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Tunneling and Nucleation Rate in the $\left(\frac{\lambda}{4!}\phi^4 + \frac{\sigma}{6!}\phi^6\right)_3$ Model

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Abstract

We evaluate both the vacuum decay rate at zero temperature and the finite temperature nucleation rate for the $\left(\frac{\lambda}{4!}\phi^4 + \frac{\sigma}{6!}\phi^6\right)_{3D}$ model. Using the thin-wall approximation, we obtain the bounce solution for the model and we were also able to give the approximate eigenvalue equations for the bounce.

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I. Introduction

The scalar field model with potential $U(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\sigma}{6!}\phi^6$ is the simplest model exhibiting a rich phase structure and for studying tricritical phenomena in both two and three dimensional systems [see, *e.g.* Refs. [1] and [2]].

In terms of the phase diagram for the model, a tricritical point can emerge whenever we have three phases coexisting simultaneously. For the above potential, in the absence of corrections due to fluctuations, one can have a second order transition in m when the scalar field mass vanishes and $\lambda > 0, \sigma > 0$.

A first order transition happens for the case of $\lambda < 0, \sigma > 0$. The tricritical point occurs when the quartic coupling constant vanishes (with $m = 0, \sigma > 0$). A study of the phase diagram for this model in $3D$, at finite temperature, was recently done [3] and it has been shown that there is a temperature $\beta^{-1}(m, \lambda, \sigma)$ for which the physical thermal mass m_β and coupling constant λ_β vanishes, thus characterizing the tricritical point.

There are many possible applications associated with the model with potential $U(\phi)$ with ϕ^6 interaction. For instance, in $D = 2$, it is known [4] that the minimal conformal quantum field theory, with central charge 7/10 (the tricritical Ising model) is in the same universality class, in the scaling region near the tricritical point, of the Landau-Ginzburg model with the above potential. This ϕ^6 potential model can then be thought of as the continuum realization of the Ising model with possible applications in, *e.g.*, the description of adsorbed helium on krypton-plated graphite [5], in understanding the statistical mechanics of binary mixtures, such as $He^3 - He^4$ [6], etc. These are just a few examples of systems exhibiting tricritical phenomena in condensed matter physics.

In field theory in general, the ϕ^6 model has been used in the study of polarons, or solitonic like field configurations on systems of low dimensionality [1]. Also, a gauged ($SU(2)$) version of the ϕ^6 potential model in Euclidean $3D$, has recently been used to study a possible existence of a tricritical point in Higgs models at high temperatures [7]. In this context the tricritical point is characterized by the ratio of the quartic coupling constant and the gauge coupling constant of the effective three-dimensional theory, obtained from the $3 + 1\,D$ high temperature, dimensionally reduced $SU(2)$ Higgs model. This can be particularly useful in the context of the study of the electroweak phase transition.

There are then many reasons that make the ϕ^6 potential model an interesting model to be studied. In this paper, we will be particularly interested in studying the regime of parameters for which:

$$m^2 > 0\,,\ \lambda < 0\,,\ \sigma > 0 \ \text{and}\ \left[\left(\frac{\lambda}{3!}\right)^2 - 4m^2\,\frac{\sigma}{5!}\right] > 0\,. \tag{1.1}$$

In this case $U(\phi)$ has three relative minima $\phi_{t\pm}$ and ϕ_f (see Fig. 1). For these parameters the system has metastable vacuum states and it may exhibit a first order phase transition. The states of the classical field theory for which $\phi = \phi_{t\pm}$ are the unique classical states of lowest energy (true vacuum) and, at least in perturbation theory, they correspond to the unique vacuum states of the quantum theory. The state of the classical field theory for which $\phi = \phi_f$ is a stable classical equilibrium state. However, it is rendered unstable by quantum effects, *i.e.*, barrier penetration (or over the barrier thermal fluctuations, at finite temperatures). ϕ_f is the false vacuum (the metastable state).

We will compute the vacuum decay (tunneling) rate at both zero and finite temperature. For calculation reasons we will restrict ourselves to the thin wall approximation for the true vacuum bubble (or bounce solution). We then consider the energy-density difference between the true and false vacuum as very small as compared with the height of the barrier of the $U(\phi)$ potential [9, 13]. From this, we are able to give the explicit expression for the bounce and also to qualitatively describe the eigensolutions for the bounce configuration. The paper is organized as follows: In Sec. II we obtain the bounce field configuration for the model and we compute the Euclidean action in the thin wall approximation.

In Sec. III we calculate the vacuum decay rate at zero temperature and we discuss the eigenvalue equations obtained for the bounce configuration within the approximations we have taken for the bounce.

In Sec. IV we compute the nucleation rate at finite temperature, following the procedure given in [14]. In Sec. V the concluding remarks are given. In this paper we use $\hbar = c = k_b = 1$.

II. The Vacuum Decay Rate and the Bounce Solution

Let us consider a scalar field model, in three-dimensional space-time, with Euclidean action given by

$$S_E(\phi) = \int d^3x_E \left[\frac{1}{2}(\partial_\mu \phi)^2 + U(\phi)\right] , \tag{2.1}$$

with

$$U(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\sigma}{6!}\phi^6 . \tag{2.2}$$

As discussed in the introduction, we are interested in the regime for which the parameters in (2.2) satisfies (1.1), such that the potential $U(\phi)$ exhibits non-degenerate local minima and then metastable states. The picture we have in mind is that once the system is prepared in the false vacuum state, it will evolve to the true vacuum state by tunneling (at zero temperature) or by bubble nucleation, triggered by thermal fluctuations over the potential barrier.

In the case of quantum field theory at zero temperature the study of the decay of false vacuum was initiated by Voloshin, Kobsarev and Okun [8] and later by Callan and Coleman [9, 10], who developed the so called bounce method for the theory of quantum decay. In this context the decay rate per unit space-time volume V_3 is given by

$$\frac{\Gamma}{V_3} = \left(\frac{\Delta S_E}{2\pi}\right)^{3/2} \left[\frac{\det'(-\Box_E + U''(\phi_b))}{\det(-\Box_E + U''(\phi_f))}\right]^{-1/2} e^{-\Delta S_E}\,(1 + \mathcal{O}(\hbar)) , \tag{2.3}$$

where $\Box_E = \frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $U''(\phi) = \frac{d^2U(\phi)}{d\phi^2}$ and $\Delta S_E = S_E(\phi_b) - S_E(\phi_f)$. $S_E(\phi_b)$ is the

Euclidean action evaluated at its extreme (specifically a saddle point), $\phi = \phi_b$, where ϕ_b is the bounce: a solution of the field equation of motion, $\delta S_E/\delta\phi|_{\phi=\phi_b} = 0$, with the appropriate boundary conditions.

The prime in the determinantal prefactor in (2.3) means that the three zero eigenvalues (the translational modes) of the $[-\Box_E + U''(\phi_b)]$ operator has been removed from it.

As it was shown by Coleman, Glaser and Martin [11], the solution that minimizes S_E is a spherical symmetric solution, $r^2 = \tau^2 + x^2 + y^2$ (in D dimensions the solution has $O(D)$ symmetry) and then ϕ_b can be written as the solution of the radial equation of motion

$$\frac{d^2\phi_b}{dr^2} + \frac{2}{r}\frac{d\phi_b}{dr} = U'(\phi_b) \ , \tag{2.4}$$

with the boundary conditions: $\lim_{r\to\infty}\phi_b(r) = \phi_f$ and $\frac{d\phi_b}{dr}|_{r=0} = 0$.

At finite temperature the calculation of the decay rate were first considered in [12] in the context of quantum mechanics and later by Linde [13], for quantum field theory. In [13], it is argued that temperature corrections to the nucleation rate are obtained recalling that finite temperature field theory (at sufficiently high temperatures) in $D = d+1$ dimensions is equivalent to d-dimensional Euclidean quantum field theory with $\hbar$ substituted by T.

At finite temperature, the bounce, $\phi_B \equiv \phi_B(\rho)$ ($\rho = |\mathbf{r}|$), is a static solution of the field equation of motion:

$$\frac{d^2\phi_B}{d\rho^2} + \frac{1}{\rho}\frac{d\phi_B}{d\rho} = U'(\phi_B) \ , \tag{2.5}$$

with boundary conditions: $\lim_{\rho\to\infty}\phi_B(\rho) = \phi_f$ and $\frac{d\phi_B}{d\rho}|_{\rho=0} = 0$.

The vacuum decay rate, or bubble nucleation rate in this case, is proportional to $e^{-\Delta E/T}$, where ΔE is the nucleation barrier, given by ($\beta = 1/T$)

$$\frac{\Delta E}{T} = \beta \int d^2\mathbf{x}\, [\mathcal{L}_E(\phi_B) - \mathcal{L}_E(\phi_f)] \ , \tag{2.6}$$

where $\mathcal{L}_E$ is the Euclidean Lagrangian density.

The problem of the computation of the nucleation rate at finite temperature was recently reconsidered by Gleiser, Marques and Ramos [14], who have used the early works of Langer [15]. In this context the nucleation rate is given by

$$(\Gamma)_\beta = -\frac{|E_-|}{\pi}\mathrm{Im}\left[\frac{\det\,[-\Box_E + U''(\phi_B)]_\beta}{\det\,[-\Box_E + U''(\phi_f)]_\beta}\right]^{-1/2} e^{-\Delta E/T}\,(1 + \mathcal{O}(\hbar)) \ , \tag{2.7}$$

where E_-^2 is the negative eigenvalue associated to the operator $[-\Box_E + U''(\phi_B)]$.

To compute the vacuum decay rates, we need, therefore, to solve Eq. (2.4) for $\phi_b(r)$ (at zero temperature), or Eq. (2.5) for $\phi_B(\rho)$ (at finite temperature). In the model studied here, we can give an approximate analytical treatment for the bounce solution in the so called thin-wall approximation, in which case the energy-density difference between the false and true vacuum states can be considered very small, as compared to the height of the potential barrier: $\epsilon_0 = U(\phi_f) - U(\phi_t) \ll U(\phi_2)$ (see Fig. 1). By interpreting ϕ_b as a position and r as the time, Eq. (2.4) can be seen as a classical equation of motion for a particle moving in a potential $-U(\phi)$ and subject to a viscous like damping force, with stokes's law coefficient inversely proportional to the time. The particle motion can then be interpreted as it was released at rest, at time zero (because of the boundary condition $\frac{d\phi_b}{dr}|_{r=0} = 0$).

From (2.4) it follows that

$$\frac{d}{dr}\left[\frac{1}{2}\left(\frac{d\phi_b}{dr}\right)^2 - U(\phi_b)\right] = -\frac{2}{r}\left(\frac{d\phi_b}{dr}\right)^2 \leq 0 \,, \tag{2.8}$$

meaning that the particle loses energy. Then, if the initial position of the particle is chosen to be at the left of some value $\phi = \phi_1$ (see Fig. 2), it will never reach the position ϕ_f.

Now, for $\phi_b(r)$ very close to ϕ_t, we can linearize Eq. (2.4) to obtain

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \mu^2\right](\phi_b - \phi_t) = 0 \,, \tag{2.9}$$

where $\mu^2 = U(\phi_t)$. The solution of (2.9) is

$$\phi_b - \phi_t = (\phi(0) - \phi_t)\frac{\sinh(\mu r)}{\mu r} \,. \tag{2.10}$$

Therefore, if we choose the position of the particle to be initially sufficiently close to ϕ_t, we can arrange for it to stay arbitrarily close to ϕ_t for arbitrarily large r.

But for sufficiently large r, $r = R$, the viscous damping force can be neglected, since its coefficient is inversely proportional to r. And if the viscous damping is neglected, the particle will reach the position ϕ_f at a finite time $R + \Delta R$. Then, by continuity, there must be an initial position between ϕ_t and ϕ_1 for which the particle will come to rest at ϕ_f, after an infinity time.

From the above arguments, we can find a general expression for ϕ_b, valid for $\epsilon_0 \ll U(\phi_2)$.

In order to not lose too much energy, we must choose $\phi_b(0)$, the initial position of the particle, very close to ϕ_t.

The particle then stays close to ϕ_t until some very large time, $r = R$. Near R (between $R - \Delta R$ and $R + \Delta R$, $\Delta R \ll R$) the particle moves quickly (according to Eq. (2.4), neglecting the viscous damping force), through the valley in Fig. 2 and it slowly comes to rest at ϕ_f, after an infinity time.

Thus, we can write for the bounce the following expression ($\phi_f = 0$)

$$\phi_b(r) = \begin{cases} \phi_t & 0 < r < R - \Delta R \\ \phi_{\rm wall}(r - R) & R - \Delta R < r < R + \Delta R \\ \phi_f & R + \Delta R < r < \infty \end{cases} , \tag{2.11}$$

where $\phi_{\rm wall}$ satisfies the equation

$$\frac{d^2\phi_{\rm wall}}{dr^2} = U'(\phi_{\rm wall}) . \tag{2.12}$$

Eq. (2.11) is the thin-wall approximation for the bounce solution. From Eq. (2.12), we obtain

$$\int_0^{\phi_{\rm wall}} \frac{d\phi}{\sqrt{2U(\phi)}} = r . \tag{2.13}$$

By rewriting $U(\phi)$ as

$$U(\phi) = \frac{\sigma}{6!}\phi^2\left(\phi^2 - \phi_0^2\right)^2 - \gamma\frac{\phi^2}{\phi_0^2} , \tag{2.14}$$

with

$$\phi_0^2 = -\frac{1}{2}\frac{6!}{\sigma}\frac{\lambda}{4!} \tag{2.15}$$

and

$$\gamma = \frac{\phi_0^2}{4}\left[\frac{6!}{\sigma}\left(\frac{\lambda}{4!}\right)^2 - 2m^2\right] , \tag{2.16}$$

then, by neglecting in (2.14) the term proportional to γ (valid in the thin-wall approximation), we obtain that

$$\int_0^{\phi_{\rm wall}} \frac{d\phi}{\sqrt{2\frac{\sigma}{6!}}\phi\left(\phi^2 - \phi_0^2\right)} = r . \tag{2.17}$$

The above integral is straightforward and the solution for $\phi_{\rm wall}(r)$ can be written as

$$\phi_{\rm wall}^2(r) = \frac{\phi_0^2}{1 + \exp\left(\sqrt{8\frac{\sigma}{6!}}\phi_0^2 r\right)} . \tag{2.18}$$

This solution is shown in Fig. 3. Using (2.18), we obtain for ΔS_E the expression

$$\begin{aligned}\Delta S_E &= 4\pi \int_0^{\infty} drr^2 \left[\frac{1}{2}\left(\frac{d\phi_b}{dr}\right)^2 + U(\phi_b)\right] \\ &= 4\pi \int_0^{R-\Delta R} drr^2 U(\phi_t) + 4\pi \int_{R+\Delta R}^{\infty} drr^2 U(0) + \\ & \quad 4\pi \int_{R-\Delta R}^{R+\Delta R} drr^2 \left[\frac{1}{2}\left(\frac{d\phi_{\rm wall}}{dr}\right)^2 + U(\phi_{\rm wall})\right] . \end{aligned} \tag{2.19}$$

Since $\Delta R \ll R$, in the last integral of (2.19) we can take $r \approx R$ and we obtain

$$\Delta S_E \simeq -\frac{4}{3}\pi \epsilon_0 R^3 + 4\pi R^2 \mathcal{S}_0 \ , \tag{2.20}$$

with $\mathcal{S}_0$, the bounce surface energy density, given by

$$\mathcal{S}_0 = \int_{\phi_t}^{\phi_f} d\phi \sqrt{2U(\phi)} \approx \int_{\phi_0}^{0} d\phi \sqrt{2\frac{\sigma}{6!}}\phi \left(\phi^2 - \phi_0^2\right) \ , \tag{2.21}$$

where we have neglected in $U(\phi)$ the term $\gamma\phi^2/\phi_0^2$ and we also used $\phi_t \approx \phi_0$. Evaluating the above integral, we obtain for S_0 the result:

$$\mathcal{S}_0 \simeq \frac{\phi_0^4}{4}\sqrt{2\frac{\sigma}{6!}} \ . \tag{2.22}$$

In the next two sections, we deal with the evaluation of the determinantal prefactor appearing in Eqs. (2.3) and (2.7) and we obtain the subsequent radiative (1-loop) corrections to (2.20).

III. The Vacuum Decay Rate at $T = 0$

Let us consider, initially, the eigenvalue equations for the differential operators appearing in (2.3):

$$\left[-\Box_E + U''(\phi_b)\right] \psi_b(i) = E_b^2(i)\psi_b(i) \tag{3.1}$$

and

$$\left[-\Box_E + U''(\phi_f)\right] \psi_f(j) = E_f^2(j)\psi_f(j) \ . \tag{3.2}$$

We then have for the determinantal prefactor of (2.3), the following

$$\begin{aligned} K &= \left[\frac{\det'\left[-\Box_E + U''(\phi_b)\right]}{\det\left[-\Box_E + U''(\phi_f)\right]}\right]^{-1/2} \\ &= \exp\left\{-\frac{1}{2}\ln\left[\frac{\det'\left[-\Box_E + U''(\phi_b)\right]}{\det\left[-\Box_E + U''(\phi_f)\right]}\right]\right\} \\ &= \exp\left\{-\frac{1}{2}\ln\left[\frac{\prod_i {}' E_b^2(i)}{\prod_j E_f^2(j)}\right]\right\} \\ &= \exp\left\{-\frac{1}{2}\left[\sum_i {}'\ln|E_b^2(i)| - \sum_j \ln E_f^2(j)\right]\right\} . \end{aligned} \tag{3.3}$$

Since the bounce can be approximated by a constant field configuration for $r < (R - \Delta R) \approx R$, we can write for K, in the thin wall approximation, the expression:

$$K = \exp\left\{-\frac{1}{2}\left[\frac{4}{3}\pi R^3 \int \frac{d^3p}{(2\pi)^3}\ln\left[\frac{E_t^2(p)}{E_f^2(p)}\right] + \left(\sum_i {}'\ln|E^2_{\rm wall}(i)| - \sum_j \ln E_f^2(j)\right)\right]\right\} , \tag{3.4}$$

where $E^2_{t(f)}(p) = p^2 + U''(\phi_{t(f)})$.

The integral in (3.4) can then be identified as the one loop correction to the classical potential, while the remaining terms represent the quantum corrections due to fluctuations around the bounce wall [16].

Then, by using Eqs. (2.20) and (3.4) in Eq. (2.3) and following [9, 10], we obtain

$$\frac{\Gamma}{V_3} \simeq 2\left(\frac{\Delta S_E}{2\pi}\right)^{3/2} \exp\left[\frac{4}{3}\pi R^3 \Delta U_{eff} - 4\pi R^2\left(\mathcal{S}_0 + \mathcal{S}_1\right)\right] , \tag{3.5}$$

where $\mathcal{S}_1$, is the term giving the 1-loop quantum corrections to fluctuations around the bounce wall,

$$\mathcal{S}_1 = \frac{1}{4\pi R^2}\left(\sum_i {}'\ln\left|E^2_{\rm wall}(i)\right| - \sum_j \ln E_f^2(j)\right) , \tag{3.6}$$

where $E^2_{\rm wall}(i)$ are the eigenvalues of $-\Box_E + U''(\phi_{\rm wall}(r - R)$. In (3.5) we have also that $\Delta U_{\rm eff} = U_{\rm eff}(\phi_t) - U_{\rm eff}(\phi_f)$, where $U_{\rm eff}$ is the one loop effective potential, given by [17]

$$U_{\rm eff}(\phi) = U(\phi) + \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\ln\left[\frac{p^2 + U''(\phi)}{p^2 + m^2}\right] . \tag{3.7}$$

The ultraviolet divergence in (3.7) can be handled in the usual way. Integrating over p_0 and by using an ultraviolet cut-off, Λ, for the space-momentum, we obtain

$$U_{\text{eff}}(\phi) = U(\phi) - \frac{1}{12}\left[\left(\mathbf{p}^2+m^2\right)^{3/2}\Big|_0^\Lambda - \left(\mathbf{p}^2+m^2+\frac{\lambda}{2}\phi^2+\frac{\sigma}{4!}\phi^4\right)^{3/2}\Big|_0^\Lambda\right] . \tag{3.8}$$

Using that $(1+x)^{3/2} = 1+3/2x+3/8x^2+\ldots$, we obtain for U_{eff} the expression

$$\begin{aligned} U_{\text{eff}}(\phi) &= U(\phi) - \frac{1}{12}\left[\left(m^2+\frac{\lambda}{2}\phi^2+\frac{\sigma}{4!}\phi^4\right)^{3/2} - m^3 - \frac{3}{2}Y(\Lambda)\left(\frac{\lambda}{2}\phi^2+\frac{\sigma}{4!}\phi^4\right)\right] \\ &\quad + \mathcal{O}\left(1/\Lambda\right) , \end{aligned} \tag{3.9}$$

where $Y(\Lambda) = \Lambda^2 + m_r^2$. The divergent terms in (3.9) are proportional to ϕ^2 and ϕ^4 but not to ϕ^6. Then, only the mass m and λ need to be renormalized ($\sigma_r = \sigma$). From the usual definition of renormalized mass m_r and coupling constant λ_r,

$$m_r^2 = \frac{d^2 U_{\text{eff}}(\phi)}{d\phi^2}\Big|_{\phi=0} \tag{3.10}$$

and

$$\lambda_r = \frac{d^4 U_{\text{eff}}(\phi)}{d\phi^4}\Big|_{\phi=0} \tag{3.11}$$

and writing the unrenormalized parameters in terms of renormalized ones, we obtain for the renormalized one loop effective potential, the following expression:

$$U_{\text{eff}}(\phi) = U_r(\phi) - \frac{1}{12}\left[U_r''(\phi)^{3/2} - m_r^3 - \frac{3}{4}m_r\lambda_r\phi_r^2 - \frac{3\lambda_r^2}{32m_r}\phi^4 - \frac{3m_r\sigma_r}{16}\phi^4\right] , \tag{3.12}$$

where U_r means the tree level potential expressed in terms of the renormalized quantities. For convenience, from now on we drop the r subscript from the expressions and it is to be understood that the parameters m, λ and σ are the renormalized ones, instead of the bare ones. Then, ΔU_{eff} can be written as (since $\phi_f = 0$)

$$\Delta U_{\text{eff}} = U(\phi_t) - \frac{1}{12}\left[U''(\phi_t)^{3/2} - \frac{3}{4}m\lambda\phi_t^2 - \frac{3\lambda^2}{32m}\phi_t^4 - \frac{3m\sigma}{16}\phi_t^4\right] \tag{3.13}$$

We now turn to the problem of evaluating the eigenvalues

$E^2_{\text{wall}}(i)$ of $-\Box_E + U''[\phi_{\text{wall}}(r-R)]$, which appears in (3.4). This is not an easy task. In fact, only in a very few examples this has known analytical solutions, as, for example, for the kink solution in the $(\lambda\phi^4)_{D=2}$ model [18]. Unfortunately, for the model studied here, we can not find analytical solutions for these differential operators.

However, we can perform an approximate analysis, and, in particular we can find explicitly the negatives and zero modes for the differential operator for the bounce wall field configuration.

By making use of the spherical symmetry of the bounce solution, we can express the eigenvalue equation:

$$[-\Box_E + U''(\phi_{\rm wall}(r-R))]\Psi_i(r,\theta,\varphi) = E^2_{\rm wall}(i)\Psi_i(r,\theta,\varphi)$$

in the form

$$\left[-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} + m^2 + \frac{\lambda}{2}\phi^2_{\rm wall}(r-R) + \frac{\sigma}{4!}\phi^4_{\rm wall}(r-R)\right]\psi_i(r) = E^2_{\rm wall}(i)\psi_i(r)\ , \tag{3.14}$$

where $l = 0, 1, 2, \ldots$. Making $\psi_i(r) = \chi_i(r)/r$ and $z = r - R$, we obtain

$$\left[-\frac{d^2}{dz^2} + \frac{l(l+1)}{(z+R)^2} + m^2 + \frac{\lambda}{2}\phi^2_{\rm wall}(z) + \frac{\sigma}{4!}\phi^4_{\rm wall}(z)\right]\chi_{n,l}(z) = E^2_{\rm wall}(n,l)\chi_{n,l}(z)\ . \tag{3.15}$$

Since $\Delta R \ll R$, we can take $l(l+1)/(z+R)^2 \approx l(l+1)/R^2$ and then

$$\left[-\frac{d^2}{dz^2} + \frac{\lambda}{2}\phi^2_{\rm wall}(z) + \frac{\sigma}{4!}\phi^4_{\rm wall}(z)\right]\chi_n(z) = \eta_n^2\chi_n(z)\ , \tag{3.16}$$

where η_n is obtained from

$$E^2_{\rm wall}(n,l) = \eta_n^2 + m^2 + \frac{l(l+1)}{R^2}\ . \tag{3.17}$$

We know that the $-\Box_E + U''(\phi_b(r))$ operator has three zero eigenvalues coming from the bounce translational invariance. Then, for $l = 1$ and to the lowest value of η_n (which can be chosen as η_1), we will have $E^2_{\rm wall}(1,1) = 0$, with multiplicity three, as expected, and $\eta_1^2 = -m^2 - 2/R^2$.

The lowest eigenvalue $E^2_{\rm wall}$ (the negative eigenvalue) will be $E^2_{\rm wall}(1,0) = -2/R^2$, with multiplicity one, just what one would expect for the metastable state, the existence of only one negative eigenvalue [16].

To evaluate the other eigenvalues, we make the following change of variable $w = \sqrt{\frac{\sigma}{6!}}\phi_0^2\, z$ and use (2.18) in (3.17). We then get

$$\left[-\frac{d^2}{dw^2} - \frac{24}{1+e^{\sqrt{8}w}} + \frac{30}{\left(1+e^{\sqrt{8}w}\right)^2} \right] \chi_n(w) = \nu_n^2 \chi_n(w) \;, \tag{3.18}$$

where $\nu_n^2 = \eta_n^2\, 6!/(\sigma\phi_0^4)$. We can then express the eigenvalues of $-\Box_E + U''(\phi_{\rm wall}(r-R))$ as

$$E^2_{\rm wall}(n,l) = \phi_0^4 \frac{\sigma}{6!}\, \nu_n^2 + m^2 + \frac{l(l+1)}{R^2} \;. \tag{3.19}$$

We were not able to find any analytical solution for Eq. (3.18), which it is even harder to solve, due to the boundary conditions. We are currently working on the numerical solution for the eigenvalues, whose results will be reported elsewhere.

IV. The Nucleation Rate at Finite Temperature

At finite temperature, the bounce solution is a static solution of the field equation of motion, $\delta S_E/\delta\phi|_{\phi=\phi_B(\rho)} = 0$, where $\phi_B(\rho)$ is given as in (2.11), with bubble radius $\bar{\rho}$ and thickness $\Delta\rho \ll \bar{\rho}$. $\phi_{\rm wall} = \phi_{\rm wall}(\rho)$ is still expressed as in (2.18).

To calculate the determinantal prefactor in (2.7), we consider the eigenvalue equations for the differential operators:

$$\left[-\Box_E + U''(\phi_B)\right] \psi_B(i) = \mu_B^2 \psi_B(i) \tag{4.1}$$

and

$$\left[-\Box_E + U''(\phi_f)\right] \psi_f(i) = \mu_f^2 \psi_f(i) \;, \tag{4.2}$$

where, in momentum space, $\mu^2 = \omega_n^2 + E^2$, where $\omega_n = \frac{2\pi n}{\beta}$ are the Matsubara frequencies ($n = 0, \pm1, \pm2, \ldots$). Using (4.1) and (4.2) in (2.7), we obtain for K the expression:

$$\begin{aligned} K_\beta &= \left[\frac{\det\left[-\Box_E + U''(\phi_B)\right]_\beta}{\det\left[-\Box_E + U''(\phi_f)\right]_\beta} \right]^{-1/2} \\ &= \exp\left\{ -\frac{1}{2} \ln \left[\frac{\det\left[-\Box_E + U''(\phi_B)\right]_\beta}{\det\left[-\Box_E + U''(\phi_f)\right]_\beta} \right] \right\} \end{aligned}$$

$$
\begin{aligned}
&= \exp\left\{-\frac{1}{2}\ln\left[\frac{\prod_{n=-\infty}^{+\infty}\prod_i\left[\omega_n^2+E_B^2(i)\right]}{\prod_{n=-\infty}^{+\infty}\prod_j\left[\omega_n^2+E_f^2(j)\right]}\right]\right\} \\
&= \exp\left\{-\frac{1}{2}\ln\left[\frac{\prod_{n=-\infty}^{+\infty}\left(\omega_n^2+E_-^2\right)\left(\omega_n^2+E_0^2\right)^2\prod_i'\left[\omega_n^2+E_B^2(i)\right]}{\prod_{n=-\infty}^{+\infty}\prod_j\left[\omega_n^2+E_f^2(j)\right]}\right]\right\} \ , \qquad (4.3)
\end{aligned}
$$

where we have separated the negative and zero eigenvalues in the numerator of Eq. (4.3), with the prime meaning that the single negative eigenvalue, E_-^2, and the two zero eigenvalues, E_0^2 (related to the now two-dimensional space), were excluded from the product. The term for $n = 0$ in $(\omega_n^2 + E_0^2)$ can be handled using collective coordinates method, as in [10, 18], resulting in the factor $V_2\left[\frac{\Delta E}{2\pi T}\right]$, where V_2 is the "volume" of the two-space. Separating the $n = 0$ modes from both the numerator and denominator of (4.3) and using the identity

$$
\prod_{n=1}^{n=+\infty}\left(1+\frac{a^2}{n^2}\right)=\frac{\sinh(\pi a)}{\pi a} \ , \qquad (4.4)
$$

we get

$$
K_\beta = V_2\left[\frac{\Delta E}{2\pi T}\right]\exp\left\{-\ln(E_-^2)^{\frac{1}{2}}-\ln\left[\frac{\sin(\frac{\beta}{2}|E_-|)}{\frac{\beta}{2}|E_-|}\right]+a+b\right\} \ , \qquad (4.5)
$$

where

$$
a=\left(-3+\sum_j-\sum_i{}'\right)\ln\prod_{n=1}^{+\infty}\omega_n^2+\left(\sum_i{}'-\sum_j\right)\ln\beta \qquad (4.6)
$$

and

$$
b=\sum_j\left[\frac{\beta}{2}E_f(j)+\ln\left(1-e^{-\beta E_f(j)}\right)\right]-\sum_i{}'\left[\frac{\beta}{2}E_B(i)+\ln\left(1-e^{-\beta E_B(i)}\right)\right] \ . \qquad (4.7)
$$

Since $\sum_i{}'$ has three eigenvalues less than $\sum_j$ and $(E_-^2)^{1/2} = i|E_-|$, we obtain

$$
K_\beta = -i\,V_2\left[\frac{\Delta E}{2\pi T}\right]\left[2\beta^2\sin\left(\frac{\beta}{2}|E_-|\right)\right]^{-1} e^b \ . \qquad (4.8)
$$

Using the above results in (2.7), we then obtain for the nucleation rate, per unit volume, the expression:

$$
\frac{\Gamma_\beta}{V_2} = 2QT^3\exp\left[-\frac{\Delta F(T)}{T}\right] \ , \qquad (4.9)
$$

with

$$Q = \left(\frac{\Delta E}{2\pi T}\right) \frac{\frac{|E_-|}{2T}}{\pi\,\sin(\frac{|E_-|}{2T})} \tag{4.10}$$

and

$$\Delta F(T) = \Delta E - \sum_j \left[\frac{1}{2}E_f(j) + \frac{1}{\beta}\ln\left(1 - e^{-\beta E_f(j)}\right)\right] + \sum_i{}' \left[\frac{1}{2}E_B(i) + \frac{1}{\beta}\ln\left(1 - e^{-\beta E_B(i)}\right)\right] . \tag{4.11}$$

In the thin-wall approximation (in an analogous way as was done in Sec. III) we can write

$$\Delta F(T) \simeq -\pi\bar{\rho}^2\,\Delta U_{\rm eff}(T) + 2\pi(\mathcal{S}_0 + \mathcal{S}_\beta)\bar{\rho}\ , \tag{4.12}$$

where $\Delta U_{\rm eff}(T)$ is given by

$$\begin{aligned}\Delta U_{\rm eff}(T) \ &= \ \epsilon_0 + \int \frac{d^2\mathbf{p}}{(2\pi)^2}\left[\frac{1}{2}\sqrt{\mathbf{p}^2 + U''(\phi_f)} - \frac{1}{2}\sqrt{\mathbf{p}^2 + U''(\phi_t)}\ \right] \\ &+ T\int \frac{d^2\mathbf{p}}{(2\pi)^2}\ln\left[1 - e^{-\beta\sqrt{\mathbf{p}^2 + U''(\phi_f)}}\ \right] - T\int \frac{d^2\mathbf{p}}{(2\pi)^2}\ln\left[1 - e^{-\beta\sqrt{\mathbf{p}^2 + U''(\phi_t)}}\ \right]\end{aligned} \tag{4.13}$$

and $\mathcal{S}_\beta$ is the 1-loop finite temperature correction to the bubble surface energy density, given by

$$\mathcal{S}_\beta = \frac{T}{2\pi\bar{\rho}}\left\{\sum_i{}' \left[\frac{\beta}{2}E_{\rm wall}(i) + \ln\left(1 - e^{-\beta E_{\rm wall}(i)}\right)\right] - \sum_j \left[\frac{\beta}{2}E_f(j) + \ln\left(1 - e^{-\beta E_f(j)}\right)\right]\right\}\ , \tag{4.14}$$

where $E^2_{\rm wall}(i)$ are the eigenvalues of $-\nabla^2 + U''(\phi_{wall}(\rho - \bar{\rho}))$. In (4.13), the first integral is divergent but it can be handled just in the same way as in the previous section, by the introduction of the appropriated counterterms of renormalization.

The remaining integrals in (4.13) are all finite and they can be reduced to integrals of the type

$$I(t) = \int_0^\infty dx\left[x\ln\left(1 - e^{\sqrt{x^2+t^2}}\ \right)\right] \tag{4.15}$$

and it is evaluated in Appendix B. The result is:

$$I(t) = I(0) + \frac{t^3}{6} + \frac{t^2}{4} - \frac{t^2}{8}\ln t - \frac{1}{2}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}\zeta(2n)}{n(n+1)(2\pi)^{2n}}\left(t^2\right)^{n+1}\ , \tag{4.16}$$

where $\zeta(n)$ is the Riemman Zeta function. Using (4.16) in (4.13), we obtain for (4.12) the following expression

$$\Delta F(T) = -\pi \bar{\rho}^2 \left[\epsilon_0 + c(\phi_f) - c(\phi_t)\right] + 2\pi \bar{\rho} \left[\mathcal{S}_0 + \mathcal{S}_\beta\right] \ , \tag{4.17}$$

where (using $t = \beta U''(\phi)^{1/2}$)

$$c(\phi) = \frac{1}{2\pi\beta^3} I\left(t\right) \simeq \frac{1}{2\pi\beta^3} \left(I(0) + \frac{t^3}{6} + \frac{t^2}{4} - \frac{t^2}{8} \ln t - \frac{\zeta(2) t^4}{16\pi^2} \right) \ . \tag{4.18}$$

The critical radius for bubble nucleation, ρ_c, is obtained by minimizing (4.17), $\delta \Delta F(T)/\delta\bar{\rho}|_{\bar{\rho}=\rho_c} = 0$.

For the eigenvalues $E^2_{\rm wall}(i)$ of $-\nabla^2 + U''(\phi_{\rm wall}(\rho - \bar{\rho}))$, using the now cylindrical symmetry of $\phi_{\rm wall}(\rho)$, the eigenvalue equation

$$[-\nabla^2 + U''(\phi_{\rm wall}(\rho - \bar{\rho}))]\Psi_i(\rho,\varphi) = E^2_{\rm wall}(i)\Psi_i(\rho,\varphi)$$

can be written as

$$\left[-\frac{d^2}{d\rho^2} - \frac{1}{\rho}\frac{d}{\rho} + \frac{s^2}{\rho^2} + m^2 + \frac{\lambda}{2}\phi^2_{\rm wall}(\rho - \bar{\rho}) + \frac{\sigma}{4!}\phi^4_{\rm wall}(\rho - \bar{\rho}) \right] \psi_{n,s}(\rho) = E^2_{\rm wall}(n,s)\psi_{n,s}(\rho) \ , \tag{4.19}$$

where $s = 0, \pm 1, \pm 2, \ldots$. Taking $\psi(\rho) = \chi(\rho)/\rho^{1/2}$, we obtain

$$\left[-\frac{d^2}{d\rho^2} + \frac{4s^2 - 1}{4\rho^2} + m^2 + \frac{\lambda}{2}\phi^2_{\rm wall}(\rho - \bar{\rho}) + \frac{\sigma}{4!}\phi^4_{\rm wall}(\rho - \bar{\rho}) \right] \chi_{n,s}(\rho) = E^2_{\rm wall}(n,s)\chi_{n,s}(\rho) \ . \tag{4.20}$$

As in the previous section, we make $z = (\rho - \bar{\rho})$ and because $\Delta\rho \ll \bar{\rho}$, then

$$\left[-\frac{d^2}{dz^2} + \frac{\lambda}{2}\phi^2_{\rm wall}(z) + \frac{\sigma}{4!}\phi^4_{\rm wall}(z) \right] \chi_n(z) = \eta_n^2 \chi_n(\rho) \ , \tag{4.21}$$

where η_n^2 is now obtained from

$$E^2_{wall}(n,s) = \eta_n^2 + m^2 + \frac{4s^2 - 1}{4\bar{\rho}^2} \ . \tag{4.22}$$

Analogously to the zero temperature case, the differential operator $-\nabla^2 + U''(\phi_B)$ has now two zero eigenvalues (related to the translational modes in the two-dimensional space). The multiplicity of (4.22) is two for all $s \neq 0$ and for $s = 0$ the multiplicity is one.

Then for $s = 1$ and lower η_n (we choose η_1) we will have $E^2_{\rm wall}(1,1) = 0$ and then $\eta_1^2 = -[m^2 + 3/(4\bar{\rho}^2)]$. For the negative eigenvalue we obtain $E^2_- = E^2_{\rm wall}(1,0) = -1/\bar{\rho}^2$. As in the previous section, by taking $w = \sqrt{\sigma/6!}\,\phi_0^2\, z$, we obtain for the eigenvalues,

$$E^2_{wall}(n,s) = \phi_0^4 \frac{\sigma}{6!}\,\nu_n^2 + m^2 + \frac{4s^2-1}{4\bar{\rho}^2}\ , \qquad (4.23)$$

where ν_n is the same as in the previous section.

V. Conclusions

In this paper we have studied the evaluation of the vacuum decay rates, at both zero temperature and at finite temperature, for the $\sigma\phi^6$ model in $D = 3$, when the parameters of the model satisfies the conditions given in (1.1). Our main results were the determination of the expression for the bounce solution, Eqs. (2.11) and (2.18), and also, despite of the difficulties for finding the solutions for the bounce's wall eigenvalue problem, by taking consistent considerations for the field equations we have at hand (like the thin-wall approximation), we were able to perform a detailed analysis of the bounce negative and zero eigenvalues. We have also given a set of eigenvalue equations, which can be useful in a more detailed analysis of this problem using, *e.g.* numerical methods.

In [1] it was analyzed the phase structure for the $\sigma\phi^6$ model in $D = 2$ in the lattice and it was also studied the possible production of topological and nontopological excitations in the model.

By remembering that at high temperatures our model in $D = 3$ resembles the $D = 2$ model at zero temperature, it will be interesting to apply the method and results we have obtained here to the problem studied in [1] for the regime of a first order phase transition.

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Appendix

We evaluate here the integral (4.15),

$$I(t) = \int_0^\infty dx \left[x \ln \left(1 - e^{\sqrt{x^2+t^2}} \right) \right] \ . \tag{A.1}$$

Using that

$$\frac{\partial I(t)}{\partial t^2} = \int_0^\infty dx \left[x \frac{\partial}{\partial t^2} \ln \left(1 - e^{\sqrt{x^2+t^2}} \right) \right] = \int_0^\infty dx \left[x \frac{\partial}{\partial x^2} \ln \left(1 - e^{\sqrt{x^2+t^2}} \right) \right] \tag{A.2}$$

and integrating by parts, we obtain that

$$\frac{dI(t)}{dt^2} = \frac{1}{2} \ln \left(1 - e^{\sqrt{x^2+t^2}} \right) \Big|_0^\infty = -\frac{1}{2} \ln \left(1 - e^{-t} \right) \ . \tag{A.3}$$

We can then write $I(t)$ in the form:

$$I(t) = I(0) - \int_0^t dt \left[t \ln \left(1 - e^{-t} \right) \right] = I(0) - \int_0^t dt \left[t \ln \sinh \left(\frac{t}{2} \right) + t \ln 2 - \frac{t^2}{2} \right] \tag{A.4}$$

If one uses (4.4) and expanding the logarithm in the above equation, we are able to perform the integral and we obtain the result shown in Sec. IV, Eq. (4.16).

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Figure Captions

Figure 1: The potential $U(\phi)$ for parameters m, λ, σ satisfying Eq. (1.1).

Figure 2: The inverted potential $-U(\phi)$.

Figure 3: The bubble field configuration $\phi_{\text{wall}}(r)$.

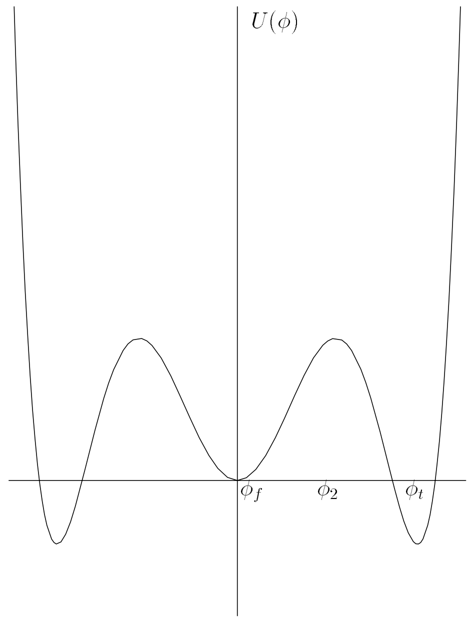

Figure 1

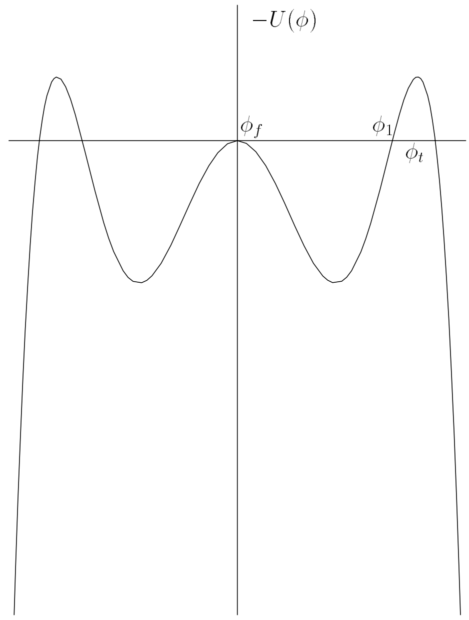

Figure 2

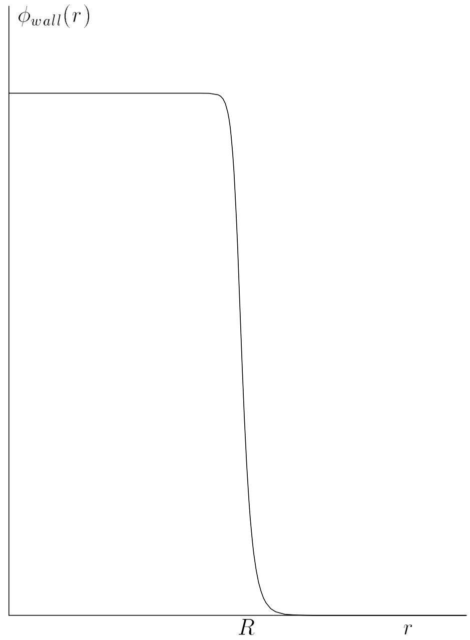

Figure 3