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THE QUANTIZATION OF CLASSICAL NON-HOLONOMIC SYSTEMS

by

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RESUMO

A quantização de sistemas clássicos sujeitos a vínculos não-holônomos é proposta e exemplos são apresentados em detalhes. Mostra-se que existem classes de hamiltonianos que descrevem por imersão no espaço de fase o mesmo sistema não-holônomo. Mostra-se ainda que os vínculos não-holônomos devem ser tratados quanticamente no sentido fraco em que sua restrição ao movimento seja obedecida somente em média sobre o estado do sistema.

ABSTRACT

The quantization of classical non-holonomic systems is proposed and examples are presented in detail. It is shown that there exist classes of hamiltonians which describe by immersion in their phase space the same non-holonomic system. It is further shown that the constraints should be treated quantically in the weak sense, i.e., their restriction to the motion is obeyed only as an average over the state of the system.

1. INTRODUCTION

In this paper we will extend the method for quantization of classical systems to non-holonomic systems. The quantization of classical systems proceeds straighforwardly if the system under consideration is hamiltonian. The determination of the corresponding quantum system is not unique but it can be made precise by postulating a well defined procedure proposed by Weyl[1] and generally accepted as the standard procedure for quantization.

If the system is not manifestly hamiltonian one has to define a procedure to put it into the hamiltonian formalism from where its quantized version is obtained. This is the case of Dirac's generalized dynamics[2] where systems described by singular lagrangians are transformed into hamiltonian systems which take into account the constraints originated from the singular nature of the lagrangian (primary constraints) and those required for the consistency of the equations of motion (secondary constraints). It sometimes happens that some of these constraints are first-class constraints. In such case somewhat arbitrary conditions are imposed upon the system (gauge conditions) in order to produce a well defined hamiltonian system equivalent to the original one, resulted from the singular lagrangian. In both cases, it is the quantization of the resulting well defined hamiltonian system that is assumed as the quantization of the original system.

The only procedure known to us in the literature for the quantization of non-holonomic systems was put forward by

Eden[3]. In the Appendix we will analyse Eden's results and we will show that its quantization is equivalent to adopt for the hamiltonian of the non-holonomic system the same hamiltonian as if the system was holonomic. This hamiltonian unfortunately gives wrong equations of motion when the constraint is not holonomic[4].

Let us consider the simplest non-holonomic system, a particle of unit mass moving in a three dimensional euclidean space subject to the constraint

$$x\,dy + dz = 0 \quad ,$$

x, y and z being the cartesian coordinates of the particle. This system has the following equations of motion:

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= -xv \\ \dot{u} &= 0 \\ \dot{v} &= -\frac{xuv}{1+x^2} \end{aligned}$$

and we observe that its phase space is of odd dimension. The only way to give to this system a hamiltonian formalism is by imbedding it in a larger system. In what follows, we will show among other results that there exists a hamiltonian system in a six-dimensional phase space, in which the subsystem characterized by a constraint $\dot{\chi} = 0$ ($\dot{\chi} = \{\chi,H\} = 0$) is equivalent to the non-holonomic system under consideration. It is this hamiltonian extension together with the constraint that gives, by correspondence, the quantization of the non-holonomic system.

In section 2 we will construct the hamiltonian formalism for the above mentioned system and for the well-known system of a disc pivoting and rolling on a horizontal plane[5]. In section 3 we quantize these systems and exhibit their eigenvalues and eigenfunctions. In section 4, the propagator for the former system is obtained and the propagation of wave packets is exhibited in the semiclassical approximation, showing that the quantum description gives back the classical laws of motion for the non-holonomic system. Section 5 contains a discussion of our results and a proposal of a general procedure for quantizing non-holonomic systems.

2. EXAMPLES

Let us consider a particle of unit mass moving in a three dimensional euclidean space, with coordinates (x,y,z), subject to the constraint

$$x\,dy + dz = 0$$

This is a non-holonomic constraint as can be easily verified and the equations of motion are

$$\begin{aligned} \ddot{x} &= 0 \\ \ddot{y} &= -\frac{x\dot{x}\dot{y}}{1+x^2} \\ \dot{z} &= -x\dot{y} \end{aligned} \tag{2.1}$$

These equations can be integrated and we obtained

$$x = \alpha_1 t + \beta_1$$
$$y = \alpha_2 \log(x + \sqrt{1+x^2}) + \beta_2 \qquad (2.2)$$
$$z = -\alpha_2 \sqrt{1+x^2} + \beta_3$$

where β_1, β_2 and β_3 are constants of integration related to the initial values of (x,y,z) while α_1 and α_2 are related to the initial values of the velocity.

The essential point of our procedure for obtaining the hamiltonian formalism for non-holonomic systems is to recognize that the above equations for the trajectories can be obtained in the Hamilton-Jacobi formalism by introducing a new constant α_3 and setting

$$\chi = \alpha_2 + \alpha_3 = 0$$
$$\frac{\partial S}{\partial \alpha_1} = x - \alpha_1 t$$
$$\frac{\partial S}{\partial \alpha_2} = y - \alpha_2 \log(x + \sqrt{1+x^2}) \qquad (2.3)$$
$$\frac{\partial S}{\partial \alpha_3} = z - \alpha_3 \sqrt{1+x^2}$$

One immediately observes that the introduction of the new constant α_3 was such as to guarantee that

$$\frac{\partial^2 S}{\partial \alpha_i \partial \alpha_j} = \frac{\partial^2 S}{\partial \alpha_j \partial \alpha_i} \quad , \quad i,j = 1,2,3$$

This shows that the function S exists and it is given by

$$S = \alpha_1 x - \frac{1}{2}\alpha_1^2 t + \alpha_2 y - \frac{1}{2}\alpha_2^2 \log(x+\sqrt{1+x^2}) + \alpha_3 z - \alpha_3^2 \sqrt{1+x^2} .$$

We interpret $S(x,y,z,\alpha_1,\alpha_2,\alpha_3,t)$ as the solution of

Hamilton-Jacobi equation for the extended system we are looking for. Under this hypothesis we define the canonical momenta by the following equations:

$$p_x = \frac{\partial S}{\partial x} = \alpha_1 - \frac{\alpha_2^2}{2}\frac{1}{\sqrt{1+x^2}} - \frac{\alpha_3^2}{2}\frac{x}{\sqrt{1+x^2}}$$

$$p_y = \frac{\partial S}{\partial y} = \alpha_2 \qquad (2.4)$$

$$p_z = \frac{\partial S}{\partial z} = \alpha_3$$

From the above eqs. we can obtain α_1, α_2, α_3 as functions* of $\vec{p}$ and $\vec{r}$.

We also have

$$H = -\frac{\partial S}{\partial t} = \frac{1}{2}\alpha_1^2 \qquad (2.5)$$

From eqs. (2.4) and (2.5) we finally obtain the hamiltonian for the system:

$$H = \frac{1}{2}\left[p_x + \frac{1}{2\sqrt{1+x^2}}(p_y^2 + xp_z^2)\right]^2 \qquad (2.6)$$

together with the constraint condition obtained from eqs. (2.3) and (2.4).

$$\chi \equiv p_y + p_z = 0 \qquad (2.7)$$

Before we proceed, let us show how the hamiltonian given by eq. (2.6) together with the constraint given by eq. (2.7) describes the motion given by eqs. (2.1). We first observe that H is independent of y and z and therefore both p_y and p_z are constants of motion as well as the constraint eq. (2.7). The eqs. for the velocities are given by:

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* $\vec{r}$ is the position vector of the particle.

$$\begin{aligned} \dot{x} &= \{x,H\} = p_x + \frac{1}{2\sqrt{1+x^2}}(p_y{}^2 + xp_z{}^2) = \sqrt{2H} \\ \dot{y} &= \{y,H\} = \frac{\dot{x}p_y}{\sqrt{1+x^2}} \\ \dot{z} &= \{z,H\} = \frac{x\dot{x}p_z}{\sqrt{1+x^2}} \end{aligned} \tag{2.8}$$

One observes from the above equations that $\dot{x}$ is a constant of motion and that

$$x\dot{y} + \dot{z} = \frac{x\dot{x}}{\sqrt{1+x^2}}(p_y + p_z) = 0$$

by the constraint eq. (2.7). The integration of eqs. (2.8) is easily performed and leads to the solutions of d'Alembert's equations given by eqs. (2.2).

We will now consider another example, the well-known system of a vertical disc pivoting and rolling on a horizontal plane. We will assume the mass, the moments of inertia and the radius of the disc are all equal to unity. Thus the kinetic energy of the system is given by

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2 + \dot{\phi}^2)$$

where θ is the coordinate associated to rolling, ϕ to pivoting, x and y to the point of contact of the disc with the horizontal plane.

The system has the two following constraints

$$\begin{aligned} dx - d\theta\cos\phi &= 0 \\ dy - d\theta\sin\phi &= 0 \end{aligned} \tag{2.9}$$

The eqs. of motion are

$$\dot{x} = \dot{\theta}\cos\phi$$
$$\dot{y} = \dot{\theta}\sin\phi$$
$$\ddot{\theta} = 0$$
$$\ddot{\phi} = 0$$

which can be easily integrated

$$\begin{aligned} \theta - \alpha_1 t &= \beta_1 \\ \phi - \alpha_2 t &= \beta_2 \\ x - \frac{\alpha_1}{\alpha_2}\sin\phi &= \beta_3 \\ y + \frac{\alpha_1}{\alpha_2}\cos\phi &= \beta_4 \end{aligned} \tag{2.10}$$

Let us now introduce two new parameters by the relations

$$\begin{aligned} \chi_1 &= \alpha_3 - \alpha_4 = 0 \\ \chi_2 &= \alpha_3\alpha_2 - \alpha_1 = 0 \end{aligned} \tag{2.11}$$

Following the same steps as in the preceding example we set:

$$\frac{\partial S}{\partial \alpha_1} = \theta - \alpha_1 t$$

$$\frac{\partial S}{\partial \alpha_2} = \phi - \alpha_2 t$$

$$\frac{\partial S}{\partial \alpha_3} = x - \alpha_3 \sin\phi$$

$$\frac{\partial S}{\partial \alpha_4} = y + \alpha_4 \cos\phi.$$

The principal Hamilton function $S(x,y,\theta,\phi,\alpha_1,\alpha_2,\alpha_3,\alpha_4,t)$ can be found from the above equations and we have

$$S = \alpha_1\theta + \alpha_2\phi + \alpha_3 x + \alpha_4 y - \frac{1}{2}(\alpha_1{}^2+\alpha_2{}^2)t - \frac{1}{2}(\alpha_3{}^2\sin\phi - \alpha_4{}^2\cos\phi).$$

The hamiltonian is found by setting

$$p_\theta = \frac{\partial S}{\partial \theta} = \alpha_1 \quad ,$$

$$p_\phi = \frac{\partial S}{\partial \phi} = \alpha_2 - (\frac{\alpha_3^{\,2}}{2} \cos\phi + \frac{\alpha_4^{\,2}}{2} \sin\phi) \quad ,$$

$$p_x = \frac{\partial S}{\partial x} = \alpha_3 \quad ,$$

$$p_y = \frac{\partial S}{\partial y} = \alpha_4 \quad ,$$

and

$$H = -\frac{\partial S}{\partial t} = \frac{1}{2} (\alpha_1^{\,2} + \alpha_2^{\,2}) \quad .$$

Thus we obtain

$$H = \frac{1}{2} [p_\theta^{\,2} + (p_\phi + \frac{p_x^{\,2}}{2} \cos\phi + \frac{p_y^{\,2}}{2} \sin\phi)^2] , \qquad (2.12)$$

while the constraint eqs. (2.9) take the form

$$\begin{aligned} \chi_1 &\equiv p_x - p_y = 0 \\ \chi_2 &\equiv (p_\phi + \frac{p_x^{\,2}}{2} \cos\phi + \frac{p_y^{\,2}}{2} \sin\phi)\, p_x - p_\theta = 0 . \end{aligned} \qquad (2.13)$$

It is easy to show that the canonical equations of motion, as derived from H given by eq. (2.12), lead to the solutions given by eqs. (2.10) when the constraint eqs. (2.11) are imposed upon. It can also be shown that χ_1 and χ_2 are constants of motion:

$$\{\chi_1, H\} = \{\chi_2, H\} = 0 .$$

One can extend these two systems by adding potential functions to the hamiltonians already found. In doing so, one has to be careful not to destroy the constancy of the constraint equations.

To illustrate this procedure, in the first example one has to introduce a potential function V(x,y,z) such as to satisfy the following equation

$$\{p_y+p_z,V\} = 0 \quad .$$

This eq. can easily be solved and we observe that in this case V must have the general form $V = V(x,y-z)$.

In the second example it is not so easy to find the general form of $V(x,y,\theta,\phi)$ but in any case we must impose

$$\{\chi_1,V\} = \{\chi_2,V\} = 0$$

in order to preserve the constancy of the constraints.

3. THE QUANTIZATION

Once the hamiltonian is found the quantization of the system may proceed by the standard procedure. We substitute, in coordinate representation, the momenta by their corresponding derivative operators*:

$$\begin{aligned} p_x &\to -i\ \partial_x \\ p_y &\to -i\ \partial_y \\ p_z &\to -i\ \partial_z \quad . \end{aligned}$$

Thus for the first example of the preceding section we have†

$$\hat{H} = -\frac{1}{2}\left[\frac{\partial}{\partial x} - \frac{i}{2\sqrt{1+x^2}}\left(\frac{\partial^2}{\partial y^2} + x\,\frac{\partial^2}{\partial z^2}\right)\right]^2 . \tag{3.1}$$

One observes that $\hat{H}$ is the square of a hermitian operator and is therefore itself hermitian.

The Schrödinger equation takes the form

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*We set $\hbar = 1$.

†The symbol ^ over the classical variable indicates the corresponding quantum operator.

$$- \frac{1}{2} \left[\frac{\partial}{\partial x} - \frac{i}{2\sqrt{1+x^2}} \left(\frac{\partial^2}{\partial y^2} + x \frac{\partial^2}{\partial z^2}\right)\right]^2 \psi = i \frac{\partial}{\partial t} \psi. \qquad (3.2)$$

The eigenfunctions of $\hat{H}$ are easily found and we obtained

$$\psi_{p_1,p_2,p_3}(x,y,z,t) = (2\pi)^{-3/2} \exp[i(p_1x+p_2y+p_3z) -$$
$$- \frac{i}{2} p_2^2 \log(x+\sqrt{1+x^2}) - \frac{i}{2} p_3^2 \sqrt{1+x^2} - \frac{i}{2} p_1^2 t]$$

for the solution of the Schrödinger equation with eigenvalues given by

$$E = \frac{1}{2} p_1^2.$$

We have normalized the eigenfunctions in the continuum using the condition

$$\int \psi^*_{\vec{p}}(x,y,z,t)\psi_{\vec{p}\,'}(x,y,z,t)dxdydz = \delta^3(\vec{p}-\vec{p}\,').$$

We must observe that among the set of eigenfunctions we have those that obey as well as those that do not obey the constraint equation ($\hat{p}_y+\hat{p}_z = 0$). If we had restricted the eigenfunctions only to those that obey the constraint equation we would not have obtained completeness and in particular it would be impossible to set as initial condition the localization of the particle in the neighbourhood of a given point of the configuration space.

As we will show in next section, in order to get back the classical non-holonomic behaviour of the particle one needs to impose the constraint equation only as an average over the initial state of the particle.

Let us now consider the second example. In this case the quantization can also proceed in a straightforward manner. The hamiltonian operator

$$\hat{H} = -\frac{1}{2}\{\frac{\partial^2}{\partial\theta^2} + [\frac{\partial}{\partial\phi} - \frac{i}{2}(\frac{\partial^2}{\partial x^2}\cos\phi + \frac{\partial^2}{\partial y^2}\sin\phi)]^2\},$$

being the sum of the squares of two hermitian operators is therefore hermitian.

The Schrödinger equation takes the form

$$-\frac{1}{2}\{\frac{\partial^2}{\partial\theta^2} + [\frac{\partial}{\partial\phi} - \frac{i}{2}(\frac{\partial^2}{\partial x^2}\cos\phi + \frac{\partial^2}{\partial y^2}\sin\phi)]^2\}\psi = i\frac{\partial}{\partial t}\psi$$

and the eigenfunctions of $\hat{H}$ can as well be easily obtained. We got

$$\psi_{p_1,p_2,m,n}(x,y,\theta,\phi) = (2\pi)^{-2}\exp[ip_1x + ip_2y +$$

$$+ im\theta + in\phi + \frac{i}{2}(p_1{}^2\sin\phi - p_2{}^2\cos\phi)]$$

with eigenvalues $E_{m,n} = \frac{1}{2}(m^2+n^2)$.

We have imposed the normalization

$$\int \psi^*_{p_1,p_2,m,n}\,\psi_{p_1',p_2',m',n'}\,dxdyd\theta d\phi = \delta(p_1-p_1')\delta(p_2-p_2')\delta_{mm'}\delta_{nn'}.$$

It is important to observe that the configuration space of this system is the cartesian product of a bidimensional euclidean space described by the coordinates x and y times the bidimensional torus described by the angular coordinates θ and ϕ . Therefore, the wave function must be cyclic in the coordinates θ and ϕ what imposes that the angular momentum for rolling and pivoting must be integer numbers (m and n respectively). The momenta associated to x and y coordinates of the contact point take the continuous values p_1 and p_2 respectively. Integration over θ and ϕ are restricted to the interval $[-\pi,\pi]$.

4. THE MOTION OF WAVE PACKETS

We will now discuss in some detail how to obtain back the motion of the classical constrained particle from the quantum description. We will exhibit the procedure making use of the first example.

Let us obtain the propagator $G(\vec{r},\vec{r}',t)$ for the particle. G must satisfy the equation

$$\left(\frac{\partial}{\partial t} + i\,\hat{H}\right)G = \delta(t)\,\delta^3(\vec{r}-\vec{r}') \qquad (4.1)$$

with the boundary condition of being zero for $t < 0$. Making use of the eigenfunctions of $\hat{H}$, we formally have

$$G(\vec{r},\vec{r}',t) = \sum_n \psi_n(\vec{r})\psi_n^*(\vec{r}')\, e^{-iE_n t} \quad , \qquad t > 0 \qquad (4.2)$$

Applying this formal solution to the system described by the hamiltonian given by eq. (3.1) we have:

$$G(\vec{r},\vec{r}',t) = \int \frac{d^3p}{(2\pi)^3} \exp[ip.(\vec{r}-\vec{r}') - \frac{i}{2}\,p_1^2\, t - \frac{i}{2}\,(p_2^2\beta + p_3^2\gamma)] \qquad (4.3)$$

where

$$\beta = \log(x + \sqrt{1+x^2}) - \log(x' + \sqrt{1+x'^2}) \qquad (4.4)$$

and

$$\gamma = \sqrt{1+x^2} - \sqrt{1+x'^2} \quad . \qquad (4.5)$$

The integrand in eq. (4.3) is gaussian and thus the integration can be performed analytically. We have:

$$G = [(2\pi i)^3\, t\beta\gamma]^{-1/2} \exp\{\frac{i}{2}\,[\frac{(x-x')^2}{t} + \frac{(y-y')^2}{\beta} + \frac{(z-z')^2}{\gamma}]\} \qquad (4.6)$$

One observes that

$$\lim_{t\to 0^+} G(\vec{r},\vec{r}',t) = \delta^3(\vec{r}-\vec{r}') \quad .$$

The quantum description for the initial conditions relative to the motion of the constrained particle is given by the wave function

$$\psi_0(x,y,z) = (\tfrac{\alpha}{\pi})^{3/4} \exp\{ - \tfrac{\alpha}{2}[(x-x_0)^2+(y-y_0)^2 + + (z-z_0)^2] + i\, p_1^0 x + i p_2^0 (y+z) \} \qquad (4.7)$$

where we imposed $\psi_0(x,y,z)$ to satisfy the constraint equation in the mean, i.e.:

$$<\psi_0|p_y+p_z|\psi_0> = 0 \qquad (4.8)$$

We therefore observe that ψ_0 depends only on five initial parameters: the initial mean position of the particle (x_0,y_0,z_0) and two initial mean momenta p_1^0 and p_2^0 corresponding to the contrained initial velocity components of the particle.

It is important to stress that one should not impose the constraint in the strong sense, i.e.:

$$(p_y+p_z)|\psi_0> = 0$$

because, if we had imposed the above equation, ψ_0 would only be a function of x and (y-z) and then it would be impossible to set as initial condition the localization of the particle in a certain neighbourhood of a given point (x_0,y_0,z_0). The consequence from the fact that the constraint equation is imposed only as an average over the initial state, is that the value of $\chi \equiv p_y+p_z$ fluctuates along the motion of the packet. We actually have

$$<\psi_0|\chi^2|\psi_0> = \alpha > 0$$

The evolution of the wave packet is given by:

$$\psi(\vec{r},t) = \int G(\vec{r},\vec{r}\,',t)\psi_0(\vec{r}\,')\, d^3r'$$

The integration in y' and z' can be performed analytically. The integration in x' is done approximately making use of the stationary phase method. This consists essentially in substituting x' by x_0 in the expressions for β and γ given by eqs. (4.4) and (4.5). Thus we obtained after some tedious calculations:

$$\psi(x,t) = \left(\frac{\alpha}{\pi}\right)^{3/4} [(1+i\alpha t)(1+i\alpha\beta_0)(1+i\alpha\gamma_0)]^{-1/2} \exp\phi$$

where

$$\beta_0(x) = \log\,(x + \sqrt{1+x^2}) - \log(x_0 + \sqrt{1+x_0{}^2})$$

$$\gamma_0(x) = \sqrt{1+x^2} + \sqrt{1+x_0{}^2}$$

and

$$\phi = \frac{i}{2}\left(\frac{x^2}{t} + \frac{y^2}{\beta_0} + \frac{z^2}{\gamma_0}\right) - \frac{\alpha}{2}(x_0{}^2+y_0{}^2+z_0{}^2) + \frac{[\alpha x_0 - \frac{i}{t}(x-p_1{}^0 t)]^2}{2(\alpha - \frac{i}{t})} + \frac{[\alpha y_0 - \frac{i}{\beta_0}(y-p_2{}^0\beta_0)]^2}{2(\alpha - \frac{i}{\beta_0})} + \frac{[\alpha z_0 - \frac{i}{\gamma_0}(z-p_2{}^0\gamma_0)]^2}{2(\alpha - \frac{i}{\gamma_0})} .$$

From this result, we finally arrive at the probability of finding the particle at $\vec{r}$ after a time interval t:

$$P(\vec{r},t) = \psi(\vec{r},t)\psi^*(\vec{r},t)$$

$$= \left(\frac{\alpha}{\pi}\right)^{3/2}(1+\alpha^2t^2)^{-1/2}(1+\alpha^2\beta_0^2)^{-1/2}(1+\alpha^2\gamma_0^2)^{-1/2} .$$

$$\exp\left\{-\frac{[x-x(t)]^2}{2[\Delta x(t)]^2} - \frac{[y-y(t)]^2}{2[\Delta y(t)]^2} - \frac{[z-z(t)]^2}{2[\Delta z(t)]^2}\right\}$$

where

$$x(t) = x_0 + p_1^0 t$$

$$y(t) = y_0 + p_2^0 [\log(x+\sqrt{1+x^2}) - \log(x_0+\sqrt{1+x_0^2})]$$

$$z(t) = z_0 - p_2^0 (\sqrt{1+x^2} - \sqrt{1+x_0^2})$$

are the solutions for the classical trajectory of the particle subject to the constraint $xdy + dz = 0$, with initial position (x_0, y_0, z_0) and initial velocity $(p_1^0, p_1^0 p_2^0/\sqrt{1+x_0^2}, -x_0 p_1^0 p_2^0/\sqrt{1+x_0^2})$.

The uncertainties in the position of the particle are given by

$$[\Delta x(t)]^2 = \frac{1}{2\alpha} + \frac{\alpha}{2} t^2$$

$$[\Delta y(t)]^2 = \frac{1}{2\alpha} + \frac{\alpha}{2} [\beta_0(x(t))]^2$$

$$[\Delta z(t)]^2 = \frac{1}{2\alpha} + \frac{\alpha}{2} [\gamma_0(x(t))]^2 \quad .$$

They have the usual interpretation. For example, $[\Delta y(t)]$ is composed of two terms: the first $(1/2\alpha)$ is the initial uncertainty squared and $\sqrt{\frac{\alpha}{2}}\beta_0(t)$ is the uncertainty in the y co-ordinate of the particle due to the initial uncertainty $\sqrt{\alpha/2}$ in the y-component of the momentum.

5. Conclusions

First we would like to point out that, though the examples considered are simple cases of non-holonomy, the procedure used is susceptible of generalizations.

Let us consider a particle of unit mass moving in a three dimensional euclidean space subject to the constraint

$$u(x,y)dy + dz = 0 \tag{5.1}$$

that generalizes the constraint of the first example of section 2. We will assume that

$$\frac{\partial u}{\partial x} \neq 0$$

what guarantees that the constraint is non-holonomic. The first integrals of motion are

$$\begin{aligned} \dot{x} &= \alpha_1 \\ \dot{y} &= \frac{\alpha_2}{\sqrt{1+u^2}} \\ \dot{z} &= -u\dot{y}\ . \end{aligned} \tag{5.2}$$

We have*

$$dy\sqrt{1+u^2} - \alpha_2\, dx = 0 \tag{5.3}$$

whose solution has the following form

$$\psi(x,y,\alpha_2) = \beta_2 \quad . \tag{5.4}$$

Solving the equation above for y, we have

$$y = f(x,\alpha_2,\beta_2) \quad . \tag{5.5}$$

Using eqs. (5.3) and (5.5) in eq. (5.2), we obtain:

$$z + \int^x \frac{\alpha_2\, u(x,f(x,\alpha_2,\beta_2))}{\sqrt{1+u^2(x,f(x,\alpha_2,\beta_2))}} = \beta_3$$

Calling

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*we have redefined α_2/α_1 as α_2.

$$g(x,\alpha_2,\beta_2) = \int^x \frac{\alpha_2\, u(x,f(x,\alpha_2,\beta_2))}{\sqrt{1+u^2(x,f(x,\alpha_2,\beta_2))}}\, dx$$

and using eqs. (5.4) we finally arrive at

$$z+g(x,\alpha_2,\psi(x,y,\alpha_2)) = \beta_3 \ .$$

We set

$$\alpha_2 = \alpha_3$$

and we obtain

$$\frac{\partial S}{\partial \alpha_1} = \beta_1 = x - \alpha_1 t$$

$$\frac{\partial S}{\partial \alpha_2} = \beta_2 = \psi(x,y,\alpha_2)$$

$$\frac{\partial S}{\partial \alpha_3} = \beta_3 = z + g(x,\alpha_3,\psi(x,y,\alpha_3))$$

From the equations above one finds $S(x,y,z,\alpha_1,\alpha_2,\alpha_3,t)$ and thus

$$p_x = \frac{\partial S}{\partial x} \quad ; \quad p_y = \frac{\partial S}{\partial y} \quad ; \quad p_z = \frac{\partial S}{\partial z}$$

and

$$H = -\frac{\partial S}{\partial t}$$

from where one proceeds in a straightforward manner for the determination of H and the constraint $X \equiv \alpha_2-\alpha_3 = 0$ in terms of $\vec{r}$ and $\vec{p}$ (*).

We think that the generalization of the procedure can be made in the following way. Let us consider a particle moving in a n-dimensional configuration space subject to m (m < n-1) non-holonomic constraints. The equation for the trajectory can be written as

$$q^i = q^i(t,\alpha_1,\ldots,\alpha_{n-m},\ \beta_1,\ldots,\beta_n) \qquad i = 1,\ldots,n$$

(*) One should observe that among the possible constraints of the form (5.1) there are non-liouvillian constraints as defined in [6].

where α_i's and β_i's are initial constants of motion respectively related to the initial velocity and position of the particle. We may rewrite the equations above as:

$$\psi_i(\alpha_1,\ldots,\alpha_{n-m},q,t) = \beta_i$$

The central issue in the procedure is to assume the existence of Hamilton's principal function from which the relations above are obtained. Thus, we introduce m new parameters $\{\alpha_{n-m+i};\ i = 1,\ldots,m\}$ and m constraint equations of the form

$$\chi_i(\alpha_1,\ldots,\alpha_n) = 0 \qquad i = 1,\ldots,m$$

We claim that by a judicious choice of the new m parameters and functions χ_i's , we can set

$$\left.\frac{\partial S}{\partial \alpha_i}\right|_{\{\chi\}=0} = \psi_i \qquad i = 1,\ldots,n$$

From the equations above we can find S and from it the momenta by:

$$p_i = \frac{\partial S}{\partial q_i}$$

We further have

$$\alpha_i = \alpha_i(q,p,t)$$

The hamiltonian is obtained by

$$H = -\frac{\partial S}{\partial t}$$

where we substitute α_i's by their expression in terms of p and q. The constraint equation are similarly obtained as

$$\chi_i(\alpha\ (p,q)) = 0$$

and we observe that

$$\{\chi_i, H\} = 0 \qquad i = 1,\ldots,m$$

because χ_i's are functions only of the α_i's.

We have only proven existence of S for the class of systems exhibited in this paper. We have not been able so far to prove this procedure for a general non-holonomic system.

Suppose that a hamiltonian H(q,p) together with the constraints $\chi_i(q,p)$, $i = 1,\ldots,m$ are found. The system can be generalized to include forces described by potential functions V(q) that obey the equations

$$\{\chi_i, V\} = 0 \quad ,$$

without going through the procedure of actually solving for the motion. Besides we observe that H is not unique. Any function h(p,q) which satisfies the condition

$$\left.\frac{\partial h}{\partial p_i}\right|_{\{\chi\}=0} = 0 \qquad ; \qquad \left.\frac{\partial h}{\partial q^i}\right|_{\{\chi\}=0} = 0$$

can be added to the hamiltonian without changing the dynamics of the non-holonomic system. We have not investigated whether this arbitrariness in fixing H reflects in a substantial way on the quantization of the non-holonomic system.

Let us now observe that, for a given configuration space, the set of holonomic constraints is closed[4] what implies the impossibility of reaching the dynamics of non-holonomic systems by continuation of the holonomic behaviour. The situation is reversed if we start from the non-holonomic behaviour. In this case every holonomic system can be reached continuously from the non-holonomic set. For example, if we take the family of constraints

$$x\,dy + \varepsilon\,dz = 0$$

labelled by the parameter ε, we have for $\varepsilon = 1$ the first example of section 2 and for $\varepsilon = 0$ the holonomic constraint

$$dy = 0 \quad .$$

Now, the hamiltonian given by eqs. (2.6) does not go continuously to the known hamiltonian for the holonomic case above, even though the laws of motion are correctly obtained. We are not sure whether this agreement is necessary for the correct quantization of the non-holonomic system.

Finally we would like to point out that the relations between p and $\dot{q}$ are not linear. This implies that the lagrangian may have an unusual dependence on the velocity of the particle. For the first example of section 2, we found

$$L(\vec{r},\dot{\vec{r}}) = \frac{1}{2}\,\dot{x}^2 + \frac{1}{2}\,\frac{\sqrt{1+x^2}}{x\dot{x}}\,(x\dot{y}^2 + \dot{z}^2)$$

while the constraint takes its usual form

$$x\dot{y} + \dot{z} = 0 \quad . \tag{5.6}$$

One may verify that Euler's equations obtained from the lagrangian above gives back d'Alembert's eqs. (2.1) when one imposes eq. (5.6) which is itself a constant of motion.

In the Appendix we have commented on Eden's method for quantization of non-holonomic systems as presented in reference [3]. We will now draw the main disagreements with what we have repported here. Eden bases his dynamics on the free hamiltonian while we actually found specific hamiltonians for each constrained system, by intermediation of Hamilton's principal function. The hamiltonians we found are substancially different from the free hamiltonians. It is an assumption in our approach that Hamilton's principal function is a regular function what is justified by the results we obtained. In reference [3], due to the use of the free hamiltonian, Hamilton's principal function comes out to be an indefinite function in the sense that mixed second derivatives do not commute. The equivalent constraint equations ($\chi_i = 0$) in our hamiltonian formalism are constants of motion. In reference [3] this is not the case resulting to be the main reason for the indefiniteness of Hamilton's principal function. For us, at the quantum level, the constraint equations have to imposed only in the weak sense, i.e., as an average value over the initial state of the system. Due to the constancy of the constraint equations along the motion, it stays valid in average at every subsequent instant of time. The opposite happens in Eden's approach: his constraint equations are imposed in the strong sense, i.e., the initial state has to be an eigenstate of the constraint, what requires the development of the system be regarded as a continuous adjustment of its apparent initial conditions.

From the fact that we take the constraints only in the weak sense, we are allowed to imagine, making use of the path integral formalism, the classical non-holonomic trajectory resulting from the coherent interference of its neighbouring trajectories that violates the constraint law but having carefully constructed phases necessary to build up the classical motion.

We may therefore conclude in a loose sense that non-

-holonomy disappears at the quantum level and only comes about when one considers semi-classical limits.

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APPENDIX

Eden's paper[3] is the only one known to us to have dealt with the problem of quantization of non-holonomic systems. As his procedure to quantize non-holonomic systems is formal and his conclusions are qualitative in nature, we will illustrate his procedure by discussing one example. In doing so, we are aware that some of the subtle points discussed by Eden may have been overlooked by us.

To fix our ideas let us consider a particle of unit mass moving in a three dimensional space subject to the constraint

$$\vec{a}(\vec{r}).d\vec{r} = 0$$

The hamiltonian H for the unconstrained motion is called by Eden the locally free hamiltonian and we have

$$H = \frac{1}{2}\,\vec{p}^{\,2}$$

with

$$\dot{\vec{r}} = \vec{p}$$

and

$$\vec{p} = 0$$

for the equations of the free motion.

The constrained variables are introduced as

$$\vec{r}^{\,*} = \vec{r}$$
$$\vec{p}^{\,*} = Q\,\vec{p} \qquad \text{(A.1)}$$

where Q is the matrix whose elements are:

$$Q_{ij} = \delta_{ij} - a_i(\vec{r})\; a_j(\vec{r}) \qquad \text{(A.2)}$$

with

$$\vec{a}(\vec{r}).\vec{a}(\vec{r}) = 1 \quad .$$

The constrained variables (r^*,p^*) do not form canonical pairs and their equations of motion are[(*)]

$$\dot{\vec{r}}^* = \{\vec{r}^*,H\} = \vec{p}^* + \vec{a}\,\Omega$$

$$\dot{\vec{p}}^* = \{\vec{p}^*,H\} = -\,\vec{a}\,\frac{\partial a_j}{\partial x_k^*}\,p_j^* p_k^* - \Omega\left(\vec{a}\,\frac{\partial a_j}{\partial x_k^*}\,p_j^* a_k + \frac{\partial \vec{a}}{\partial x_k^*}\,p_k^*\right)$$

with $\Omega = \vec{a}.\vec{p}$.

If we set $\Omega = 0$ in the above equations then $\vec{r}^*$ and $\vec{p}^*$ obey the dynamics of the non-holonomic system as given by d'Alembert's principle

$$\dot{\vec{r}}^* = \vec{p}^*$$

$$\dot{\vec{p}}^* = -\,\vec{a}\,\frac{\partial a_j}{\partial x_k^*}\,p_j^* p_k^* \qquad \text{(A.3)}$$

and

$$\vec{a}.\dot{\vec{r}}^* = 0$$

Calculating $\dot{\Omega}$ we obtain

$$\dot{\Omega} = \{\Omega,H\} = \frac{\partial a_j}{\partial x_k^*}\,p_j^* p_k^* + \Omega\,\frac{\partial a_j}{\partial x_k^*}\,p_j^* a_k$$

from which we see that for Ω to be a constant of motion one must have

$$\frac{\partial a_j}{\partial x_k^*}\,p_j^* p_k^* = 0$$

i.e, the strength of the reaction force must be zero. This is a rather trivial case and in general the reaction force does not vanish. Eden suggests that at every instant of time we have to reset $\Omega = 0$ in order that the system continue to obey eqs. (A.3).

The quantization is done by using a coordinate representation and writing the Schrödinger equation as

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(*) $\{x_i\}$, $i = 1,2,3$ are the components of the vector $\vec{r}$.

$$i \frac{\partial}{\partial t} <\vec{r}|\phi = -\frac{1}{2} \frac{\partial^2}{\partial x_i^2} <\vec{r}|\phi \tag{A.4}$$

supplemented by the equation(*)

$$a_i(\vec{r}) \frac{\partial}{\partial x_i} <\vec{r}|\phi = 0 \tag{A.5}$$

in analogy to the classical procedure. From eqs. (A.4) and (A.5) one may conclude that in general we must have

$$\frac{\partial^2}{\partial x_i \partial x_j} <\vec{r}|\phi \neq \frac{\partial^2}{\partial x_j \partial x_i} <\vec{r}|\phi$$

what shows that $<\vec{r}|\phi$ is an indefinite function of $\vec{r}$, i.e., the difference of $<\vec{r}|\phi$ taken at two points depends on the path connecting these two points.

In order to avoid the indefiniteness in the probability itself, Eden assumes that

$$<\vec{r}|\phi = e^{i\beta} <\vec{r}|\psi \tag{A.6}$$

where $<\vec{r}|\psi$ is a regular function of $\vec{r}$ and β is a hermitian operator found by Eden to be given by

$$\frac{\partial \beta}{\partial x_i} = - a_i(\vec{r})(\vec{a}(\vec{r}).\vec{p}^*) \ . \tag{A.7}$$

From eqs. (A.6) and (A.7) we obtain

$$\vec{p}<\vec{r}|\phi = e^{i\beta} [\vec{p}-\vec{a}(\vec{a}.\vec{p})]<\vec{r}|\psi \tag{A.8}$$

Thus we have

$$\Omega<\vec{r}|\phi = e^{i\beta} \vec{a}.[\vec{p}-\vec{a}(\vec{a}.\vec{p})]<\vec{r}|\psi = 0$$

what shows that eq. (A.5) is identically satisfied. We obtain the equation for $<\vec{r}|\psi$ from eqs. (A.4) and (A.8):

(*) Symmetrization to make dynamical variables hermitian will not be shown explicitly.

$$i \frac{\partial}{\partial t} \langle \vec{r} | \psi = - \frac{1}{2} \frac{\partial}{\partial x_i} (Q_{ij} \frac{\partial}{\partial x_j}) \langle \vec{r} | \psi \quad . \tag{A.9}$$

This equation shows that the particle described by $\langle \vec{r} | \phi$ has the same classical motion as that described by the classical hamiltonian

$$H^* = \frac{1}{2} Q_{ij} \, p_i p_j$$

which does not give eqs. (A.3) that are the correct equations of motion[4].

LIST OF REFERENCES

[1] H. Weyl, Gruppentheorie und Quantenmechanik 2nd. ed. Hirzel, Leipzig 1931. (English translation, Dover, New York).

[2] P.A.M. Dirac, Can. J. Math. 2, 129 (1950).

[3] R.J. Eden, Proc. Roy. Soc. A 205, 564 (1951).
R.J. Eden, Proc. Roy. Soc. A 205, 583 (1951).

[4] L.C. Gomes, R. Lobo, F.R.A. Simão, Rev. Bras. Fís., 11,241 (1981).

[5] H. Goldstein, Classical Mechanics, Addison-Wesley Pub. Co., Inc., Reading, Massachusetts, 1950.

[6] P. Carrilho Soares, L.C. Gomes, Rev. Bras. Fís., 12, 261 (1982).