y = x$, and

3. repeat steps 1, 2, etc.

A graphical representation of the $su(2)$ algebra can be seen in fig. (1) where we plot the function $f(\alpha) = \alpha - 1$ versus α for $\alpha_j = j = 2$. We also plot the vertical line representing the cut condition given by eq. (31), i.e., $\alpha_{j-d} = -\alpha_j - 1 = -j - 1 = -3$.

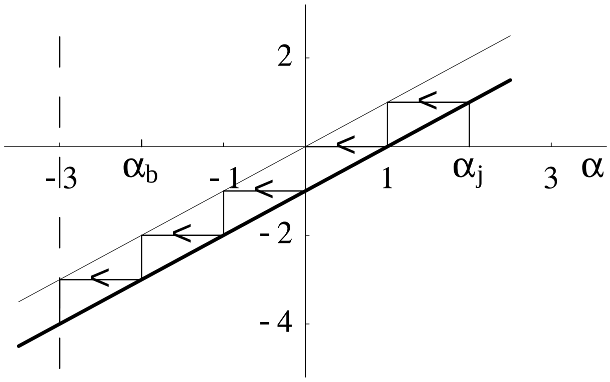

Figure 1: Five dimensional representation of the $su(2)$ algebra for $\alpha_j = 2$. The dashed line ($\alpha_{j-d} = -3$) is the cut condition. The iterations are indicated by the arrows. The thinner line is the function $f(\alpha) = \alpha$ and the thicker one is the function $f(\alpha) = \alpha - 1$.

We see that the iterations of j through f reach exactly the intersection of the vertical line given by the cut condition and the function itself. If the starting point α_j is not an integer or semi-integer, the future iterations of this value will never reach the intersection with the cut line and the iterations will evolve forever. There is no lower bound. The dimension of this representation is infinite. Note, also, that there is no fixed point in this case. In this way, a graphical analysis of the function f gives us a quickly and very useful information about the representation of the algebra, without need to use extensive calculation. It is also easy to see in this graph that the eigenvalues always decrease in absolute value when the iterations of the function f increase.

3.2 Linear deformations of $su(2)$

Let us consider the function f as

$$f = rJ_0 - s\,, \tag{36}$$

where r and s are real numbers. The eqs. (1-3) can be written as:

$$[J_0, J_-]_r = -sJ_- \tag{37}$$

$$[J_0, J_+]_{r^{-1}} = (s/r)J_+ \tag{38}$$

$$[J_+, J_-] = (1-r^2)J_0^2 + (1+2rs-r)J_0 - s(1-s)\,, \tag{39}$$

where $[A, B]_r = AB - rBA$ is the r-deformed commutation of two operators A and B.

There are three important cases to be analysed: (I) $r = 1$ and $s > 0$; $s \neq 1$, (II) $r > 1$ and (III) $|r| < 1$. In the first case we consider only positive (unlike $s = 1$) values of s because negative values of s will not satisfy the condition given by eq. (25). The r-commutator in eqs. (37-39) turns out to be real commutators and the eigenvalues of J_0 will be:

$$\alpha_{j-m} = \alpha_j - ms\,. \tag{40}$$

The cut condition given by eq. (31) leads to the following value of α_j once the dimension (d) of the representation is chosen:

$$\alpha_j = \frac{sd-1}{2}\,. \tag{41}$$

For a fixed value of s and for each different dimension of the representation we want, we have a different initial value allowed. As s can be any positive real number ($\neq 1$), this implies that α_j can also be a real number. Note that the transformed operators $\tilde{J}_\pm = J_\pm/s$ and $\tilde{J}_0 = J_0/s + (s-1)/(2s^2)$ obey the $su(2)$ algebra, while $J_\pm$ and J_0 satisfy the algebra given by eqs. (37-39) for $r = 1$ and $s > 0$; $s \neq 1$. The graphical representation of this case is similar to the $su(2)$ case, with constant spacing s between the eigenvalues. The generalized Casimir operator for this case is

$$C_s = (1/2)\{J_+J_- + J_-J_+ + 2J_0^2 + (1-s)J_0 + s(s-1)\}\,. \tag{42}$$

In case (II), $r > 1$ and thus exists a fixed point α^* where the function f crosses the diagonal $y = \alpha$,

$$\alpha^* = s/(r-1)\,. \tag{43}$$

This fixed point is unstable, $((\partial f/\partial\alpha)|_{\alpha^*} = r > 1)$, showing that only points below α^* are allowed if we obey the condition given by eq. (25). Also, eq. (32) shows that $\alpha_j \geq -1/2$, a necessary condition to have $\alpha_j \geq \alpha_{j-d}$. Then, s and r should satisfy the inequality $\alpha^* \geq -1/2$ and α_j lies within $-1/2 < \alpha_j < \alpha^*$. But even in this interval only those values of α_j that satisfies eq. (31) are allowed. The eigenvalues of J_0 can be written as:

$$\alpha_{j-m} = f^m(\alpha_j) = r^m\alpha_j - s\,[m]_r\ , \tag{44}$$

where $[m]_r \equiv (r^m-1)/(r-1)$ is the Gauss number. The cut condition given by eq. (31) and the eq. (44) yield us an expression for α_j once the function f has been given (this means that r and s are given) and the dimension of the representation is chosen. We have for α_j:

$$\alpha_j = (s[d]_r - 1)/(r^d + 1)\,. \tag{45}$$

For each dimension we want, we have a different starting point, generally a real number. In fig. (2) we show an example of this case.

There is also a marginal two dimensional representation for $r = -1$. The case $r < -1$ is not allowed because it is not possible to obtain a highest weight representation.

In case (III), there is also a fixed point with the same formal expression for α^*, but in this case $|r| < 1$. This fixed point is stable $(\partial f/\partial\alpha)|_{\alpha^*} = r$; $|r| < 1$), indicating that only the region with $\alpha_j > \alpha^*$ is allowed (since α_j is the highest value). The formal expressions given by eqs. (44) and (45) are still valid here, but with $r < 1$. However, as in this case we are only considering $\alpha_j > \alpha^*$, the iterations of f will approach the fixed point. Yet, note that in order to have a finite dimension representation we must have $\alpha^* < -\alpha_j - 1$, otherwise the dimension of the representation will be infinite. Also, the eq.

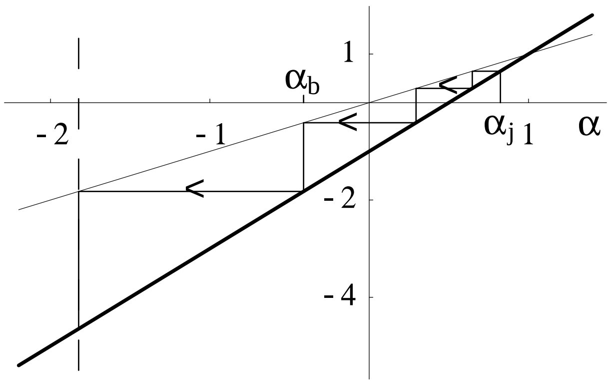

Figure 2: Case II; 4-dimensional representation for $r = 2$, $s = 1$, $\alpha_j = 14/17$ and $\alpha^\star = 1$. The dashed line is the cut condition $\alpha_{j-d} = -31/17$. The thinner line is the function $f(\alpha) = \alpha$ and the thicker one is the function $f(\alpha) = 2\,\alpha - 1$.

(45), under the restriction given by eq. (32) yields, in this case, $\alpha^* < -1/2$. Obviously, the cut condition given by eq. (31) should be obeyed by the allowed values of α_j. For a fixed function, there are infinitely countable possible values of α_j, one for each respective dimension. A graphical representation of this case can be seen in fig. (3).

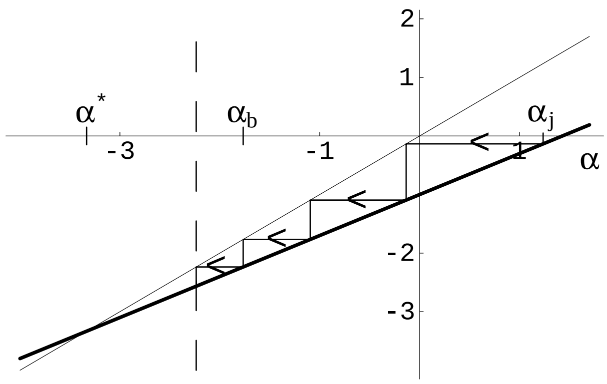

Figure 3: Case III; 4-dimensional representation for $r = 0.7$, $s = 1$, $\alpha_j = 1.23619\ldots$ and $\alpha^\star = -3.33333\ldots$. The dashed line is the cut condition $\alpha_{j-d} = -2.23619\ldots$. The thinner line is the function $f(\alpha) = \alpha$ and the thicker one is the function $f(\alpha) = 0.7\,\alpha - 1$.

3.3 Connection with $su_q(2)$

In this section we are going to show the connection of the generalized $su(2)$ algebra for linear $f(J_0)$, eqs. (1-3), with the deformation of $su(2)$ found in the literature as $su_q(2)$ [1]. The $su_q(2)$ operators, let us call them S_3 and $S_\pm$, have the following commutation relations among them [1]:

$$[S_3, S_\pm] = \pm S_\pm \tag{46}$$

$$[S_+, S_-] = [S_3]\ , \tag{47}$$

where $[x] \equiv (q^x - q^{-x})/(q - q^{-1})$. The parameter q is a real number and is called the deformation parameter of the algebra. When $q \to 1$ the above commutation relations recover the $su(2)$ relations. A simple transformation shows that $[x] = q^{-x+1}[x]_{q^2}$, where $[x]_{q^2} = (q^{2x} - 1)/(q^2 - 1)$. The action of these operators on the states of an irreducible representation of the $su_q(2)$ algebra, whose dimension is $2j + 1$, can be written as [1]:

$$\begin{aligned} S_\pm |j, j-m\rangle &= \sqrt{q^{-2j+1}[j \mp (j-m)]_{q^2}[j \pm (j-m) + 1]_{q^2}}\, |j, j-m \pm 1\rangle \\ S_3|j, j-m\rangle &= (j-m)|j, j-m\rangle \end{aligned} \tag{48}$$

where $2j + 1$ is a positive integer and $m = 0, 1, 2, \ldots, 2j$.

In general it can be found explicit functionals that map generators of an specific algebra to another one [10]. In particular, there is a specific map that convert the $su(2)$ generators into the generators of $su_q(2)$ [10]. The relevance of these maps is that they allow to construct different co-algebra structures based on a known co-algebra.

In our case, if we remember the action of the J_0 operator in a space of dimension $2j + 1$, eq. (27), and use the expression of α_{j-m} given by eq. (44) with $r \equiv q^2$, we see that expressing the operator J_0 as:

$$J_0 = q^{2(j-S_3)}\alpha_j - s\,[j - S_3]_{q^2} \ , \tag{49}$$

this operator acts on the $2j + 1$ states of the representation of the $su_q(2)$ algebra exactly as it does on its own space of same dimension.

If we identify

$$J_+ = \frac{\sqrt{(Q_1\alpha_j - Q_2[j - S_3 + 1]_{q^2})(Q_3\,\alpha_j + 1 + Q_2[j - S_3 + 1]_{q^2})}}{\sqrt{q^{-2j+1}[j - S_3 + 1]_{q^2}[j + S_3]_{q^2}}}\, S_+ \ , \tag{50}$$

where $Q_1 \equiv (q^2 - 2)/(q^2 - 1)$, $Q_2 \equiv (q^2 - 1)\alpha_j - s$ and $Q_3 \equiv q^2/(q^2 - 1)$, this operator also acts on the $2j+1$ states of the $su_q(2)$ algebra exactly the same way it does on its own $2j+1$ space of states as given by eqs. (28) and (44). As $J_- = J_+^\dagger$, the transformations given by eqs. (49, 50) connect the $su_q(2)$ algebra with the $r \neq 1$ linear case of our formalism.

Applying the same procedure just described above, we can compute the inverse map, i.e., to express the $su_q(2)$ generators in terms of $J_{(\pm,0)}$, that could be used to obtain the co-algebra structure of the generalized $su(2)$ algebra given by eqs. (37-39).

Therefore, we have shown that this linear case is connected to the $su_q(2)$ algebra. Moreover, this formalism allows generalizations of $su(2)$ to more complex algebras obtained by considering non-linear functions f in eqs. (1-3). These algebras, depending on the function f, will not be simply deformations of $su(2)$. For each non-linear function f the representation theory should be constructed following the scheme presented in [5].

4 Non-linear functions

In this section we consider some elementary aspects of the representation theory of the algebra defined by eqs. (1-3) for $f(x) = t\,x^2 + r\,x - s$. In this case the algebra becomes

$$[J_0, J_+]_{r^{-1}} = -r^{-1}\,(tJ_0^2 - s)\,J_+ \quad , \tag{51}$$

$$[J_0, J_-]_r = J_-\,(tJ_0^2 - s) \quad , \tag{52}$$

$$\begin{aligned}[J_+, J_-] = & -t^2J_0^4 - 2trJ_0^3 + (1-(1-s)t - r^2 + st)J_0^2 \\ & +(1 - r(1-2s))J_0 - s(1-s) \quad .\end{aligned} \tag{53}$$

Of course, for $t = 0$ we recover the linear (or r-deformed) $su(2)$ algebra given in eqs. (37-39). For $t = 0$ and $r = s = 1$ we recover the standard $su(2)$ algebra.

We focus now on the analysis of the finite dimensional representations of the above quadratic $su(2)$ algebra [1]. To this aim we have to look for the finite dimensional representation solutions of eqs. (27-30). Since we are starting from a highest weight vector and in order to have a finite representation we must find the conditions where the eq. (29) is identically null. This is obtained analysing the zeros of the equation

$$N_{d-1}^2 = (\alpha_j - \alpha_{j-d})(\alpha_j + \alpha_{j-d} + 1) = 0\,. \tag{54}$$

In eq. (54), the zeros of the term $\alpha_j - \alpha_{j-d}$ can be obtained through the analysis and the stability of the fixed points of $f(x) = t\,x^2 + r\,x - s$ and their composed functions [5]. The other term, $\alpha_j + \alpha_{j-d} + 1$, is the cut condition.

In order to analyse the stability of the fixed points of $f(x)$ it is convenient to discriminate three cases: (I) $\Delta < 0$, (II) $\Delta = 0$ and (III) $\Delta > 0$, for $\Delta = (r-1)^2 + 4\,t\,s$. In the first case there is no fixed point and we see, by a graphical analysis similar to that discussed in subsection 3.1, that there is no finite dimensional representation; in case (II), we have one fixed point given by $\alpha^\star = (1-r)/2t$. This fixed point corresponds to a trivial one-dimensional representation of the algebra for $\alpha_j = \alpha^\star$ since $N_0 = 0$ ($\alpha_{j-1} = \alpha_j = \alpha^*$).

Case (III) is less trivial. In this case it is also possible to have attractors of period 1, 2, 4, $\cdots$ and even a chaotic region in the space of parameters (t, r, s, α_0) where, as it is well-known, there are cycles associated with all integer numbers.

For $0 < \Delta < 4$ there is only trivial one-dimensional representations associated to the fixed points $\alpha^\star = f(\alpha^\star)$, with highest weight:

$$\alpha_\pm^\star = \frac{1 - r \pm \sqrt{\Delta}}{2\,t} \quad . \tag{55}$$

At $\Delta = 4$ the one-cycle looses stability and a stable two-cycle appears, solution of $\beta^\star = f^2(\beta^\star)$, where it is considered solutions different from the previous one (attractors of period 1), where $f^2(x)$ means $f(f(x))$. They are

$$\beta_\pm^\star = \frac{-1 - r \pm \sqrt{\Delta_1}}{2\,t} \quad , \tag{56}$$

[1]From now one we are considering $t > 0$; the analysis of negative values of t is similar.

where $\Delta_1 = -3 - 2r + r^2 + 4ts$. In this case, $\Delta > 4$, we have a two-dimensional representation of the algebra simply choosing the highest weight, α_j, as the highest element of the two cycle, i.e, $\beta_+^\star$. Note that in eq. (54) the term $\alpha_j - \alpha_{j-d}$ becomes, in this two-dimensional case, $\beta_+^\star - f^2(\beta_+^\star)$, that is identically zero. The matrix representation is given by

$$J_0 = \begin{pmatrix} \beta_+^\star & 0 \\ 0 & \beta_-^\star \end{pmatrix}, \; J_+ = \begin{pmatrix} 0 & 0 \\ N_0 & 0 \end{pmatrix}, \; J_- = J_+^\dagger \,, \tag{57}$$

where N_0 is computed for $\Delta > 4$ and $\alpha_j = \beta_+^\star$.

For $\Delta > 6$ we will have other cycles, of length $4, 8, \ldots, 2^k \ldots$, entering then in the chaotic region where all cycles will be present. In general, for a d-cycle we have a d-dimensional representation where the highest weight of the representation is the largest element of the cycle. The term $\alpha_j - \alpha_{j-d}$ of eq. (54) is identically null for this cycle.

There are also finite dimensional representations coming from the zeroes of the cut condition $\alpha_j + \alpha_{j-d} + 1 = 0$ in the expression of N_{d-1}. For example, the regions associated with the possible highest weight vector solutions for the first cycles are better understood studying the corresponding Δ intervals. For the one-cycle, the following region

$$0 < \Delta < 4 \;\; , \;\; \alpha_-^* < \alpha_j < \alpha_+^* \,, \tag{58}$$

is the only region in the α real axis where it is possible to find highest weight vectors, apart the trivial one-dimensional case already mentioned. In order to select finite d-dimensional representations we must pick up the points (if they exist) in this interval that satisfy the cut condition $\alpha_j + \alpha_{j-d} + 1 = 0$.

For the two-cycle, the following region

$$4 < \Delta < 6 \,, \;\; \beta_+^\star < \alpha_j < \alpha_+^\star \,, \tag{59}$$

is the only region in the α real axis where it is possible to find highest weight vectors, apart the two-dimensional representation ($\alpha_j = \beta_+^*$). In order to select finite dimensional representation we have to find the points (if they exist) in this interval that satisfy the cut condition. For higher cycles the analysis is similar. In all cycles the allowed regions to get possible finite dimensional representations range from the largest element of the cycle up to α_+^*.

5 Conclusion

We have developed and extended the study of the algebraic structure of a $su(2)$ generalized algebra introduced in [9]. We have constructed the representations of this generalized algebra for the linear case, $f(J_0) = rJ_0 - s$, and also for the quadratic case, $f(J_0) = tJ_0^2 + rJ_0 - s$. These representations are constructed using their dependence on the fixed points of f, and their stability properties.

The connection with the $su_q(2)$ has also been established. We have constructed a map connecting the generators of the generalized algebra for the linear case, to the generators of the well-known deformation $su_q(2)$. This map could be useful to construct the co-multiplication rule for the generalized algebra in the linear case $f(J_0) = rJ_0 - s$.

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