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What do very nearly flat detectable cosmic topologies look like?

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Abstract

Recent studies of the detectability of cosmic topology of nearly flat universes have often concentrated on the range of values of Ω_0 given by current observations. Here we study the consequences of taking a range of bounds satisfying $|\Omega_0 - 1| \ll 1$, which include those expected from future observations such as the Planck mission, as well those predicted by inflationary models. We show that in this limit, a generic detectable non-flat manifold is locally indistinguishable from either a cylindrical ($\mathbb{R}^2 \times \mathbb{S}^1$) or toroidal ($\mathbb{R} \times \mathbb{T}^2$) manifold, irrespective of its global shape, with the former being more likely. Importantly this is compatible with some recent indications based on the analysis of high resolution CMB data. It also implies that in this limit an observer would not be able to distinguish topologically whether the universe is spherical, hyperbolic or flat. By severely restricting the expected topological signatures of detectable isometries, our results provide an effective theoretical framework for interpreting cosmological observations, and can be used to confine any parameter space which realistic search strategies, such as the 'circles in the sky' method, need to concentrate on.

Key-words: Detection of cosmic topology; Inflationary models; Cosmic topology; Observational cosmology; Nearly flat universes; CMB data; FLRW models.

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Two fundamental questions regarding the nature of the universe concern its geometry and topology. Regarding geometry, recent high precision data by WMAP [1] have provided strong evidence suggesting that the universe is nearly flat, with the ratio of its total matter–energy density to the critical value, Ω_0, very close to one. They, however, do not fix the sign of its curvature.[1] The near flatness of the universe is compatible with inflationary models which further predict this ratio to be extremely close to one. Regarding topology, most recent studies of the detectability of the cosmic topology have so far concentrated on the range of values of Ω_0 given by current observations [5, 6]. These works have demonstrated that although the detectability of any given topology depends on the curvature radius, there still remain infinite topologies that are in principle detectable for any given Ω_0.

Here we take a range of bounds satisfying $|\Omega_0 - 1| \ll 1$, and examine what would a detectable topology look like in such nearly flat universes. We study the local nature of detectable hyperbolic and spherical manifolds and discuss the significance of our results for search strategies for the detection of the topology of the universe.

We assume that the universe can be locally modelled by a Friedmann-Lemaître-Robertson-Walker (FLRW) metric and consider the possibility that the 3-space may be a multiply connected manifold, $M = \widetilde{M}/\Gamma$, where $\widetilde{M} = \mathbb{S}^3$ or $\mathbb{H}^3$, and Γ is a fixed-point free and discrete group of isometries of $\widetilde{M}$.

Since we study the topology of universes with curved spatial sections, we need to express the redshift–distance relation in units of the curvature radius, which depends on the matter–energy content of the universe as well as Ω_0. Even though current observations favor the content to be well approximated by dust plus a dark energy, its actual details do not affect our considerations below, and thus in this sense our results are robust. This is important since the precise nature of the dark energy is at present not known.

A natural way to study the detectability of topology is through the lengths of closed geodesics. For an isometry $g \in \Gamma$ and a point $x \in M$, the length of the closed geodesic generated by g is given by its distance function $d(x, gx)$, i.e. the distance between x and its image gx. This readily allows the definition of the injectivity radius $r_{inj}(x)$ as half the length of the smallest closed geodesic passing through x. One can also define an injectivity radius for the whole manifold M as $r_{inj} = \inf_{x\in M} r_{inj}(x)$, which is the radius of the smallest sphere inscribable in M. A necessary condition for detectability of cosmic topology is then given by $r_{inj}(x) < \chi_{obs}$, where χ_{obs} is the redshift–distance relation evaluated at the maximum redshift ($z = z_{max}$) of the survey used. Similarly, a sufficient condition for detectability is $r_{max} < \chi_{obs}$, where $r_{max} = \sup_{x\in M} r_{inj}(x)$ is the maximum injectivity radius.

To study the local shape of the detectable non–flat manifolds in the density limit suggested by inflationary models, $|\Omega_0 - 1| \ll 1$, we make three physically motivated assumptions: (i) the observer is at a position x where topology is detectable, (ii) the survey depth is very small in units of the curvature radius, and (iii) the topology is not excludable, i.e. it does not produce too many images so as to make it already observationally excludable. [2] Therefore in the following our main physical assumption will be

$$r_{inj}(x) \lesssim \chi_{obs} \ll 1 \; . \qquad (1)$$

We note that although in general smaller values of χ_{obs} increase the number of unde-

[1] The most recent estimates of the density parameters specify this region to be $\Omega_0 \in [0.99, 1.05]$ and $\Omega_\Lambda \in [0.69, 0.79]$ with a 2σ confidence, which still leaves open the sign of the curvature of the cosmic geometry.

[2] A sufficient condition for a topology to be excludable is $r_{max} \ll \chi_{obs}$.

tectable topologies, nevertheless for any value of χ_{obs}, no matter how small, there will always remain an infinite number of topologies which are in principle detectable.

An important class of isometries are the Clifford translations (CT), defined as those with constant distance functions. We shall encounter isometries whose distance functions do not vary appreciably inside the observable sphere of radius χ_{obs}. This motivates the definition of an isometry g as CT–like inside a given (here the observable) sphere if the difference between the maximum and minimum lengths of all closed geodesics generated by g that lie within the sphere is sufficiently small ($\ll 1$). In the following we consider the hyperbolic and spherical spaces separately, and prove that in the limit (1) all isometries that generate observable images generically behave CT–like.

It is known that the set of compact orientable hyperbolic manifolds can be ordered in a sequence of sequences of manifolds. The manifolds in each individual sequence are ordered by increasing volume, with a cusped manifold as its limit. This implies that for a sufficiently large index i a compact manifold M_i looks like a cusped manifold, and we shall therefore refer to such manifolds as cusped–like. An important feature of such cusped and cusped-like manifolds is that they possess regions (namely the cusp regions) where $r_{inj}(x)$ takes small values (zero in the limiting case). Therefore for a given χ_{obs}, in any sequence we can always choose i such that for all $j > i$, the cusped–like regions of the manifolds M_j satisfy (1). Thus the compact hyperbolic manifolds that have detectable topologies are the cusped-like ones in their cusp regions.

Such cusped–like regions are well approximated by pure cusp manifolds, whose covering group is generated by two parabolic isometries with the same fixed point at infinity. Since these isometries commute, the fundamental group of a pure cusp is $\mathbb{Z}\times\mathbb{Z}$, thus, topologically, a pure cusp is equivalent to $\mathbb{R}\times\mathbb{T}^2$. In principle, the length scales of both generators are mutually independent, however if one of the length scales is larger than the diameter of the observable sphere, then the detectable part of the topology will be that of a horn, which is equivalent to $\mathbb{R}^2\times\mathbb{S}^1$. Within all known cusped–like manifolds, pure cusps generically have one length scale much smaller than the other, which therefore makes the latter case more likely.

We can formulate the previous argument more precisely by studying the CT–like behavior of a parabolic isometry in the limit (1). A convenient model of a hyperbolic space for our analysis is the upper–half space model, $\mathbb{H}^3 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_3 > 0\}$. The distance between two points $x, y \in \mathbb{H}^3$ in this model is given by

$$\cosh(d(x,y)) = 1 + \frac{|x-y|^2}{2x_3y_3} \,, \tag{2}$$

and a parabolic isometry can always be put in the form $gx = (x_1 + L, x_2, x_3)$, with $L > 0$, so that, under the condition (1), its distance function reduces to $d(x, gx) = \frac{L}{x_3}$.

We can obtain a bound on the variation of this distance function inside a detectable sphere of radius $\chi_{obs} \ll 1$ by computing the difference between the maximum and minimal lengths of closed geodesics associated with g inside this sphere. Up to second order this gives

$$\frac{\Delta d}{d_0} < \frac{1}{2}\,\frac{d_{\max} - d_{\min}}{d_0} \simeq \chi_{obs} \,, \tag{3}$$

where d_0 is the distance function of g evaluated at the center of the sphere, and $\Delta d = |d(x, g(x)) - d_0|$. This demonstrates that under condition (1), parabolic isometries behave

CT–like inside the observable universe, thus an observer living in a horn or cusped region, may not distinguish, even topologically, her universe as being hyperbolic or flat.

We shall now consider the spherical spaces and recall that any spherical 3-manifold is a quotient $\mathbb{S}^3/\Gamma$, where Γ is a finite subgroup of $SO(4)$ acting freely on the 3-sphere. The classification of spherical 3-dimensional manifolds is well known [7]. Since Γ is a finite group, any element $g \in \Gamma$ is of finite order, thus Γ contains cyclic subgroups. Generically, the injectivity radius is determined by the cyclic subgroup of largest order of Γ [8]. We shall therefore initially consider the action of cyclic groups on $\mathbb{S}^3$.

A cyclic group $\mathbb{Z}_p$ may act on $\mathbb{S}^3$ in different ways parametrized by an integer q such that p and q are relatively prime, and $1 \leq q < p/2$. These actions give rise to the lens spaces $L(p,q)$, whose global injectivity radii depend only on p and are given by $r_{inj} = \frac{\pi}{p}$. The lens spaces $L(p,1)$ are globally homogeneous, thus in the following we shall only consider inhomogeneous lens spaces ($q \geq 2$ and $p \geq 5$). We note that for any given Ω_0 there will always exist a p_* such that the infinite set of lens spaces with $p > p_*$ remain detectable.

Representing the 3-sphere as the subset of pairs $z = (z_1, z_2)$ of complex numbers of unit length, $|z|^2 = |z_1|^2 + |z_2|^2 = 1$, let $g_{(p,q)}$ be the generator of the covering group of $L(p,q)$, and $g^n z$ the image of z by its n-th iteration. The distance between z and its n-th image is then given by

$$\cos(d(z, g^n z)) = \cos\left(\frac{2\pi n}{p}\right)|z_1|^2 + \cos\left(\frac{2\pi n q}{p}\right)|z_2|^2 \,. \tag{4}$$

In order to determine the local nature of a lens space we shall require some results from number theory (see e.g. [9]).[3] We start by computing the iterate of $g_{(p,q)}$ which generates the smallest geodesic at a given point z. For points on the great circle $(z_1, 0)$ the smallest geodesic is the one corresponding to the generator (or its inverse). This, however, is not the case for points on the great circle $(0, z_2)$. In fact, in this case the n-th iteration of $g_{(p,q)}$ will generate the smallest geodesic if n is chosen such that $nq = \pm 1 \mod (p)$. The existence of this integer is guaranteed by Euler's theorem, according to which $q^{\phi(p)} = 1 \mod (p)$, where $\phi(p)$ is the number of integers which are relative prime to and smaller than p. Thus, the smallest geodesic for any point on $(0, z_2)$ will be generated by the choice of $n = \pm q^{\phi(p)-1} \mod (p)$. Moreover, the only points for which $r_{inj}(z) = \pi/p$ lie on the great circles given by $(z_1, 0)$ and $(0, z_2)$. These circles are the equators of minimum injectivity radius, and will be important in the following considerations.

It can be shown that the following generalization holds: For any point in the lens space $L(p,q)$ there is a convergent $\frac{q_j}{p_j}$ such that the isometry that generates the smallest geodesic is g^{p_j}. An immediate consequence of this is that the number of candidate isometries that may generate the smallest geodesic is k. Furthermore, $k \leq \log_2 4q^2$ and thus k is small even

[3] Any rational number $\frac{q}{p} < 1$, with q and p relative primes, can be written as a continued fraction

$$\frac{q}{p} = \frac{1}{a_2 + \frac{1}{a_3 + F_4}} \,,$$

with $F_n < 1$, thus allowing the representation $\frac{q}{p} \equiv [0, a_2, a_3, ..., a_k]$ with $a_i \in \mathbb{N}$, which is unique if we demand $a_k > 1$. The convergents are defined as $c_i = \frac{q_i}{p_i} = [0, a_2, a_3, ..., a_i]$ for $i \leq k$, where q_i and p_i which are relative primes can be obtained recursively, the sequences $\{q_i\}$ and $\{p_i\}$ are strictly increasing and $p_{k-1} < \frac{p}{2}$. It can further be shown that

$$\frac{q}{p} - \frac{q_i}{p_i} = \frac{(-1)^{i+1}}{p_i(p_{i+1} + p_i F_{i+2})} \equiv (-1)^{i+1}\frac{G_i}{p_i} \,. \tag{5}$$

for lens spaces with large p and q. Therefore, this result provides an effective procedure for obtaining the length of the smallest geodesic at any point. Moreover, using Eq. (5) from the footnote, and recalling that for any integer m, $\cos(2\pi m + s) = \cos(|s|)$, we have

$$\cos\left(2\pi \frac{p_i q}{p}\right) = \cos\left(\frac{2\pi}{p}\,|p_i q - p q_i|\right) = \cos(2\pi G_i)\ , \tag{6}$$

with $i = 1, ..., k$. Thus, since $G_k = 1/p$, for points on the equator $(0, z_2)$ the isometry which generates the smallest geodesics is $g^{p_{k-1}}$, and therefore $q^{\phi(p)-1} \equiv \pm p_{k-1} \bmod(p)$. Now For any $n = p_j$, we can rewrite (4) as

$$\cos(d(z, g^{p_j} z)) = \cos\left(\frac{2\pi p_j}{p}\right)|z_1|^2 + \cos(2\pi G_j)|z_2|^2\,. \tag{7}$$

Choosing j from now on such that $p_j \leq \sqrt{p} \leq p_{j+1}$ we have $\frac{2\pi p_j}{p} \leq \frac{2\pi}{\sqrt{p}}$ and $2\pi G_j \leq \frac{2\pi}{\sqrt{p}}$. To obtain an upper bound on the length of the smallest closed geodesics at any point, we substitute these inequalities into (7) obtaining $\cos(d(z, g^{p_j} z)) \geq \cos\left(\frac{2\pi}{\sqrt{p}}\right)$, for all z in the 3-sphere. We then have

$$r_{\max} \leq \frac{1}{2}\sup_{z\in M} d(z, g^{p_j} z) \leq \frac{\pi}{\sqrt{p}} = r_{inj}\sqrt{p} \tag{8}$$

since $r_{\max}$ is the supremum of the smallest geodesics. A similar bound can be obtained if we restrict our analysis to geodesics generated by $g_{(p,q)}$ (see [5]). The minimum length of such geodesics is $\leq \frac{2\pi q}{p}$, and this can be viewed as an alternative upper bound on $2r_{\max}$. It is clear that $r_{\max} \leq \frac{\pi q}{p}$ is a better bound than (8) if and only if $q < \sqrt{p}$. Moreover, in this case, since $p_2 = \mathrm{int}[\frac{p}{q}]$, we have $p_2 > \sqrt{p}$, thus $p_j = p_1 = 1$ and therefore the generator $g_{(p,q)}$ gives the smallest geodesic at least for $|z_2| \leq |z_1|$, which corresponds to half of the lens space.

In the above analysis no constraints were imposed on the lens spaces, i.e. the values of p and q. We wish now to concentrate on values of these parameters which lead to observable isometries in the limit (1). In order to show that the isometry that generates the smallest geodesic behaves CT–like, it suffices to show that g^{p_j} behaves CT–like in the observable universe. Recall that any detectable lens space will have $p > \frac{\pi}{\chi_{obs}}$ [5], thus from (8) we have that $d(z, g^{p_j} z)$ is small. Expanding (7) and using the fact that $|z_1|^2 + |z_2|^2 = 1$ we obtain

$$d(z, g^{p_j} z) = 2\pi\left[\left(\frac{p_j}{p}\right)^2 + \left(G_j^2 - \left(\frac{p_j}{p}\right)^2\right)|z_2|^2\right]^{1/2}\ , \tag{9}$$

thus we need to determine the variation of $|z_2|$ inside the sphere of radius χ_{obs} and centered at z.

Let an observer be located at $z \in \mathbb{S}^3$, and let w be another point at a distance $\chi_{obs} \ll 1$, then we have $\chi_{obs} > |\eta|$, where $\eta = z - w$. Denoting by $\delta_1 = ||z_1| - |w_1||$, and correspondingly for δ_2, up to the first order in χ_{obs}, one has $|z_1|\delta_1 + |z_2|\delta_2 = 0$, which then gives $\chi_{obs}^2 > |\eta|^2 \geq \delta_1^2 + \delta_2^2 = \left(\frac{\delta_2}{|z_1|}\right)^2$, where the equality is achieved whenever z_1 is parallel to w_1, and z_2 is parallel to w_2. Thus one has

$$\delta_2 < |z_1|\chi_{obs}\ . \tag{10}$$

Now in order to analyze the behavior of the isometry g^{p_j}, we have to consider two different cases: (I) when the observer is not close to an equator of minimum injectivity radius, and (II) when she is. We consider each case separately.

Case I. $|z_1|$ and $|z_2| \gg \chi_{obs}$. The maximum and minimum lengths of the observable geodesics associated to g^{p_j} are

$$d_{ext} = d_0 \pm (2\pi)^2 \left| G_j^2 - \left(\frac{p_j}{p}\right)^2 \right| \frac{|z_2||z_1|}{d_0} \chi_{obs} , \tag{11}$$

where the plus and the minus signs correspond to $d_{\max}$ and $d_{\min}$ respectively. We then have

$$\frac{\Delta d}{d_0} \leq 2 \frac{1}{|z_2||z_1|} \chi_{obs} \ll 1 . \tag{12}$$

This is a bound on the maximum variation of the lengths of the smallest geodesics in the observable universe, which means that g^{p_j} behaves CT–like. In the special case $q < \sqrt{p}$, we have seen that $p_j = 1$, thus eqs. (11) yield

$$\frac{\Delta d}{d_0} \leq 4r_{inj}^2 \left(q^2 - 1\right) \frac{|z_2||z_1|}{d_0^2} \chi_{obs} . \tag{13}$$

For $q = 1$ the global homogeneity is manifest from this expression.

Case II. $|z_1|$ or $|z_2| \sim \chi_{obs}$. If $|z_1|$ or $|z_2| \sim 1$, then as we have seen the shortest geodesics will be generated by either $g_{(p,q)}$ or $g^{p_{k-1}}$ respectively. Both cases are very similar. Let z_l be $|z_1|$ or $|z_2|$ respectively, we then have

$$\begin{aligned} d_0^2 &\simeq 4r_{inj}^2 + 2[1 - \cos(2\pi\mu)]z_l^2 , \\ d_{ext}^2 &\simeq 4r_{inj}^2 + 2[1 - \cos(2\pi\mu)](z_l \pm \chi_{obs})^2 , \end{aligned}$$

where μ is respectively $\frac{q}{p}$ or $\frac{p_{k-1}}{p}$. In these cases the isometries $g_{(p,q)}$ and $g^{p_{k-1}}$ do not look like translations, unless $\mu \ll 1$. Importantly, however, from an observational point of view, the set of observers which detect such non–CT–like isometries is small. This can be estimated by calculating the ratio of the volumes of the 3-sphere, V_M, and the region where $|z_l| \sim \chi_{obs}$, V_R, to give

$$\frac{V_R}{V_M} \sim \frac{3}{2} \chi_{obs}^2 ,$$

which clearly is very small. In this way, the likelihood of an observer detecting a non–CT–like isometry is very small.

To summarize, we have shown that subject to condition (1), the detectable isometries of lens spaces behave CT–like for generic observers. Although our results were obtained for lens spaces, they are far more general and apply to generic spherical manifolds. This is because typically only the largest cyclic subgroup of the covering group is detectable, resulting in the universe ‘looking like a lens space’, no matter what its true topology may be [6].

We now briefly discuss the consequences of our results. Recent high resolution CMB observations are making it possible, for the first time, to seriously look for the possible signatures of a nontrivial cosmic topology [1, 2]. They have also produced strong support for the central predictions of inflationary cosmology, among them the near–flatness of the universe. Together, these provided the motivation for our study here.

Most detection methods rely on pattern repetition, and in this context detectable isometries are those which generate closed geodesics shorter than the maximum survey depth χ_{obs}. So the subset of detectable isometries depend crucially on both χ_{obs} and the position of the observer. We have found that in the limit $\chi_{obs} \ll 1$, the subset of such isometries in hyperbolic and generic spherical spaces turn out to be CT–like for a typical observer. This has the important observational consequence that a detectable manifold is generically locally indistinguishable from cylindrical $\mathbb{R}^2 \times \mathbb{S}^1$ (or more rarely toroidal $\mathbb{R} \times \mathbb{T}^2$) manifolds, irrespective of its global shape. We also note that importantly the topological signatures expected from our results are compatible with recent indications coming from the analyses of high resolution CMB data, such as the alignment of the quadrupole and octopole moments of CMB anisotropies [3, 4], as well as the surprisingly low amplitude of the CMB quadrupole [1, 3].

Given the infinite number of possible candidate manifolds any realistic search strategy must rely on a theoretical framework which can radically restrict the expected possibilities. Our results provide precisely such a framework by severely restricting the detectable isometries, and thereby confining the expected topological signatures as well as any parameter space which realistic search strategies, such as the (computationally intensive) 'circles in the sky' method, need to concentrate on. This is particularly important in the inflationary limit, where the order of any detectable cyclic subgroup as well as the number of candidate isometries which generate small geodesics are extremely large. We have also obtained bounds on r_{max} which can in principle observationally rule out a large quantity of spherical manifolds.

An important geometrical consequence of the bound $\chi_{obs} \ll 1$ is that curvature effects are undetectable in the observable universe, resulting in a geometrical degeneracy in this limit. Our results show that a parallel degeneracy exists with respect to the topological structure of the universe, in the sense that it is also impossible to distinguish topologically the universe as being hyperbolic, flat or spherical — in the inflationary limit the precise nature of cosmic topology becomes undecidable.

Our results regarding the minimal lengths of geodesics are quite general and apply to any cyclic subgroup. The extent to which detectable isometries are CT–like is on the other hand dependent on conditions imposed on χ_{obs}. Although we have used the condition (1) as synonymous to the inflationary limit, this limit is sufficient but not necessary for (1) to hold. Even for far less restrictive conditions (such as the more precise observational bounds expected from the Planck mission), these results still provide useful constraints on the nature of detectable isometries which are not strictly CT–like. We note that the question of likelihood of detectability does not concern us here as we are dealing with the subset of topologies that are detectable.

Finally, our results could in principle provide a topological test for inflation, in the sense that if observations were to detect isometries with significant deviations from CT–likeness, they would set bounds on the amount of inflation the primordial universe would have undergone.

Further investigations of some of the questions raised here are in progress and will be presented elsewhere.

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