

SUPERSYMMETRIC CLASSICAL MECHANICS: FREE CASE

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ABSTRACT

We present a review work on Supersymmetric Classical Mechanics in the context of a Lagrangian formalism, with $N = 1$ -supersymmetry. We show that the $N = 1$ supersymmetry does not allow the introduction of a potential energy term depending on a single commuting supercoordinate, $\phi(t; \Theta)$.

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I. INTRODUCTION

Supersymmetry (SUSY) in classical mechanics (CM) [1,2] in a non-relativistic scenario is investigated. SUSY first appeared in relativistic theories in terms of bosonic and fermionic fields[†], and the possibility was early observed that it can accommodate a Grand-Unified Theory (GUT) for the four basic interactions of Nature (strong, weak, electromagnetic and gravitational). However, after a considerable number of works investigating SUSY in this context, confirmation of SUSY as high-energy unification theory is missing. Furthermore, there exist phenomenological applications of the $N = 2$ SUSY technique in quantum mechanics (QM) [3]. In the literature, there exist four excellent review articles about SUSY in quantum mechanics [4]. Recently a general review on the SUSY QM algebra and the procedure on how to build a SUSY Hamiltonian hierarchy in order of a complete spectral resolution it is explicitly applied for the Pöschl-Teller I potential [5].

We must say that, despite being introductory, this work is not a mere scientific exposition. It is intended for students and teachers of science and technology. The pre-requisites are differential and integral calculus of two real variables functions and classical mechanics. Recently, two excellent mini-courses were ministered of introduction to the theory of fields with the aim of presenting the fundamental basics of the theory of fields including the idea of SUSY with emphasis on basic concepts and a pedagogical introduction to weak scale supersymmetry phenomenology, in which the reader may use for different approaches and viewpoints include [6].

Considering two ordinary real variables x and y , it is well known that they obey the commutative property, $xy = yx$. However, if \tilde{x} and \tilde{y} are real Grassmann variables, we have: $\tilde{x}\tilde{y} = -\tilde{y}\tilde{x} \Rightarrow \tilde{x}^2 = -\tilde{x}^2 = 0$.

In this work, we will use a didactic approach about the transformations in the su-

[†]A bosonic field (associated with particles of integral or null spin) is one particular case obeying the Bose-Einstein statistic and a fermionic field (associated to particles with semi-integral spin) is that obey the Fermi-Dirac statistic.

perspace, showing the infinitesimal transformation laws of the supercoordinate and of its components, denominated by even and odd coordinates, in the unidimensional space-time $D=(0+1) = 1$. We will see that by making an infinitesimal variation in the even coordinate, we generate the odd coordinate and vice-versa. This approach is done in this work using the right derivative rule. We will distinguish that property of supersymmetry in which the action is invariant with the translation transformations in the superspace ($\delta S = 0$), noting that the same does not occur with the Lagrangian ($\delta L \neq 0$).

In the construction of a SUSY theory with $N > 1$, referred to as extended SUSY, for each spatial commuting coordinate, representing the degrees of freedom of the system, we associate one anticommuting variable, which are known that Grassmannian variables. However, we consider only the $N = 1$ SUSY for a non-relativistic point particle, which is described by the introduction of only one real Grassmannian variable Θ , in the configuration space, but all the dynamics are putted in the time t . In this case, we have two degrees of freedom. The generalized anticommuting coordinate (odd magnitude) will be represented by $\psi(t)$. The new real coordinate defined in the superspace will be called supercoordinate. It will have the following more general possible Taylor expansion: $\phi(t; \Theta) = q(t) + i\Theta\Psi(t)$. Note that the first term is exactly the ordinary real commuting coordinate $q(t)$ and, like the next term, must to be linear in Θ , because $\Theta^2 = 0$. In this case, the time dependent part multiplying Θ is necessarily one Grassmannian variable $\Psi(t)$, which need the introduction of i for warranty that the supercoordinate $\phi(t; \Theta)$ will be real[‡].

We would like to highlight to the readers who know field theory, but have never seen the supersymmetrical formalism in the context of classical mechanics that in this work we will present the ingredients for implenting N=1 SUSY, namely, superspace, supertranslation, supercoordinate, SUSY covariant derivative and super-action. Indeed, the steps adopted in this approach are the same as used in the supersymmetrizations of quantum field theories in the quadrimensional space-time of the special relativity ($D = (3+1)$, three position coordinates and one temporal coordinate).

[‡]Like will be see below.

This work is organized as follow: in section II we construct a finite supercoordinate transformation and the infinitesimal transformations on the supercoordinate and its components via the translation in the superspace. In section III, we investigate the superparticle using the Lagrangian formalism in the superspace, noting the fact that the N=1 SUSY does not allow the introduction of a potential term for only one supercoordinate and we indicate the quantizing procedure. In section IV, we present the conclusion.

II. TRANSLATIONS IN SUPERSPACE

We will consider the N=1 supersymmetry i.e. SUSY with only one anticommuting variable. Supersymmetry in classical mechanics unify the even coordinate $q(t)$ and the odd coordinate $\Psi(t)$ in a superspace characterized by the introduction of a Grassmannian variable, Θ , not measurable [1,2,7].

$$\text{Superspace} \rightarrow (t; \Theta), \quad \Theta^2 = 0, \quad (1)$$

where t and Θ act, respectively, like even and odd elements of the Grassmann algebra.

The anticommuting coordinate, Θ , will parametrize all points of superspace, but all dynamics will be put in the time coordinate t . SUSY in classical mechanics is generated by a translation transformation in the superspace, viz.,

$$\begin{aligned} \Theta &\rightarrow \Theta' = \Theta + \epsilon, \Rightarrow \delta\Theta = \Theta' - \Theta = \epsilon \\ t &\rightarrow t' = t + i\epsilon\Theta \Rightarrow \delta t = t' - t = i\epsilon\Theta, \end{aligned} \quad (2)$$

where Θ and ϵ are real Grassmannian parameters,

$$[\Theta, \epsilon]_+ = \Theta\epsilon + \epsilon\Theta = 0 \Rightarrow (\Theta\epsilon)^* = (\epsilon^*\Theta^*) = (\epsilon\Theta) = -(\Theta\epsilon). \quad (3)$$

This star operation of a product of two anticommuting Grassmannian variables ensures that the product is a pure imaginary number and for this reason must insert the $i = \sqrt{-1}$ in (2) to obtain the real character of time. SUSY is implemented for maintain the line element invariante[§]:

[§]That properties of Grassmannian magnitudes, necessary for one better comprehension of this section will be introduced gradatively.

$$dt + i\Theta d\Theta = \text{invariant}, \quad (4)$$

where one again we introduce an i for the line element to become real.

The supercoordinate for $N = 1$ is expanded in a Taylor series in terms of even $q(t)$ and odd $\psi(t)$ coordinates:

$$\phi \equiv \phi(t; \Theta) = q(t) + i\Theta\psi(t). \quad (5)$$

Now, we need to define the derivative rule with respect to one Grassmannian variable. Here, we use the right derivative rule i.e. considering $f(\Theta_1, \Theta_2)$ a function of two anticommuting variables, the right derivative rule is the following:

$$\delta f = \frac{\partial f}{\partial \Theta_1} \delta \Theta_1 + \frac{\partial f}{\partial \Theta_2} \delta \Theta_2, \quad (6)$$

where $\delta \Theta_1$ and $\delta \Theta_2$ appear on the right side of the partial derivatives.

One infinitesimal transformation of supercoordinate that obey the SUSY transformation law given by (2) results in:

$$\delta \phi(t; \Theta) = \phi(t'; \Theta') - \phi(t; \Theta) = (\partial_t \phi) \delta t + (\partial_\Theta \phi) \delta \Theta = i\epsilon \Theta \dot{q}(t) - i\epsilon \psi(t), \quad (7)$$

On the other hand, making an infinitesimal variation of (5) i.e. $\delta \phi(t; \Theta) = \delta q(t) + i\Theta \delta \psi(t)$ and comparing with (8) we obtain the following SUSY transformation law for the components of the supercoordinate:

$$\delta q(t) = i\epsilon \psi(t), \quad \delta \psi(t) = -\epsilon \dot{q}(t). \quad (8)$$

Therefore making a variation in the even component we obtain the odd component and vice versa i.e. SUSY mixes the even and odd coordinates.

Note from (5) and (7), that the infinitesimal SUSY transformation law can be written in terms of the supercoordinate $\phi(t; \Theta)$:

$$\delta \phi(t; \Theta) = \epsilon Q \phi(t; \Theta), \quad Q = -\partial_\Theta + i\Theta \partial_t, \quad (9)$$

where $\partial_\Theta \equiv \frac{\partial}{\partial \Theta}$, $\partial_t \equiv \frac{\partial}{\partial t}$.

Therefore any coordinate which obey equation (9) will be interpreted as supercoordinate**. The differential operator Q , called the supercharge, is a representation of the translation generator in the superspace. In fact one finite translation can be easily obtained (9) which has an analogous form as that of translation in the ordinary space

$$U(\epsilon)\phi(t; \Theta)U^{-1}(\epsilon) = \phi(t'; \Theta'), \quad U(\epsilon) = \exp(\epsilon Q), \quad U^{-1}(\epsilon) = U(-\epsilon), \quad (10)$$

with the operator Q doing a similar role as that of the linear momentum operator in ordinary space.

III. COVARIANT DERIVATIVE AND THE LAGRANGIAN

Now, we build up a covariant derivative (with respect to Θ) which preserves the supersymmetry of super-action i.e. we will see that the derivative with respect to Θ ($\partial_\Theta \Phi$) does not itself transform like a supercoordinate. So it is necessary to construct a covariant derivative.

SUSY really possesses a peculiar characteristic. As the anticommuting parameter ϵ is a constant we see that SUSY is a global symmetry.

In general local symmetries are the ones which require covariant derivatives. For example the gauge theory $U(1)$ with local symmetry requires covariant derivatives. But because of the fact that $\partial_\Theta \phi(t; \Theta)$ is not a supercoordinate, SUSY will require a covariant derivative for us to write the super-action in a consistent form. To prove this fact we use (10) so as to obtain the following variations:

$$\begin{aligned} \delta \partial_\Theta \phi(t; \Theta) &= \partial_\Theta \delta \phi(t; \Theta) = -i\epsilon \dot{\phi}(t; \Theta) + i\epsilon \Theta \partial_\Theta \dot{\phi}(t; \Theta) \neq \epsilon Q \partial_\Theta \phi(t; \Theta) \\ \delta \partial_\Theta \phi(t; \Theta) &= \epsilon \Theta \dot{\psi}(t) \neq \epsilon Q \partial_\Theta \phi(t; \Theta) = -\epsilon \Theta \dot{\psi}. \end{aligned} \quad (11)$$

On the other hand, making an infinitesimal variation of the partial temporal derivative we find:

**Note that $\delta \tau(t; \Theta) = \epsilon Q \tau(t; \Theta)$ (where τ is a supercoordinate) give us a way for test if τ is a supercoordinate indeed. If this equality not is true, τ is not a supercoordinate.

$$\delta\partial_t\phi(t; \Theta) = \epsilon Q\partial_t\phi(t; \Theta). \quad (12)$$

So we conclude that $\partial_t\phi$ obeys the SUSY transformation law and therefore it is a supercoordinate. The covariant derivative of supersymmetric classical mechanics is constructed so that it obeys the anticommutativity with Q i.e. $[D_\Theta, Q]_+ = 0$. It is easy to verify that one representation for a covariant derivative is given by:

$$D_\Theta = -\partial_\Theta - i\Theta\partial_t \Leftrightarrow \delta D_\Theta\phi(t; \Theta) = \epsilon Q D_\Theta\phi(t; \Theta). \quad (13)$$

Another interesting property which occurs when the SUSY generator (Q) is realized in terms of spatial coordinates on in the configuration representation [3,4] is the fact that the anti-commutator of the operator Q with itself gives us the SUSY Hamiltonian:

$$[Q, Q]_+ = -2i\partial_t = -2H, \quad Q^2 = -H, \quad (SUSY)^2 \propto H, \quad (14)$$

i.e. two successive SUSY transformations give us the Hamiltonian. This is an algebra of left supertranslations and time-translations. The corresponding right-supertranslations satisfy the following algebra:

$$[D_\Theta, D_\Theta]_+ = 2i\partial_t = 2H, \quad D_\Theta^2 = H. \quad (15)$$

Before we construct the Lagrangian for a superpoint particle, we introduce the Berezin integrals [7] for an anticommuting variable:

$$\int d\Theta\Theta = 1 = \partial_\Theta\Theta, \quad \int d\Theta = 0 = \partial_\Theta 1. \quad (16)$$

Now we are in conditions to analyse the free superpoint particle in one dimension and to construct a manifestly supersymmetric action. We will see that SUSY is a superaction symmetry but does not let the Lagrangian invariant. A super-action for the free superpoint particle can be written as the following double integral^{††}

^{††}In this section about supersymmetry we use the unit system in which $m = 1 = \omega$, where m is the particle mass and ω is the angular frequency.

$$\begin{aligned}
S &= \frac{i}{2} \int \int dt d\Theta (D_\Theta \phi) \dot{\phi} \\
&= \frac{i}{2} \int \int dt d\Theta \{-i\psi \dot{q} - \Theta \psi \dot{\psi} - i\Theta \dot{q}^2\} \\
&= -\frac{i}{2} \int dt \{i\psi \dot{q} \int d\Theta + \psi \dot{\psi} \int d\Theta \Theta + i\dot{q}^2 \int d\Theta \Theta\} \\
&\equiv \int dt L.
\end{aligned} \tag{17}$$

Indeed after integrating in the variable Θ , we obtain the following Lagrangian for the superpoint particle:

$$L = \frac{1}{2} \dot{q}^2 - \frac{i}{2} \psi \dot{\psi}, \tag{18}$$

where the first term is the kinetic energy associated with the even coordinate in which the mass of the particle is unity. The second term is a kinetic energy piece associated with the odd coordinate (particle's Grassmannian degree of freedom) dictated by SUSY and is new for a particle without potential energy. Thus we see that the Lagrangian is not invariant because it's variation result in a total derivative and consequently is not zero, which can be obtained from δS , $D_\Theta |_{\Theta=0} = Q_\Theta |_{\Theta=0}$:

$$\delta S = \frac{i}{2} \int dt d\Theta \delta\{(D_\Theta \phi) \dot{\phi}\} \Rightarrow \delta L = \frac{1}{2} \epsilon \frac{d}{dt} (D_\Theta \phi) \dot{\phi} |_{\Theta=0} = \frac{i}{2} \epsilon \frac{d}{dt} \{\psi \dot{q}\} \neq 0. \tag{19}$$

Because of the fact that the Lagrangian is a total derivative, we obtain $\delta S = 0$ i.e. the super-action is invariant under N=1 SUSY transformation.

Note that for $N = 1$ SUSY and with only one coordinate ϕ , we can't introduce a potential term $V(\phi)$ in the super-action because it conduces to non-invariance i.e. ($\delta S \neq 0$). There are even two more inconsistency problems. First we note that the super-action S acts like an even element of the Grassmann algebra and for this reason any additional term must be an even element of this algebra. Indeed, analysing the terms present in the super-action we see that the line element has one $d\Theta$ and one dt which are respectively odd and even. As the supercoordinate is even, the potential $V(\phi)$ must also be even, which when acts with the line element $dt d\Theta$ becomes odd which fact will let the super-action odd and this is not admissible. The other inconsistency problem can be traced form the dimensional analysis. In the system of natural units the super-action must be non-dimensional. In such a system of units, the time and the even component

$q(t)$ of the supercoordinate have dimension of $[massa]^{-1}$. In this way, starting from the supertranslation, we will see that Θ will have dimension $[massa]^{-\frac{1}{2}}$. Consequently the supercoordinate ϕ has dimension of $[massa]^{-1}$ and $\dot{\phi}$ is non-dimensional. Because of this when we introduce a potential term $V(\phi)$ we would obtain a super-action with inconsistent dimension.

The canonical conjugate momentum associated to the supercoordinate is given by

$$\Pi(t, \Theta) = \frac{\partial}{\partial \frac{d\phi}{dt}} L = \frac{i}{2} \frac{\partial}{\partial \frac{d\phi}{dt}} \int d\Theta (D_{\Theta}\phi) \dot{\phi} = \frac{1}{2} \int d\Theta \{ \Theta \dot{\phi} + i D_{\Theta}\phi \}, \quad \dot{\phi} \equiv \frac{d\phi}{dt}, \quad (20)$$

which leads to the following Poisson brackets:

$$\{ \phi(t, \Theta), \phi(t, \Theta') \} = 0 = \{ \Pi(t, \Theta), \Pi(t, \Theta') \} \quad (21)$$

$$\{ \phi(t, \Theta), \Pi(t, \Theta') \} = \delta(\Theta - \Theta'). \quad (22)$$

We can not implement the first canonical quantizing method directly because there exist constraints: the primary obtained from the definition of canonical momentum and the secondary obtained from the consistency condition. In this case we must to construct the modified Poisson parentheses called Dirac brackets. These aspects have been considered in the quantization of the superpoint particle with extended $N = 2$ SUSY and is out of the scope of this work [9].

We finalize by writing another manifestly supersymmetric action which can be constructed for the case with $N = 1$ SUSY using the generator of right supertranslation D_{Θ} :

$$S_2 = \frac{i}{2} \int \int d\Theta dt D_{\Theta}(D_{\Theta}\phi). \quad (23)$$

It is left as an exercise for the reader to demonstrate that it is possible to effect the integral in Θ and encounter the $N = 1$ SUSY Lagrangian for the case.

IV. CONCLUSION

After the introduction of a real Grassmannian anticommuting variable, we consider a translation in superspace and implement the transformation laws of the supercoordinate and its components. We show that an infinitesimal variation of the even coordinate generates the odd coordinate and vice-versa, characterizing N=1 SUSY. We introduce a covariant derivative for writing the super-action in a consistent way. We verify that an interesting property occurs when the SUSY generator (Q) is realized in terms of Grassmannian coordinates: the anticommutator of Q with itself results in the Hamiltonian i.e. two successive SUSY transformations generate the Hamiltonian. If the reader considers two successive supertranslations, will obtain exactly the Hamiltonian as result i.e. $D_{\Theta}^2 = H$. In the original works about supersymmetric in classical mechanics [1,2], the respective authors do not justify as to why in the case of N=1 SUSY is not allowed to put a potential term in the Lagrangian. Therefore, the main purpose in this work has been to make an analysis of this question in the context of a Lagrangian formalism in superspace with N=1 SUSY. In synthesis from the fact as to how the super-action must be even and the line element $dt d\theta$ in its construction be odd, we show that it is not possible to introduce a potential energy term $V(\phi)$, because which a potential term would conduce in a super-action with inconsistent dimension i.e. the super-action itself becomes odd too. Therefore when we have only one supercoordinate ϕ , the $N = 1$ SUSY exists only for a free superpoint particle. The equations of motion for the superpoint particle with N=1 SUSY are first order for odd coordinate ($\dot{\Theta} = \frac{d\Theta}{dt} = 0$) and second order for even coordinate ($\ddot{x} = \frac{d^2}{dt^2}x = 0$).

In conclusion we must stress that the super-action must always be even but the Lagrangian may eventually be odd. Nonetheless, the same analysis can be implemented for the case with $N = 2$ SUSY so that one may put a potential term in the superaction. In this case, considering only one supercoordinate ϕ of commuting nature, is allowed the introduction of a potential term in the Lagrangian [9,10]. On the other hand, one can introduce an odd supercoordinate of anticommuting nature so that the $N = 1$ SUSY is ensured and the main consequence is to obtain the unharmonic oscillator potential.

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