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Triality of Majorana-Weyl Spacetimes with Different Signatures

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Abstract

Majorana-Weyl spacetimes offer a rich algebraic setup and new types of space-time dualities besides those discussed by Hull. The triality automorphisms of $Spin(8)$ act non-trivially on Majorana-Weyl representations and Majorana-Weyl spacetimes with different signatures. In particular relations exist among the $(1+9) \leftrightarrow (5+5) \leftrightarrow (9+1)$ spacetimes, as well as their transverse coordinates spacetimes $(0+8) \leftrightarrow (4+4) \leftrightarrow (8+0)$. Larger dimensional spacetimes such as $(2+10) \leftrightarrow (6+6) \leftrightarrow (10+2)$ also show dualities induced by triality. A precise three-languages dictionary is here given. It furnishes the exact translations among, e.g., the three different versions (one in each signature) of the ten-dimensional $N = 1$ superstring and superYang-Mills theories. Their dualities close the six-element permutation group S_3. Bilinear and trilinear invariants allowing to formulate theories with a manifest space-time symmetry are constructed.

Key-Words: Triality; Duality; Supersymmetry.

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1 Introduction.

Physical theories formulated in different-than-usual spacetimes signatures have recently found increased attention. One of the reasons can be traced to the conjectured F-theory [1] which supposedly lives in $(2+10)$ dimensions [2]. The current interest in AdS theories motivated by the AdS/CFT correspondence furnishes another motivation. Two-time physics e.g. has started been explored by Bars and collaborators in a series of papers [3]. For another motivation we can also recall that a fundamental theory is expected to explain not only the spacetime dimensionality, but even its signature (see [4]). Quite recently Hull and Hull-Khuri [5] pointed out the existence of dualities relating different compactifications of theories formulated in different signatures. Such a result provides new insights to the whole question of spacetime signatures. Other papers (the most recent is [6]) have remarked the existence of space-time dualities.

Majorana-Weyl spacetimes (i.e. those supporting Majorana-Weyl spinors) are at the very core of the present knowledge of the unification via supersymmetry, being at the basis of ten-dimensional superstrings, superYang-Mills and supergravity theories (and perhaps the already mentioned F-theory). A well-established feature of Majorana-Weyl spacetimes is that they are endorsed of a rich structure. A legitimate question one can ask oneself is whether they are affected, and how, by space-time dualities. The answer is quite surprising (in fact it should not be so, with afterthought), the structure of dualities is much richer than expected and potentially able to shed a complete new light on the subject. Indeed all different Majorana-Weyl spacetimes which are possibly present in any given dimension are each-other related by duality transformations which are induced by the $Spin(8)$ triality automorphisms. The action of the triality automorphisms is quite non-trivial and has far richer consequences than the $\mathbf{Z}_2$-duality (its most trivial representative) associated to the space-time $(s,t) \leftrightarrow (t,s)$ exchange discussed in [4]. It corresponds to S_3, the six-element group of permutations of three letters, identified with the group of congruences of the triangle and generated by two reflections. The lowest-dimension in which the triality action is non-trivial is 8 (not quite a coincidence), where the spacetimes $(8+0)-(4+4)-(0+8)$ are all interrelated. They correspond to the transverse coordinates of the $(9+1)-(5+5)-(1+9)$ spacetimes respectively, where the triality action can also be lifted. Triality relates as well the 12-dimensional Majorana-Weyl spacetimes $(10+2)-(6+6)-(2+10)$, i.e. the potentially interesting cases for the F-theory, and so on.
As a consequence of triality, supersymmetric theories formulated with Majorana-Weyl spinors in a given dimension but with different signatures, are all dually mapped one into another. A three-language dictionary is here furnished with the exact translations among, e.g., the different versions of the 10-dimensional MW supersymmetric theories, formulated in the Majorana-Weyl representation.

The strategy here followed is based in three steps. At first it is shown that Majorana-Weyl spacetimes in dimensions $d > 8$ can be recovered from the properties of the 8-dimensional Majorana-Weyl spacetimes and Γ-matrices representations. Next, working in $d = 8$, we construct, for each one of the three Majorana-Weyl spacetimes $(8+0)$, $(4+4)$, $(0+8)$, the "bridge transformations" relating the corresponding Majorana-Weyl representations to the representations (called "VCA" in the text) which exhibit manifest triality among vectors, chiral and antichiral spinors. As a final step new "bridge transformations" of spacetime kind, relating among them the VCA representations constructed in each one of the Majorana-Weyl spacetimes above, are given.

We emphasize that, contrary to Hull [5], the dualities here discussed are already present for the *uncompactified* theories and in this respect look more fundamental.

Moreover, bilinear and trilinear invariants under the S_3 permutation group of the three Majorana-Weyl spacetimes are constructed. They can be possibly used to formulate supersymmetric Majorana-Weyl theories in a manifestly triality-invariant form which presents an explicit symmetry under exchange of space and time coordinates.

The present paper is intended to be an abridged version, suitable for a letter-size, of a forthcoming extended version which presents in full detail the construction and where extra results which are outside the scope of this letter are also furnished.

The scheme of this work is as follows. In the next section we recall, following [7] and [8], the basic properties of Γ-matrices and Majorana conditions needed for our construction. Majorana-type representations are analyzed in section 3. We show how to relate the Majorana-Weyl representations in $d > 8$ to the 8-dimensional Majorana-Weyl representations. In section 4 we introduce, for $d = 8$, the set of data necessary to define a supersymmetric Majorana-Weyl theory, i.e. the set of "words" of our three-languages dictionary. The Cartan's [9] triality among vectors, chiral and antichiral spinors is presented in section 5. The main result is furnished in section 6, where spacetime triality is discussed. In the Conclusions we furnish some comments and point out some perspectives.

2 Preliminary results.

Here we limit ourselves to introduce the basic ingredients needed for our constructions. Further information is found in [7] and [8].

We denote as g_{mn} the flat (pseudo-)euclidean metric of a $(t + s)$-spacetime. Time (space) directions in our conventions are associated to the $+$ (respectively $-$) sign.

The Γ's matrices are assumed to be unitary (the chosen normalization is for the square of time-like Γ-matrices being $+1$). The three matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are the generators of the three conjugation operations (hermitian, complex conjugation and transposition respectively) on the Γ's. In particular

$$\mathcal{C}\Gamma^m\mathcal{C}^\dagger = \eta(-1)^{t+1}\Gamma^{m\,T} \tag{1}$$

where $\eta = \pm 1$ in even-dimensional spacetimes label inequivalent choices of the charge conjugation matrix $\mathcal{C}$.

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ are related by the formula

$$\mathcal{C} = \mathcal{B}^T\mathcal{A} \tag{2}$$

Up to an inessential phase, $\mathcal{A}$ is specified by the product of all the time-like Γ matrices. An unitary transformations U applied on spinors act on Γ^m, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ according to [8]

$$\begin{aligned}
\Gamma^m &\mapsto U\Gamma^m U^\dagger \\
\mathcal{A} &\mapsto U\mathcal{A}U^\dagger \\
\mathcal{B} &\mapsto U^*\mathcal{B}U^\dagger \\
\mathcal{C} &\mapsto U^*\mathcal{C}U^\dagger
\end{aligned} \tag{3}$$

A Majorana representation for the Γ's can be defined as the one in which $\mathcal{B}$ is set equal to the identity. Spinors can be assumed real in this case.

In even dimensions we can also introduce the notion of Weyl representation, i.e. when the "generalized Γ^5 matrix" is symmetric and block diagonal and with no loss of generality can be assumed to be the direct sum of the two equal-size blocks $\mathbf{1} \oplus (-\mathbf{1})$. The compatibility of both Majorana and Weyl conditions constraints the spacetime $(t+s)$ to satisfy

$$s - t \;=\; 0 \quad mod \quad 8, \quad for \quad both \quad values \quad \eta = \pm 1 \tag{4}$$

In even dimensions Majorana representations, but not of Weyl type, are also found for

$$\begin{aligned} s - t &= 2 \quad mod \quad 8 \quad for \quad \eta = -1; \\ s - t &= 6 \quad mod \quad 8 \quad for \quad \eta = +1. \end{aligned} \tag{5}$$

For $d < 8$ the only spacetimes supporting Majorana-Weyl spinors have signatures $(n+n)$. At $d = 8$ a new feature arises, Majorana-Weyl spinors can be found for different signatures.

3 Majorana-type representations.

It is convenient to introduce the notion of Majorana-type representation (or shortly MTR) for the Γ matrices as one in which all the Γ's have a definite symmetry. For $d = p + q$ a MTR with p symmetric and q antisymmetric Γ's will be denoted as (p_S, q_A) in the following.

For such representations the $\mathcal{C}$ matrix introduced in the previous section is given by either the product of all the symmetric Γ matrices, denoted as $\mathcal{C}_S$, or all the antisymmetric ones ($\mathcal{C}_A$). In even dimensions $\mathcal{C}_S$, $\mathcal{C}_A$ correspond to opposite values of η in (1).

A Majorana representation in a given signature spacetime is a MTR. Conversely, given a MTR, we can find a spacetime signature for which the representation is Majorana. The admissible couples of (p_S, q_A) values for a MTR are immediately read from the Majorana tables given above (4) and (5). The construction is such that $\mathcal{C}$ must correspond to the correct value of η in the tables.

The list of all possible MTR's in any given dimension is easily computed. In order just to give an example one can check that in $d = 6$ there exists a MTR (not of Weyl kind) with 6 anticommuting Γ matrices plus an anticommuting Γ^7 $(0_S, 6_A, {\Gamma^7}_A)$. It gives the Majorana basis in the Euclidean 6-dimensional space.

In $d = 8$ the MTR's of Weyl type are $(8_S, 0_A)$, $(4_S, 4_A)$, $(0_S, 8_A)$, associated to the corresponding Majorana-Weyl spacetimes.

Different MTR's belong to different classes under similarity transformations of the Γ's representations. Indeed in, let's say, an euclidean (all + signs) space, the index

$$I = tr(\Gamma^m \cdot {\Gamma_m}^T) = (p_S - q_A) \cdot tr\mathbf{1} \tag{6}$$

takes different values for different MTR's.

We computed explicitly all MTR's up to $d = 12$ and Majorana-Weyl representations up to $d = 14$ (the results will be furnished elsewhere) by using a recursive algorithm presented in [10]. It allows producing Weyl representations in d dimensions from any given couple of representations in r and s dimensions, for even-dimensional d, r, s satisfying $d = r + s + 2$. The only MTR up to $d = 12$ which does not directly fit into this scheme, the above-mentioned 6-dimensional $(0_S, 6_A)$, is however constructed from the $(3_S, 3_A)$ representation (this one directly produced from the 2-dimensional Pauli matrices for $r = s = 2$) after computing the value of the symmetric matrix $\mathcal{B}$ in the euclidean 6-dimensional space, and later finding the transformation (3) which maps it into the unity.

The algorithm is given by the formula

$$\begin{aligned}
{\Gamma_d}^{i=1,\ldots,s+1} &= \sigma_x \otimes \mathbf{1}_L \otimes {\Gamma_s}^{i=1,\ldots,s+1} \\
{\Gamma_d}^{s+1+j=s+2,\ldots,d} &= \sigma_y \otimes {\Gamma_r}^{j=1,\ldots,r+1} \otimes \mathbf{1}_R
\end{aligned} \tag{7}$$

where $\mathbf{1}_{L,R}$ are the unit-matrices in the respective spaces, while $\sigma_x = e_{12} + e_{21}$ and $\sigma_y = -ie_{12} + ie_{21}$ are the 2-dimensional Pauli matrices. ${\Gamma_r}^{r+1}$ corresponds to the "generalized Γ^5-matrix" in $r+1$ dimensions. In the above formula the values $r, s = 0$ are allowed. The corresponding ${\Gamma_0}^1$ is just 1.

With the help of the above formula we have a very efficient tool to reduce the analysis of Majorana-Weyl representations for $d \geq 10$ to the 8-dimensional case, since we can always set either r or s equal to 8 (or possibly both, which corresponds to the $d = 18$ case). Up to $d = 14$ Majorana-Weyl spacetimes exist for three different signatures and the same situation of $d = 8$ is repeated. A new feature arises for $d \geq 16$. A careful analysis of the consistency conditions is needed in this case, since the triality transformations that we later discuss no longer preserve the similarity classes of MTRs; stated otherwise, different representatives of MTR's in the same similarity class and for the same couple of values (p_S, q_A) are mapped under a given triality transformation into representatives of MTR belonging to different similarity classes. This feature is likely to be related with the problems encountered in defining supersymmetric theories in dimensions greater than 14, which have been e.g. pointed out in [11].

In any case the construction here discussed is suited to analyze and works perfectly well for the range $d = 8, \ldots, 14$, i.e. the cases which are of interest for, among the others, the superstrings and the F-theory (notice that the above scheme can find applications to dualities also for odd-dimensional Majorana spacetimes as the 11-dimensional ones supporting the M-theory, we will comment more on that in the conclusions). We postpone to the extended version of this paper the presentation of the full set of reconstruction formulas which make explicit the construction of MW-spacetimes in dimensions $d > 8$ in terms of the 8-dimensional ones. For the purpose of this paper is sufficient to recall that such reconstruction formulas can be given. In particular the higher dimensional $\mathcal{C}$ charge conjugation matrices for $d > 8$ can be expressed in terms of the eight-dimensional matrices $\mathcal{C}_8$.

4 The set of data for $d = 8$.

In this section we present the set of data needed to specify a Majorana-Weyl supersymmetric theory formulated in 8-dimensions. As we have stated in the previous section, this set of data can be "lifted" to define Majorana-Weyl supersymmetric theories formulated in higher-dimensions. The results here furnished therefore have a more general validity.

At first we recall that we have three different Majorana-Weyl spacetimes $(8+0)-(4+4)-(0+8)$ and two choices for $\eta=\pm 1$, so in total $3\times 2=6$ inequivalent theories (i.e. inequivalent versions of some given supersymmetric theory) that can be formulated in $d=8$.

Each one of these versions is characterized by the following set of data *all expressed in the corresponding Majorana-Weyl representation*, being this one the most suitable for analyzing supersymmetry. Such data will play the roles of the "words" in the three-languages dictionary that will be later furnished:

i) the bosonic (and/or vector-fields) coordinates x_m, with vector index $m=1,...,8$;

ii) the fermionic coordinates (and/or spinorial fields) ψ_a, $\chi_{\dot{a}}$, with chiral and antichiral indices $a=1,...,8$, $\dot{a}=1,...,8$ respectively;

iii) the diagonal (pseudo-)orthogonal spacetime metric $(g^{-1})^{mn}$, g_{mn};

iv) the $\mathcal{A}$ matrix of section 2, used to introduce barred spinors, which is now decomposed in an equal-size block diagonal form such as $\mathcal{A}=A\oplus\tilde{A}$, with structure of indices ${(A)_a}^b$ and ${(\tilde{A})_{\dot{a}}}^{\dot{b}}$ respectively;

v) the charge-conjugation matrix $\mathcal{C}$, always symmetric, also put in equal-size block diagonal form $\mathcal{C}=C^{-1}\oplus\tilde{C}^{-1}$. Since $\mathcal{C}$ is invariant under bispinorial transformations it can be promoted to be a metric for the space of chiral (and respectively antichiral) spinors, used to raise and lower spinorial indices. Indeed we can set $(C^{-1})^{ab}$, $(C)_{ab}$, and $(\tilde{C}^{-1})^{\dot{a}\dot{b}}$, $(\tilde{C})_{\dot{a}\dot{b}}$;

vi) finally we have the upper-right σ and the lower-left $\tilde{\sigma}$ blocks in the Γ's matrices with structure of indices ${(\sigma_m)_a}^{\dot{b}}$ and ${(\tilde{\sigma}_m)_{\dot{a}}}^b$ respectively.

The $\mathcal{B}$ matrix is automatically set to be the identity ($\mathcal{B}=\mathbf{1}$) due to our choice of working in the Majorana-Weyl representation.

In order to work in the Majorana-Weyl basis for each one of the six different versions of the theory, the correct Majorana-type representation must be picked up. In $(4+4)$ the $(4_S,4_A)$ representation must be chosen for both values of η, while in $(8+0)$ the $(8_S,0_A)$ works for $\eta=+1$ and the $(0_S,8_A)$ works for $\eta=-1$ (and conversely in the $(0+8)$ case).

For later purpose it is convenient to present the matrix $\mathcal{C}$ for each one of the six versions. We have

$$\begin{array}{lllr}
\mathcal{C} &=& \Gamma^9=\mathbf{1}_8\oplus(-\mathbf{1}_8) & in\quad (8+0)\\
\mathcal{C} &=& \mathbf{1}_{16} & in\quad (0+8)\\
\mathcal{C} &=& (C^{-1})\oplus(\eta C^{-1}) & in\quad (4+4)
\end{array}\tag{8}$$

where in the last case C^{-1} can be chosen to be the 8×8 matrix with $(+--++--+)$ entries in the antidiagonal and 0 entries in any other position.

5 The V-C-A triality.

The outer automorphisms of the D_4 Lie algebra is responsible for the triality property among the 8-dimensional vector, chiral and antichiral spinor representations of $SO(8)$ which has been first discussed by Cartan [9].

In our language the triality property can be restated as follows. The C^{-1}, $\tilde{C}^{-1}$ matrices introduced in the previous section, for each one of the six different cases we discussed, are symmetric and with the same set (up to an overall sign) of eigenvalues (± 1) as the corresponding g^{-1} spacetime metric.

In each one of the above cases one can simultaneously map both C^{-1}, $\tilde{C}^{-1}$ (with different similarity transformations $\mathcal{G}$, $\tilde{\mathcal{G}}$) into the corresponding spacetime g^{-1} metric:

$$\begin{array}{llll} \mathcal{G}: & C^{-1} & \mapsto & g^{-1} = G\cdot C^{-1}\cdot G^T \\ \tilde{\mathcal{G}}: & \tilde{C}^{-1} & \mapsto & g^{-1} = \tilde{G}\cdot \tilde{C}^{-1}\cdot \tilde{G}^T \end{array} \qquad (9)$$

The structure of indices for G and its inverse G^{-1} is $(G)^m{}_a$ and $(G^{-1})^a{}_m$ (an analogous structure holds for $\tilde{G}$, and $\tilde{G}^{-1}$). Therefore G, $\tilde{G}$ can be used to transform chiral (antichiral) indices in vector indices allowing to work with, let's say, vector indices alone.

The concrete 8×8-dimensional matrices G, $\tilde{G}$ are of course not uniquely defined since any other matrix of the kind $L_C\cdot G\cdot L_g$, with L_C, L_g preserving by similarity the C^{-1} and the g^{-1} metrics respectively, are equally well suited. For our purposes however it is sufficient to furnish a concrete representative for the G, $\tilde{G}$ matrices.

In all the above cases we can choose the concrete matrices G, $\tilde{G}$ to be square root of unity:

$$G^2 = \tilde{G}^2 \;\; = \;\; \mathbf{1}_8 \qquad (10)$$

Indeed the only case in the previous section discussion where C^{-1} ($\tilde{C}^{-1}$) is not diagonal is the $(4+4)$ case, with C^{-1} given in (8) and $\tilde{C}^{-1} = \pm C^{-1}$. A matrix G which sets C^{-1} to the diagonal form $(++++----)$ is

$$G = \frac{1}{\sqrt{2}}\cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad (11)$$

In all the other cases we have to flip a number n of signs in the diagonal, with $n = 0$ *mod* 4. Instead of working with the standard Wick rotation prescription, which is the only one applicable for *odd* numbers of signs to be flipped, a smarter choice is allowed for *even* numbers of flipping: the passage e.g. from $(++)\mapsto(--)$ can be produced via similarity with the help of the σ_y Pauli matrix $\sigma_y = -ie_{12} + ie_{21}$, through

$$\sigma_y\cdot \mathbf{1}_2\cdot {\sigma_y}^T \;\; = \;\; -\mathbf{1}_2 \qquad (12)$$

Of course σ_y satisfies ${\sigma_y}^2 = \mathbf{1}_2$ and is antisymmetric. The bridge matrices G which flip the euclidean $(++++++++)$ metric into the $(++++----)$ and the $(--------)$ metrics are therefore respectively given by

$$\begin{array}{lll} G_1 & = & \mathbf{1}_4\oplus\sigma_y\oplus\sigma_y \\ G_2 & = & \sigma_y\oplus\sigma_y\oplus\sigma_y\oplus\sigma_y \end{array} \qquad (13)$$

which are both square root of unity.

The above given bridge operators G, $\tilde{G}$ in a given Majorana-Weyl spacetime allows to pass from the Majorana-Weyl representation to another representation, that we can call VCA, where triality is manifest and only vector-like indices are present. Please notice that, as far as transformation properties alone are concerned, the commuting or anticommuting nature of spinors is not taken into account. For commuting spinors a more radical property holds. Bilinear and trilinear invariants under the S_3 permutation group of vectors, chiral and antichiral spinors, can be constructed. The procedure is as follows. At first the three bilinear scalars

$$B_V = V^T \eta^{-1} V, \qquad B_C = \psi^T C^{-1} \psi, \qquad B_A = \chi^T \tilde{C}^{-1} \chi \tag{14}$$

and the trilinear one

$$T \;\; = \;\; \psi^T C^{-1} \sigma \eta^{-1} V \chi \tag{15}$$

are constructed. Applying the above bridge transformations in a passive way we can set

$$\hat{\psi} = (G^T)^{-1} \psi, \qquad\qquad \hat{\chi} = (\tilde{G}^T)^{-1} \chi \tag{16}$$

Therefore

$$B_V = V^T \eta^{-1} V, \qquad B_C = \hat{\psi}^T \eta^{-1} \hat{\psi}, \qquad B_A = \hat{\chi}^T \eta^{-1} \hat{\chi} \tag{17}$$

and their sum

$$B \;\; = \;\; B_V + B_C + B_A \tag{18}$$

is by construction invariant under the S_3 exchange of $V, \hat{\psi}, \hat{\chi}$.

As for the trilinear term T, it reads as follows

$$T \;\; = \;\; \hat{\psi}^T M V \hat{\chi} \tag{19}$$

where the trivector M is given by

$$M^{mnp} \;\; = \;\; (\eta^{-1})^{mr} {(G^{T^{-1}})_r}^a {(\sigma_q)_a}^{\dot{b}} (\eta^{-1})^{qn} {(\tilde{G}^T)_{\dot{b}}}^p \tag{20}$$

A trilinear scalar, invariant under S_3, is constructed through

$$\sum_{perm} M^{mnp} V_m \hat{\psi}_n^T \hat{\chi}_p \tag{21}$$

where the sum is extended over all the permutations of $V, \hat{\psi}, \hat{\chi}$.

The action of the S_3 permutation group on the original vectors, chiral and antichiral spinors is given by the pull-back of the bridge transformations. It is just sufficient to write it down for two of the generators, called P, R, of S_3, where

$$P^2 = R^2 = \mathbf{1} \qquad\qquad (PR)^3 = \mathbf{1} \tag{22}$$

We have, symbolically,

$$\begin{array}{lll} P: & V \mapsto V, & \psi \leftrightarrow \chi \\ R: & \chi \mapsto \chi, & V \leftrightarrow \psi \end{array} \quad (23)$$

6 The spacetime triality.

The triality discussed in the previous section is the Cartan's $V - C - A$ triality, which connects vectors, chiral and antichiral spinors of the *same* spacetime. However the procedure which has been used so far can be repeated to connect vectors belonging to spacetime metrics with different signatures. In the case of our interest three spacetimes, denoted as X, Y, Z, possess vector-indices $(m, \tilde{m}, \overline{m})$ which are referred to the metrics $(++++++++)$, $(++++----)$, $(--------)$ respectively.

The passage from one of the above metrics to another one can be done by employing the same bridge matrices introduced in (13). The construction straightforwardly repeat the one already encountered. It should be clear that an enormous technical advantage is offered by performing the connection between two different Majorana-Weyl spacetimes working in both cases with the respective VCA representations. The "spacetime bridge matrices" in this case only see vector indices. The connection between Majorana-Weyl representations is then reconstructed in terms of the bridge matrices (introduced in the previous section) linking, in each one of the two spacetimes, the Majorana-Weyl with the VCA representation.

There is no need to repeat here the formulas presented in section 5. Each one finds its "mirror spacetime" equivalent. They have just to be reinterpreted in the light of the spacetime triality.

We just point out that under spacetime triality $Y_{\tilde{m}}$, $Z_{\overline{m}}$ are mapped into

$$Y_{\tilde{m}} \mapsto \hat{Y}_m, \qquad Z_{\overline{m}} \mapsto \hat{Z}_m \quad (24)$$

carrying a $(++++++++)$ vector index-structure.

The bilinear invariant under the S_3 group of permutations is

$$B = X^T {\eta_X}^{-1} X + \hat{Y}^T {\eta_X}^{-1} \hat{Y} + \hat{Z}^T {\eta_X}^{-1} \hat{Z} \quad (25)$$

while the trilinear one is

$$T = \sum_{perm} M^{mnp} X_m \hat{Y}_n^T \hat{Z}_p \quad (26)$$

where the trivector M^{mnp} is constructed in full analogy with (20).

We remark that for what concerns spacetime triality invariances it is likely that we do not have to bother about the anticommuting character of spinors as it is the case for invariances under $V - C - A$ triality. For instance, in the simplest example, the bosonic supersymmetric composite vector $X^m = x^m - i\overline{\theta}\Gamma^m\theta$ is the building block to introduce the superparticle and we do not need to worry about Grassmann variables.

Let us finally comment that the same bridge transformations already encountered can be used to produce dualities relating the opposite values of η. Indeed just an overall $-$ sign for the $\tilde{C}^{-1}$ matrix distinguishes the two cases.

7 Conclusions.

In this paper we have shown that the triality automorphisms of $Spin(8)$ not only induce dualities among different-signatures Majorana-Weyl spacetimes, but also furnish bilinear and trilinear invariants which can be possibly used to formulate supersymmetric theories possessing a space-time triality invariance.

Triality relations connect the various Majorana-Weyl spacetimes in a given dimension.

The basic strategy of our construction consists in the fact that even for dimensions $d > 8$ the different signatures of Majorana-Weyl spacetimes can be encoded in the 8-dimensional Γ matrices, used as building blocks in the construction of the higher-dimensional Γ's.

The bridge operators connecting in each given Majorana-Weyl spacetime the Majorana-Weyl representation to a Cartan-type representation in which the 8-dimensional $V-C-A$ triality is manifest are helpful in linking together different-signatures Majorana-Weyl spacetimes. Indeed in a Cartan-like basis the problem of relating different space-time signatures is considerably simplified since we have to worry about just how to connect vector-like indices and spacetime metrics.

A complete and extensive list of the results here outlined will be presented in a forthcoming paper.

The range of possible applications for the methods and the ideas here discussed is vast. We limit ourselves to mention that we are currently investigating the web of dualities connecting the 12-dimensional Majorana-Weyl spacetimes which should support the F-theory (3×2, taking into account of η), with the 6 versions of the 11-dimensional Majorana spacetimes (for the M-theory) in $(10+1)$, $(9+2)$, $(6+5)$, $(5+6)$, $(2+9)$, $(1+10)$ signatures and with the different (again 3×2) versions of the 10-dimensional Majorana-Weyl spacetimes.

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