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An application of supersymmetric quantum mechanics to a planar physical system

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Abstract

Supersymmetry (SUSY) in non-relativistic quantum mechanics (QM) is applied to a 2-dimensional physical system: a neutron in an external magnetic field. The superpotential and the two-component wave functions of the ground state are found out.

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The algebraic technique of supersymmetry in quantum mechanics (SUSY QM) was first introduced by Witten [1]. The essential idea of this formulation is based on the Darboux procedure on second-order differential equations, which has been successfully utilized to achieve a supersymmetric generalization of the harmonic-oscillator raising and lowering operators for shape-invariant potentials [2,3]. The SUSY algebra has also been applied to construct a variety of new one-parameter families of isospectral supersymmetric partner potentials in quantum field theory [4]. The shape-invariance conditions in SUSY have been independently generalized for systems described by two-component wave functions [5]. Recently, we have found a two-by-two matrix superpotential associated to the linear classical stability from the static solutions for a system of two coupled real scalar fields in (1+1)-dimensions [6].

We also presented an integral representation for the momentum space Green's function for a neutron in interaction with a static magnetic field of a straight current carrying wire, which is also described by two-component wave functions [7]. The SUSY QM formalism was also applied to this planar physical system in the momentum [8] and coordinate [9] representations.

In this letter, we consider the notation of Ref. [8]. However, according to our developments, we can realize the supersymmetric algebra in coordinate representation, introducing some transformations in the original system corresponding to a neutron interacting with the magnetic field of a linear current carrying conductor, so that we are able to implement a comparasion with both superpotentials for the cases corresponding to currents located along x and z directions.

Now, let us consider an electrically neutral spin-$\frac{1}{2}$ particle of mass $M = 1$ and magnetic moment $\mu\vec{\sigma}$ (a neutron) interacting with an infinite straight wire carrying a current I and located along the z-axis. The magnetic field generated by the wire is given by (we use units with $c = \hbar = 1$)

$$\vec{B} = 2I\frac{(-y, x, 0)}{(x^2 + y^2)}, \tag{1}$$

where x and y are Cartesian coordinates of the plane perpendicular to the wire.

The Hamiltonian associated with the physical system is given by

$$H=\frac{\vec{p}^{\,2}}{2M}+\mu\vec{\sigma}.\vec{B}=\frac{\vec{p}^{\,2}}{2}+2I\mu\frac{(-y\sigma_1+x\sigma_2)}{(x^2+y^2)}, \tag{2}$$

where $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ are Pauli matrices. The motion along the z-axis is free and will be ignored in what follows and in this way we get a two-dimensional problem.

Due to the translational symmetry in the z-direction, the two-component wave function $\psi(\rho,k)$ can be written as

$$\begin{aligned}\psi^{(n_\rho)}(\rho,k)&=\frac{1}{\sqrt{4\pi L}}\left\{\tilde{\psi}_1^{(n_\rho,m)}(\rho,k)e^{im\phi}\begin{pmatrix}1\\0\end{pmatrix}+\tilde{\psi}_2^{(n_\rho,m)}(\rho,k)e^{i(m+1)\phi}\begin{pmatrix}0\\1\end{pmatrix}\right\}e^{i\frac{2\pi}{L}kz}\\&\equiv\begin{pmatrix}\psi_1^{(n_\rho)}(\rho,k)\\ \psi_2^{(n_\rho)}(\rho,k)\end{pmatrix},\end{aligned} \tag{3}$$

where $n_\rho=0,1,2,\cdots$ is the radial quantum number; $k=0,1,2,\cdots$; $m=0,\pm1,\pm2,\cdots$; ρ,ϕ,z are the usual cylindrical coordinates and the parameter L is the macroscopic length of the conductor.

Therefore, the Schrödinger equation splits up into a system of two coupled second order differential equations as follows

$$\begin{aligned}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\tilde{\psi}_1^{(n_\rho,m)}\right)-\frac{m^2}{\rho^2}\tilde{\psi}_1^{(n_\rho,m)}+2\tilde{E}\tilde{\psi}_1^{(n_\rho,m)}+\frac{2F}{\rho}\tilde{\psi}_2^{(n_\rho,m)}&=0\\ \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\tilde{\psi}_2^{(n_\rho,m)}\right)-\frac{(m+1)^2}{\rho^2}\tilde{\psi}_2^{(n_\rho,m)}+2\tilde{E}\tilde{\psi}_2^{(n_\rho,m)}+\frac{2F}{\rho}\tilde{\psi}_1^{(n_\rho,m)}&=0\end{aligned} \tag{4}$$

where

$$F=-\frac{\mu_0\mu I}{2\pi} \tag{5}$$

and

$$\tilde{E}=E-\frac{2\pi k^2}{L^2}. \tag{6}$$

Note that Eq. (4) is exactly Eq. (2.8) given in [8]. Now, using the relation

$$\tilde{\psi}_i^{(n_\rho,m)}=\rho^{-\frac{1}{2}}\phi_i^{(n_\rho,m)}\quad(i=1,2), \tag{7}$$

we can write the system in (4) in the matrix form as

$$\begin{pmatrix}-\frac{d^2}{d\rho^2}+\frac{m^2-\frac{1}{4}}{\rho^2}-2\tilde{E} & \frac{-2F}{\rho}\\ \frac{-2F}{\rho} & -\frac{d^2}{d\rho^2}+\frac{(m+1)^2-\frac{1}{4}}{\rho^2}-2\tilde{E}\end{pmatrix}\begin{pmatrix}\phi_1^{(n_\rho,m)}\\ \phi_2^{(n_\rho,m)}\end{pmatrix}=0, \tag{8}$$

which corresponds to a one-dimensional Schrödinger-like equation associated with the two-component wave function. Therefore, we get the eigenvalue equations

$$\mathbf{H}_1\mathbf{\Phi}_1^{(n_\rho,m)} = \tilde{E}_1^{(n_\rho,m)}\mathbf{\Phi}_1^{(n_\rho,m)}, \quad E_1^{(n_\rho,m)} = 2\tilde{E}^{(n_\rho,m)}, \tag{9}$$

where

$$\mathbf{\Phi}_1^{(n_\rho,m)} = \mathbf{\Phi}_1^{(n_\rho,m)}(\rho,k) = \begin{pmatrix} \phi_1^{(n_\rho,m)}(\rho,k) \\ \phi_2^{(n_\rho,m)}(\rho,k) \end{pmatrix}, \quad \mathbf{H_1} = -\mathbf{1}\frac{d^2}{d\rho^2} + \begin{pmatrix} \frac{m^2-\frac{1}{4}}{\rho^2} & \frac{-2F}{\rho} \\ \frac{-2F}{\rho} & \frac{(m+1)^2-\frac{1}{4}}{\rho^2} \end{pmatrix}. \tag{10}$$

Defining

$$\mathbf{H}_1 \equiv \mathbf{A}^+\mathbf{A}^- + \mathbf{1}\tilde{E}_1^{(0)}, \quad \mathbf{A}^\pm = \pm\frac{d}{d\rho} + \mathbf{W}(\rho), \tag{11}$$

we obtain the following Riccati equation in matrix form

$$\mathbf{W}'(\rho) + \mathbf{W}^2(\rho) + \mathbf{1}\tilde{E}_1^{(0)} = \begin{pmatrix} \frac{m^2-\frac{1}{4}}{2\rho^2} & \frac{-2F}{\rho} \\ \frac{-2F}{\rho} & \frac{(m+1)^2-\frac{1}{4}}{2\rho^2} \end{pmatrix}, \tag{12}$$

where $\mathbf{W}(\rho)$ is a two-by-two superpotential matrix. The hermiticity condition allows us to write

$$\mathbf{W} = \mathbf{W}^\dagger = \begin{pmatrix} f(\rho) & g(\rho) \\ g(\rho) & h(\rho) \end{pmatrix} \tag{13}$$

where f, g and h are real functions and satisfy the nonlinear system of differential equations

$$\begin{cases} f' + f^2 + g^2 + E_1^{(0)} &= \frac{m^2-\frac{1}{4}}{\rho^2} \\ fg + hg + g' &= \frac{-2F}{\rho} \\ h' + h^2 + g^2 + E_1^{(0)} &= \frac{(m+1)^2-\frac{1}{4}}{\rho^2}. \end{cases} \tag{14}$$

Now, let us try a solution for equation (14) assuming that g is constant. Then, we have

$$f + h = -\frac{2F}{g\rho}, \tag{15}$$

which gives

$$f' - h' - \frac{2F}{g\rho}(f-h) + \frac{2m+1}{\rho^2} = 0. \tag{16}$$

Solving the last equation and imposing finiteness condition on the solutions, we get

$$f(\rho) = \frac{b}{\rho},$$
$$h(\rho) = \frac{c}{\rho}, \tag{17}$$

where b and c are arbitrary constants. Substituting these solutions into the system (14), we find that a consistent solution is possible only if

$$g = -\frac{F}{m+1} \tag{18}$$

where F is defined in Eq. (5). Then, turning to Eq. (16) and substituting Eqs. (17) and (18) we find constants b and c. Putting these results back into Eq. (17), we have that

$$f(\rho) = \frac{m+\frac{1}{2}}{\rho}$$
$$h(\rho) = \frac{m+\frac{3}{2}}{\rho}. \tag{19}$$

In this case, the two almost isospectral Hamiltonians are given by

$$\mathbf{H}_1 = \mathbf{A}^+\mathbf{A}^- - \frac{F^2}{2(m+1)^2}\mathbf{1} \tag{20}$$
$$\mathbf{H}_2 \quad = \mathbf{A}^-\mathbf{A}^+ - \frac{F^2}{2(m+1)^2}\mathbf{1}. \tag{21}$$

Since $\mathbf{A}^+\mathbf{A}^-$ is positive semidefinite, according to (11) and (20) the energy eigenvalue of the ground state is

$$\tilde{E}^{(0)} = -\frac{F^2}{2(m+1)^2}, \tag{22}$$

with the annihilation conditions

$$A^-\Phi_1^{(0)} = 0 \tag{23}$$

and

$$A^+\Phi_2^{(0)} = 0 \tag{24}$$

and the new superpotential

$$\mathbf{W}_m = \begin{pmatrix} \frac{m+\frac{1}{2}}{\rho} & -\frac{F}{m+1} \\ -\frac{F}{m+1} & \frac{m+\frac{3}{2}}{\rho} \end{pmatrix}. \tag{25}$$

The energy eigenvalues of magnetically bound excitated states in terms of the radial quantum number n_ρ, for $m \geq m_0$ becomes

$$\tilde{E}^{(n_\rho)} = -\frac{F^2}{2(n_\rho + m_0 + 1)^2}. \tag{26}$$

Now let us to determine the eigenfunction associated with the ground state given by Eq.(23). To do this let us consider the transformations

$$\phi(\rho) = \chi^{(0)}\rho^{m+\frac{1}{2}}, \quad \rho = 2(m+1)\eta, \quad F = -\frac{1}{2}, \tag{27}$$

which implies that Eq. (23) turns into the following matrix diferential equation

$$\mathbf{1}\frac{d}{d\eta}\chi(\eta) = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{\eta} \end{pmatrix} \chi^{(0)}(\eta), \quad \chi^{(0)}(\eta) = \begin{pmatrix} \chi_1^{(0)} \\ \chi_2^{(0)} \end{pmatrix}, \tag{28}$$

so that we obtain the following equations for the components $\chi_1^{(0)}$ and $\chi_2^{(0)}$:

$$\begin{aligned} \frac{d}{d\eta}\chi_1^{(0)}(\eta) &= \chi_2^{(0)}(\eta), \\ \frac{d}{d\eta}\chi_2^{(0)}(\eta) &= \chi_1^{(0)}(\eta) + \frac{1}{\eta}\chi_2^{(0)}(\eta), \end{aligned} \tag{29}$$

which leads us a second-order differential equation for $\chi_2^{(0)}(\eta)$, viz.,

$$\frac{d^2}{d\eta^2}\chi_2^{(0)}(\eta) - \frac{1}{\eta}\frac{d}{d\eta}\chi_2^{(0)}(\eta) + \left(\frac{1}{\eta^2} - 1\right)\chi_2^{(0)}(\eta) = 0. \tag{30}$$

From equations (3), (29) and (30) we obtain the m-dependent normalizable ground state

$$\Psi^{(0)}(\rho) = C_m \rho^{m+1} \begin{pmatrix} e^{im\phi} K_1\left(\frac{\rho}{2m+2}\right) \\ e^{i(m+1)\phi} K_0\left(\frac{\rho}{2m+2}\right) \end{pmatrix} e^{i\frac{2\pi}{L}kz} \tag{31}$$

where C_m is the normalization constant, and $K_1\left(\frac{\rho}{2m+2}\right)$ and $K_0\left(\frac{\rho}{2m+2}\right)$ are the modified Bessel functions. The eigenfunction $\Psi^{(0)}(\rho)$ is in according with the result found via momentum representation in Ref. [8]. Note that the complete solution of Eq. (30)

$$\chi_2^{(0)}(\eta) = \eta\left(c_1 K_0(\eta) + c_2 I_0(\eta)\right), \tag{32}$$

where c_1 and c_2 are arbitrary constants, are non-normalizable. Therefore, in order to get a normalizable solution, we choose $c_2 = 0$ and in this way we drop $I_0(\eta)$ which is divergent when $\eta \to \infty$.

It is worthy noticing that under a unitary transformation, $\mathbf{U}\mathbf{W}_m\mathbf{U}^{-1} = \tilde{\mathbf{W}}_m$, this superpotential, together with the interchange of m by $m+\frac{1}{2}$, and taking $F = -\frac{1}{2}$ becomes that superpotential matrix ($\mathbf{W}_{LJM}$) shown in [9], viz., $\tilde{\mathbf{W}}_{m+\frac{1}{2}} = -\mathbf{W}_{LJM}$. This minus sign that connects $\tilde{\mathbf{W}}_{m+\frac{1}{2}}$ and $\mathbf{W}_{LJM}$ is associated to the fact that we have chosen the first-order differential operator $\mathbf{A}^-$ with the opposite sign in the derivative term of the operator A_m considered in Ref. [9].

Using the coordinate representation, we investigate the SUSY in non-relativistic quantum mechanics with two-component eigenfunctions and find a new realization of supersymmetry in a planar physical system of a neutron in interaction with a straight current-carrying wire.

The $N = 2-$SUSY superalgebra has the following representation

$$\mathbf{H}_{SUSY} = [Q_-, Q_+]_+ = \begin{pmatrix} \mathbf{A}^+\mathbf{A}^- & 0 \\ 0 & \mathbf{A}^-\mathbf{A}^+ \end{pmatrix}_{4X4} = \begin{pmatrix} \mathbf{H}_- = \mathbf{H} & 0 \\ 0 & \mathbf{H}_+ \end{pmatrix}, \quad (33)$$

where the supersymmetric partners are given by $\mathbf{H}_- = \mathbf{H}_1 - \mathbf{1}E_1^{(0)}$, $\mathbf{H}_+ = \mathbf{H}_2 - \mathbf{1}E_1^{(0)}$ and the supercharges $Q_\pm$ are 4 by 4 matrix differential operators of first order and can be given by

$$Q_- = \begin{pmatrix} 0 & 0 \\ \mathbf{A}^- & 0 \end{pmatrix}_{4X4}, \quad Q_+ = \begin{pmatrix} 0 & \mathbf{A}^+ \\ 0 & 0 \end{pmatrix}_{4X4}. \quad (34)$$

We have seen that, in non-relativistic quantum mechanics applied to two-component eigenfunctions, if $\Phi_1^{(0)}$ is a normalizable two-component eigenfunction, one cannot write $\Phi_2^{(0)}$ in terms of $\Phi_1^{(0)}$, as in the case of ordinary supersymmetric quantum mechanics. This may be shown in the system considered here of a neutron interacting with an external magnetic field with the current of the conductor in the z direction. Only in the case of 1-component wave functions one may write the superpotential as $W(x) = \frac{d}{dx}\ell n\,(\psi_0(x))$.

The hermiticity condition satisfied by the superpotential, in the general case, leads us to a method that permits to solve the matrix Riccati equation. As a final remark, we would like to draw the attention to the fact that our result, for a superpotential corresponding to a neutron in an external magnetic field in the coordinate representation, is related by the following unitary transformation, $\mathbf{U} = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3)$, where σ_1 and σ_3 are the Pauli matrix, with a new superpotential so that, after the substituition m by $m + \frac{1}{2}$ (the total angular momentum along the wire direction) it reduces to the superpotential recently found in [9], where a current I along the x axis of a Cartesian system is considered.

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