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SCALING FACTORS ASSOCIATED WITH M-FURCATIONS OF THE $1-a|x|^z$ MAP

by

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ABSTRACT

We study numerically the scaling behavior associated with M-furcations ($M = 3, 4, 5$) in the map $x_{t+1} = 1 - a|x_t|^z$ ($z > 1$). The scaling constants δ and α, as function of z, are calculated, as well as the more general scaling functions σ and $f(\alpha)$.

Key-words: Chaos; One-dimensional maps; Multifractality.

I. INTRODUCTION

The one-dimensional iterative map

$$x_{t+1} = f(x_t) \equiv 1 - \mu |x_t|^z \quad , \quad (z > 1) \tag{1}$$

which maps the interval $x \in [-1,1]$ into itself, displays a very rich dynamical behavior [1,2]. This map is generic for all single-hump one-dimensional maps which have (locally around the maximum) a leading nonlinearity of order z. The $z=2$ case is by far the most common in experiments [3], but also other values of z are found [4].

When the parameter μ in Eq. (1) is raised (starting from $\mu=0$) the attractors (or long-time solutions) of the map show a sequence of periodic orbits with period $2^k (k=0,1,2...)$. The k-th period appears at μ_k through a pitchfork bifurcation of the (k-1)-th period, and the sequence $\{\mu_k\}$ accumulates $(k \to \infty)$ at $\mu_\infty(z)$, where the system enters into chaos. In the chaotic region aperiodic attractors are present as well as an infinite number of periodic windows, which always appear in the same order, independently of z. When these windows are taken in an appropriate order, they form sequences of M-furcations, with period M^k $(M>2)$, which generalize the bifurcations $(M=2)$. The windows of the M-furcations are not adjacent on the parameter axis and they are very narrow. However, the period-tripling sequence $(M=3)$ has been observed experimentally [5]. Above $\mu=2$ no finite attractors exist and x_t is driven to infinity.

All the sequences of M-furcations present scaling factors that converge and define universality classes determined by z, in the sense that the factors do not change if higher order terms are included in Eq. (1). In the μ-direction there is the scaling factor δ and in the x-direction there is a whole set of scaling factors (the principal ones being α and α^z), which together form the scaling function σ. The presence of a set of scaling indices in the attractor at the accumulation point of the M-furcation, characterizes the presence of a multifractal, which can be studied through the function f(a).

In the paper by van der Weele et al. [6] they studied δ, α, σ and f(a) as function of z for the bifurcations (M = 2), and Shau-Jin Chang et al. [7] studied α, δ and fractal dimensions for the z = 2 case and M ≤ 7. The Refs. 8-14 deal also with scaling factors for the M-furcations in the map (1).

The aim of the present communication is to study numerically as function of z, the scaling factors α and δ, and the scaling functions σ and f(a) for M = 3, 4 and 5, which correspond respectively to trifurcations, tetrafurcations and pentafurcations. The paper is organized as follows: in the next section we study the scaling function δ; in sections II and III we study the functions σ anf f(a), respectively; the last section is dedicated to the conclusions.

II. THE SCALING FACTOR δ

In this section let us initially fix upon notations to introduce the method used in the numerical calculations. For every periodic orbit in the map (1) there is one value of the control parameter for which the orbit includes the critical point (peak) of the map. At this value of the parameter the cycle is called surperstable. Following the images of the peak at the superstable cycle it is possible to form a word of R and L according to whether the subsequents iterates in the orbit are on the right or the left of the peak. This word is called U-sequence of the cycle [15]. In the case of the tri-furcations and tetrafurcations the basic 3-cycle and 4-cycle have U-sequences RL and RLL (or RL^2). The pentafurcations have three types of sequences for the basic 5-cycle, namely RLR^2, RL^2R and RL^3. The U-sequences related to the higher order periods in the M-furcations are constructed following the rules described in Refs. 1 and 15.

For each family of cycles related to a given U-sequence the set $\{\tilde{\mu}_k\}$ where superstable cycles occur converges geometrically at a rate given by

$$\delta = \lim_{k\to\infty}\left[\frac{\tilde{\mu}_k - \tilde{\mu}_{k-1}}{\tilde{\mu}_{k+1} - \tilde{\mu}_k}\right] = \text{const} \tag{2}$$

and it accumulates at $\tilde{\mu}_\infty$.

To determine the set $\{\tilde{\mu}_k\}$ where the cycles belonging to a family of M-furcations are superstable, we use the method introduced by Hao Bai-Lin [16]. Let us explain the method with the same type of U-sequence chosen by him, namely, RLRR. At the superstable cycle the iterates of the map proceed from $x=0$ to $x=0$, i.e.,

$$f_R(f_R(f_L(f_R(f(\mu,0))))) = 0 \tag{3}$$

where the subscript R or L indicates which branch Right or Left of the map has been used at each iteration. Since the inverse mapping is two-valued, we define

$$R(x) = f_R^{-1}(\mu,x) = +\left[(1-x)/\mu\right]^{1/z} \tag{4a}$$

$$L(x) = f_L^{-1}(\mu,x) = -\left[(1-x)/\mu\right]^{1/z} \tag{4b}$$

depending on which half of the mapping is used. If successive inverses of Eq. (3) are taken we obtain for the word RLRR the functional relation

$$R(L(R(R(0)))) = 1 \tag{5}$$

which is an equation for μ. If we multiply this equation by $\beta \equiv \frac{1}{\mu}$, we obtain

$$\beta\left[\beta\left(1+\left[\beta\left(1-\left[\beta\left(1-\beta^{1/z}\right)\right]^{1/z}\right)\right]^{1/z}\right)\right]^{1/z} = \beta \tag{6}$$

This equation can be solved by iterations, as suggested by Kaplan [17], i.e., replacing the equation by

$$\beta_{n+1} = \beta_n\left[\beta_n\left(1+\left[\beta_n\left(1-\left[\beta_n\left(1-\beta_n^{1/z}\right)\right]^{1/z}\right)\right]^{1/z}\right)\right]^{1/z} \tag{7}$$

and than iterating it for a suitable β_0.

This method is quite simple and can be used for any type of map whose inverse is calculable in closed form and for any type of U-sequence. However, we observe that the convergence of Eq. (7) becomes slower in the $z \to 1$ and $z \to \infty$ limits.

In table I we display the values of the accumulation points $\tilde{\mu}_\infty$ of the superstable values and of δ for $z = 1.5, 2, 3, 4, 6, 8$ and 10 and five types of sequences, namely $(RL)^{*n}$, $(RL^2)^{*n}$, $(RLR^2)^{*n}$, $(RL^2R)^{*n}$ and $(RL^3)^{*n}$ (see Ref. 1 for this notation), which correspond to M = 3,4 and 5. The numerical values of δ as a function of z are ploted in Fig. 1. We observe that δ diverges in the limit $z \to 1$, for all cases considered and presents a minimum near $z = 2$. For the bifurcations $(M = 2)$, $\delta(z)$ is a monotonously increasing function of z and $\delta(1) = 2$ [1]. There is a controversy about the behavior of $\lim_{z\to\infty} \delta(z)$. Eckmann et al. [11] and van der Weele et al. [6,8] claim that $\lim_{z\to\infty} \delta(t) \lesssim 30$, whereas Bhattachargee et al. [13] claim that $\delta(z)$ diverges in the $z \to \infty$ limit (this result coincides with that one obtained via Renormalization Group [12]). For M > 2 we do not know any conjecture about the behavior of $\delta(z)$ in the $z \to \infty$ limit. Numerical calculations in this limit are very difficult, since the convergence of δ becomes very slow.

III. THE FUNCTION σ

In the x-direction there is a whole set of scaling indices associated with the attractors at the accumulation point of the M-furcations; this fact characterizes the presence of a multifractal. The principal indices are α and α^z, which are related to central (near $x = 0$) and top (near $x = 1$) distances of the M-furcation tree, respectively. To determine the function σ let us consider the superstable M^k-cycle $\{x_0, x_1, \ldots x_{M^k-1}\}$ with $x_0 = 0$. The distances between x_m and $x_{m+M^{k-1}}$ are given by

$$d_{k,m} = |x_{m+M^{k-1}} - x_m| = |f_{\tilde{\mu}_k}^{(m+M^{k-1})}(0) - f_{\tilde{\mu}_k}^{(m)}(0)| \qquad (8)$$

For $m > M^{k-1}$ we consider $d_{k,nM^{k-1}+p} = d_{k,p}$ with $n,p = 1,2\ldots$. Therefore the scaling function σ can be defined by

$$\sigma(t) = \lim_{k\to\infty} \frac{d_{k,q}}{d_{k+1,q}} \, , \; t = \frac{q}{M^{k+1}} \qquad (9)$$

where $q = 1,2\ldots M^k$.

The scaling factors α and α^z are associated with the greatest and smallest value of σ, respectively, and are given by the following relations $\sigma(\frac{1}{M}) = \frac{1}{\alpha}$ and $\sigma(0^+) = \frac{1}{\alpha^z}$. In table I we display the values of α for $z = 1.5, 2, 3, 4, 6, 8, 10$ and $M = 3, 4$ and 5. In Fig. 2(a) we plot α versus z. We observe that the qualitative behavior of $\alpha(z)$ is the same for all sequences with $M = 2, 3, 4$ and 5, i.e., it is a monotonously decreasing function of z and goes to infinity when $z \to 1$ (see Ref. 6 for the $M = 2$ case). The scaling α^z, is shown in Fig. 2(b). We observe a qualitative behavior very similar between δ and α^z; both of them have a minimum near $z = 2$, diverge for $z \to 1$ and in the limit $z \to \infty$ the relation $\delta \lesssim \alpha^z$ is verified (this relation was observed for $M = 2$ in Ref. 8). Therefore, the question if δ has or not a limiting value when $z \to \infty$ is transformed in a similar question for α^z.

The function σ calculated for larger values of q does not give any further information since $d_{k+1,q+M^k} = d_{k+1,q}$ and therefore $\sigma(t + \frac{1}{M}) = \sigma(t)$. In Fig. 3 we show $\sigma(t)$ for $z = 1.5, 2$ and 10 for the trifurcations ($M = 3$). In every rational value of $t (0 < t < \frac{1}{M})$ there exist a jump in the function σ, but we observe that the discontinuities decrease rapidly as the binary expansion of the rational increases. In a crude approximation there are M plateaus, which are divided in subplateaus. We observe that the discontinuities of the subplateaus become more and more pronounced when z increases, and they can be calculated using approximative methods (see Ref. 6 for the $M = 2$ case).

IV. THE FUNCTION $f(\alpha)$

The scaling function $f(\alpha)$ is another way to characterize the multifractal set associated with the x-direction. It is more convenient than the function σ, in the theoretical and experimental points of view, since it is a smooth function.

The formalism introduced by Halsey et al. [18] consists in convering the attractor with boxes, indexed by i, of size ℓ_i, and assume that the probability density scales like $p_i \propto \ell_i^{\alpha}$ in the limit $\ell_i \to 0$. The next step is to form the normalized partition function

$$\Gamma(q,\tau) = \sum_i \frac{p_i^{\,q}}{\ell_i^{\,\tau}} = 1 \qquad (10)$$

The function $\tau(q)$ gives the function $f(\alpha)$ through a Legendre transformation.

To study the multifractal set present at the attractor of the M-furcations, we have chosen $p_i \equiv p = \frac{1}{M^{k-1}}$ for the K-cycle. Therefore the partition function becomes

$$\Gamma_k = \left(\frac{1}{M^{k-1}}\right)^q \sum_{m=1}^{M^{k-1}} d_{k,m}^{-\tau} \qquad (11)$$

where $d_{k,m}$ is given by Eq. (8). The function $f_k(\alpha)$ obtained by Eq. (11) converges, for k large enough, to the universal function $f(\alpha)$. The minimal and maximal values of α, which respectively characterize the most concentrated and most rarified regions of the attractor, are given by $\alpha_{min} = \ell n M/\ell n\, \alpha^z$ and $\alpha_{max} = \ell n M/\ell n\, \alpha$. Consequently $\alpha_{max} = z\alpha_{min}$, for all kinds of sequence, which is a useful relation to determine the order of the maximum of the map in physical experiments.

In Fig. 4 we show the function $f(\alpha)$ for $z = 1.5, 2, 4, 10$ and for the sequences $(RL^2)^{*n}$ and $(RL^3)^{*n}$. In the limit $z \to 1$ the curve $f(\alpha)$ reduces to a sharp peak at $\alpha = 0$ since $\alpha_{min} = \alpha_{max} = 0$, and the Hausdorff dimension D_0 (which coincides with the maximum of $f(\alpha)$) goes to zero. For increasing z, D_0 monotonously increases and goes to 1 in the limit $z \to \infty$. The behavior of α_{min} and α_{max} for increasing z is directly related to the behavior of α^z and α, respectively. Therefore we observe that α_{min} first grows, until reaches a maximum near $z = 2$ and then decreases, whereas a_{max} is a monotonously increasing function of z, and goes to infinity in the limit $z \to \infty$.

V. CONCLUSIONS

We have studied numerically the scaling factors associated with the M-furcations (M = 3, 4 and 5) for single-hump one-dimensional maps given by $x' = 1 - a|x|^z$. The numerical data were obtained by observing the level-by-level convergence of the scalings in the M-furcation tree. When z is varied we have found that the factors δ and α^z have a similar qualitative behavior for M = 3, 4 and 5, i.e., they diverge for $z \to 1$ and have a minimum near $z = 2$. In the limit $z \to \infty$ we verify $\delta \lesssim \alpha^z$, but if these scalings diverge or not in this limit is a question to be worked out. The scaling α is a monotonously decreasing function of z for all sequences studied. We also have calculated the functions σ and $f(a)$ related to the multifractal set present at the accumulation points of the M-furcations.

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CAPTION FOR FIGURES

Fig. 1. The scaling factor δ as a function of z for the sequences $(RL)^{*n}$, $(RL^2)^{*n}$, $(RLR^2)^{*n}$, $(RL^2R)^{*n}$ and $(RL^3)^{*n}$.

Fig. 2. The scaling factors (a)α and (b)α^z as function of z for the sequences $(RL)^{*n}$, $(RL^2)^{*n}$, $(RLR^2)^{*n}$, $(RL^2R)^{*n}$ and $(RL^3)^{*n}$.

Fig. 3. The scaling function σ for $z = 1.5, 2$ and 10 for the sequence $(RL)^{*n}$.

Fig. 4. The function $f(\alpha)$ for $z = 1.5, 2, 4$ and 10 for the sequences (a) $(RL^2)^{*n}$ and (b) $(RL^3)^{*n}$.

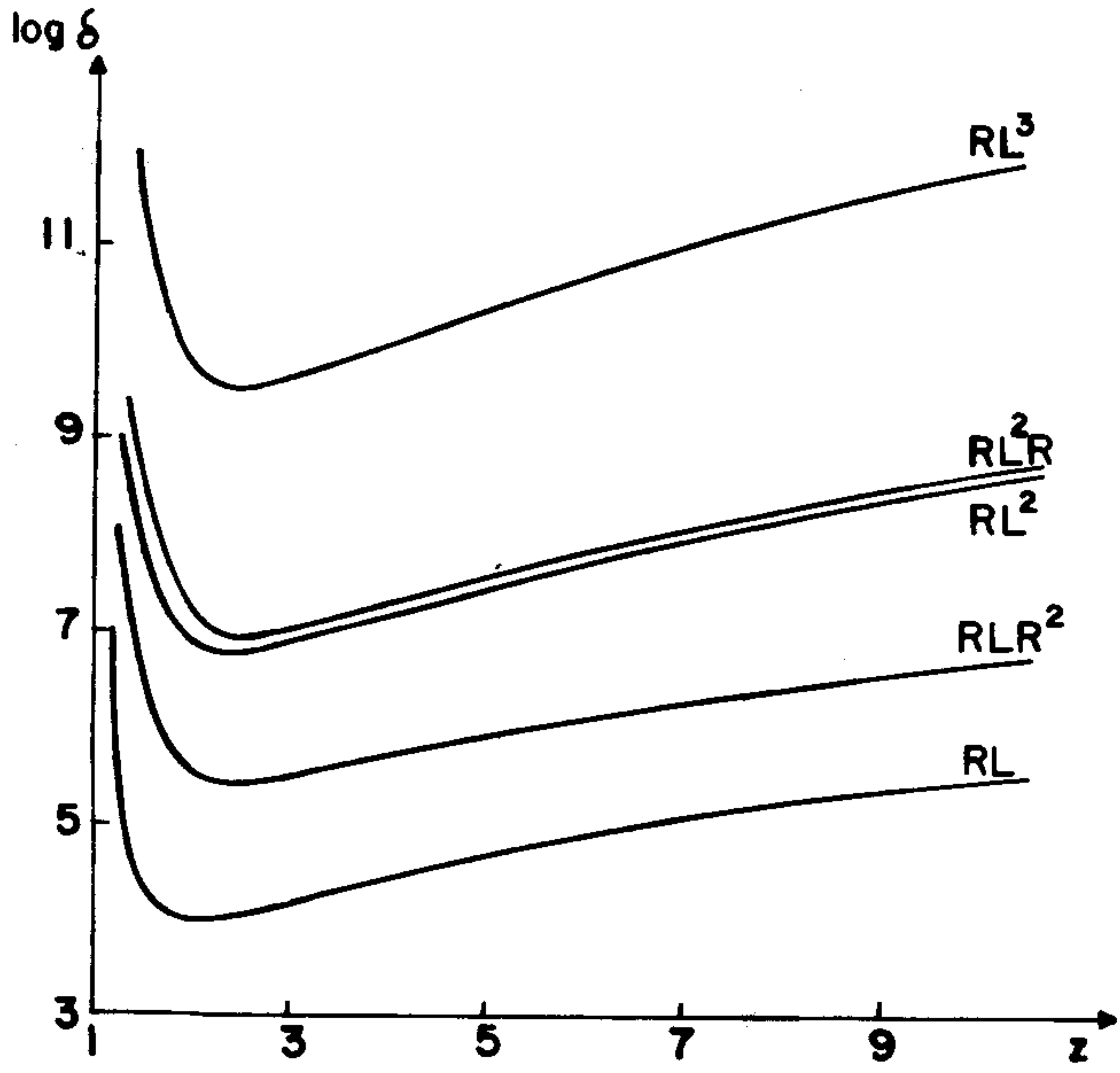

Fig. 1

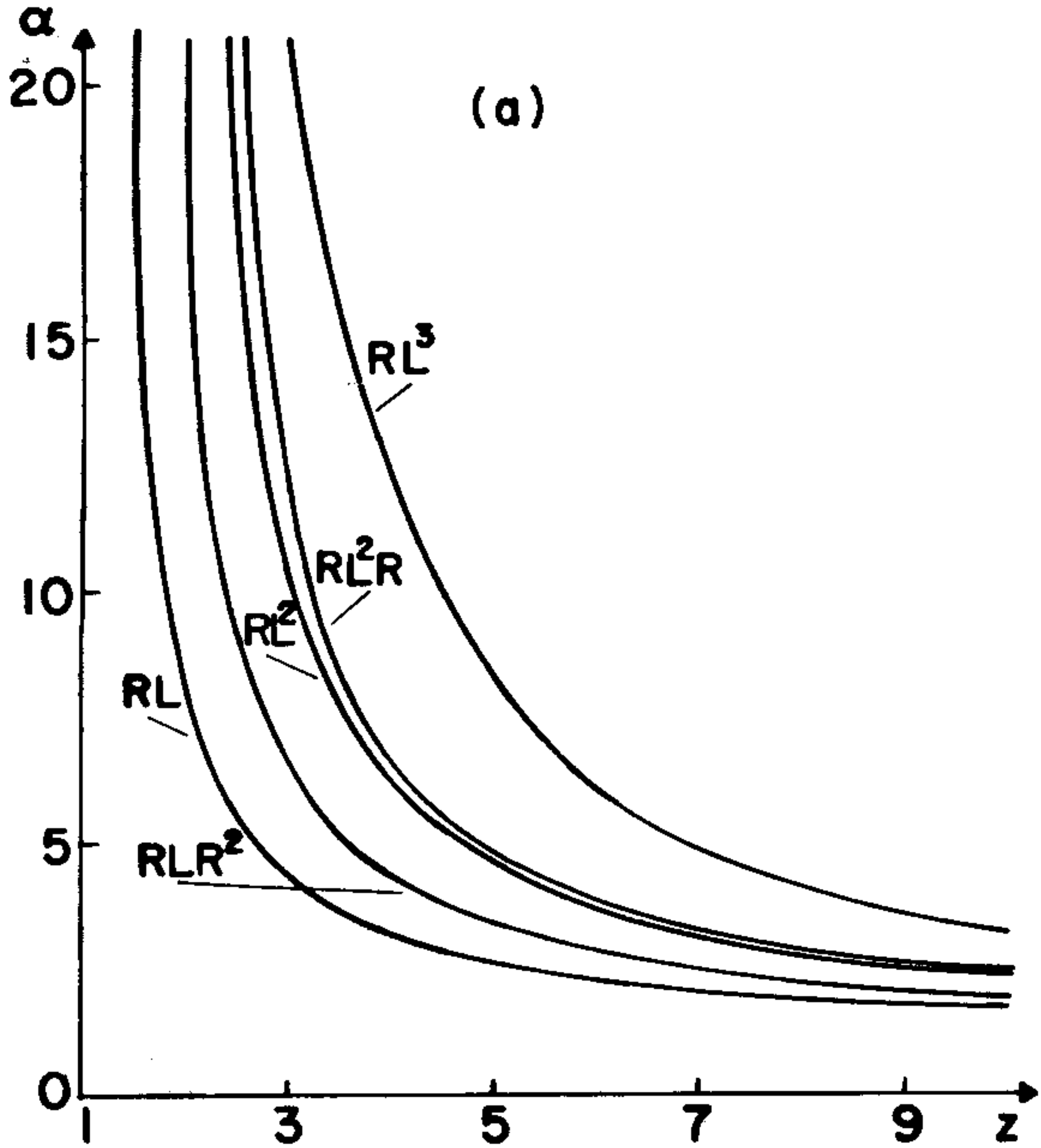

Fig. 2(a)

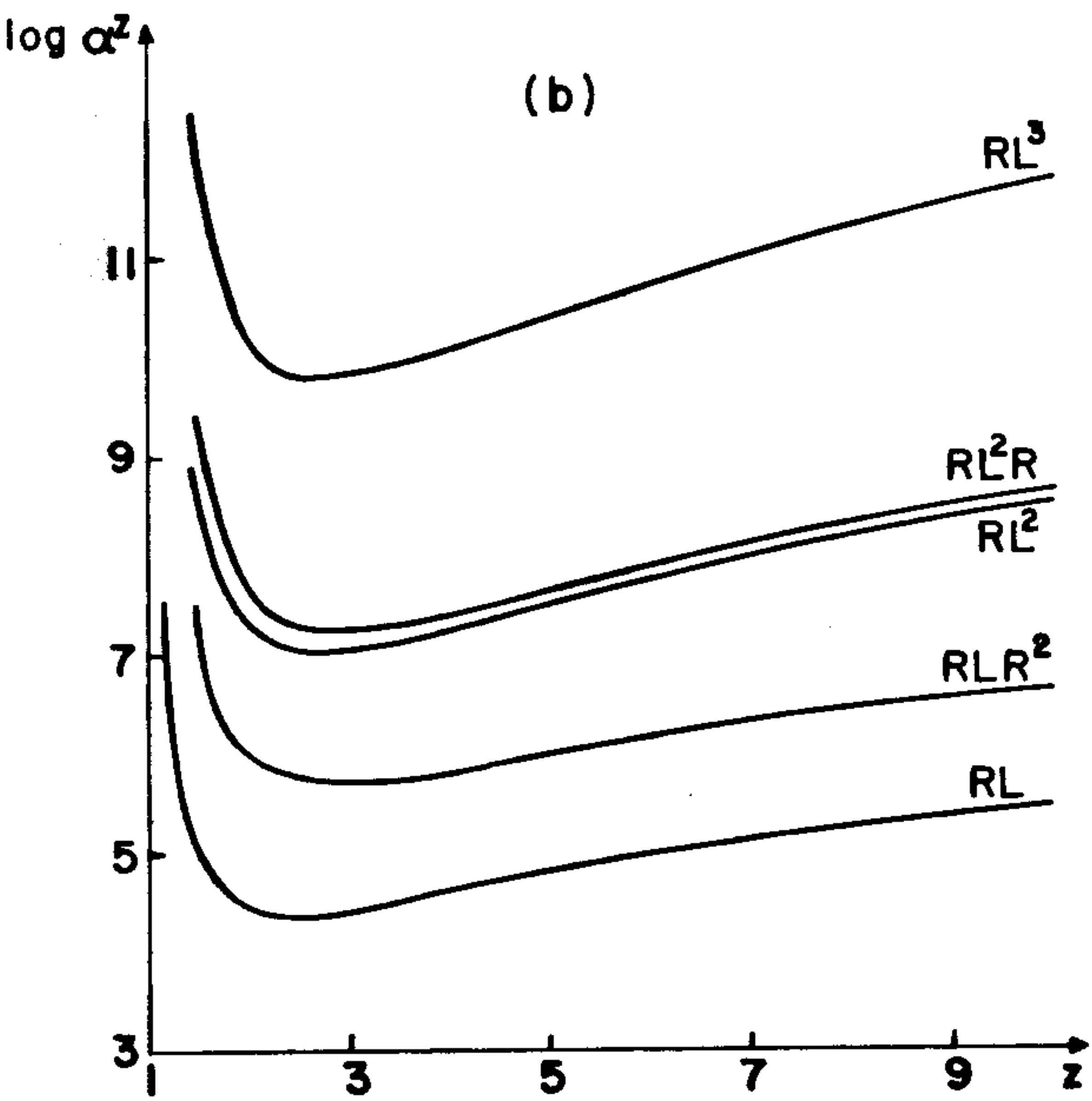

Fig. 2(b)

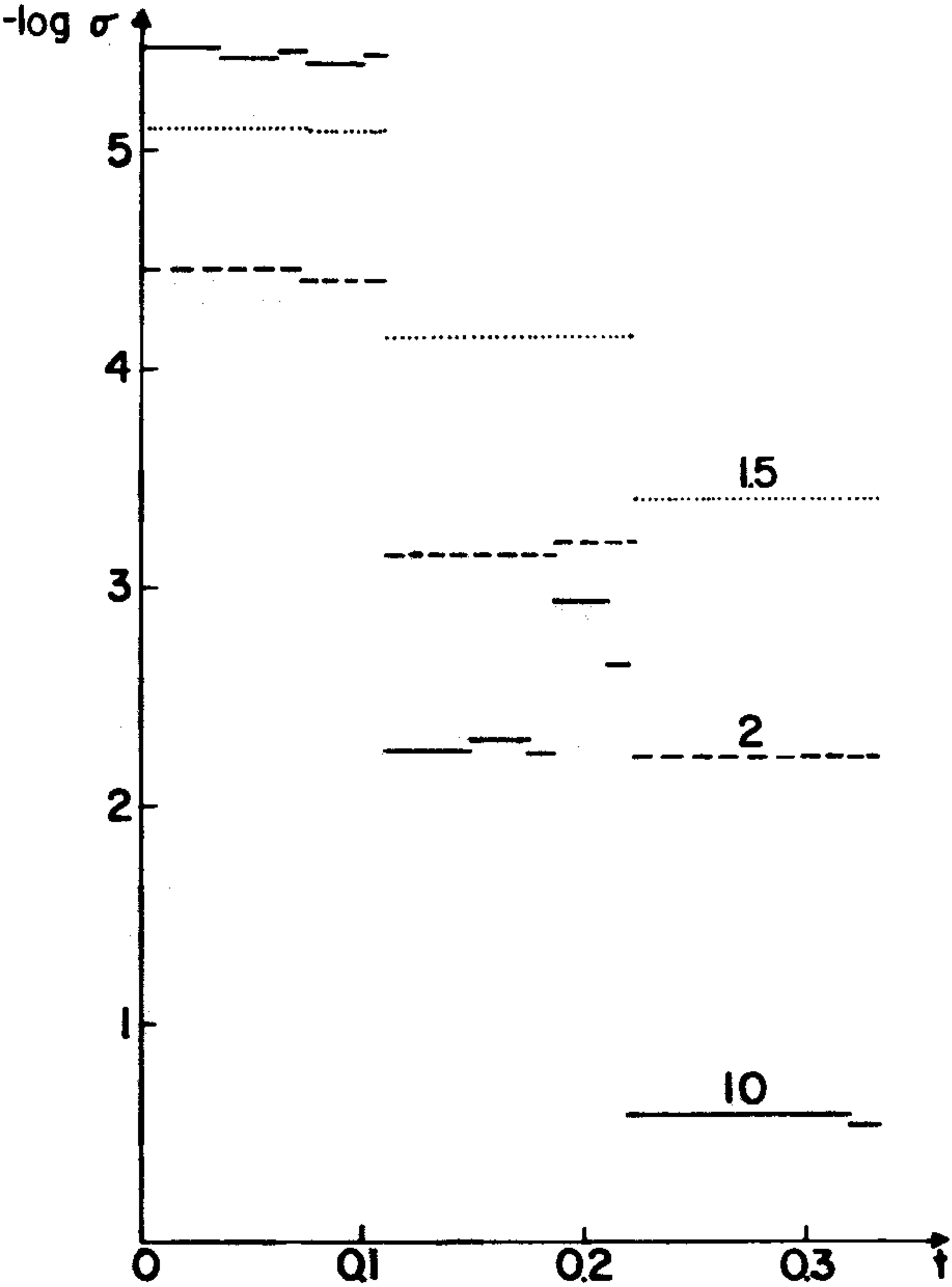

Fig. 3

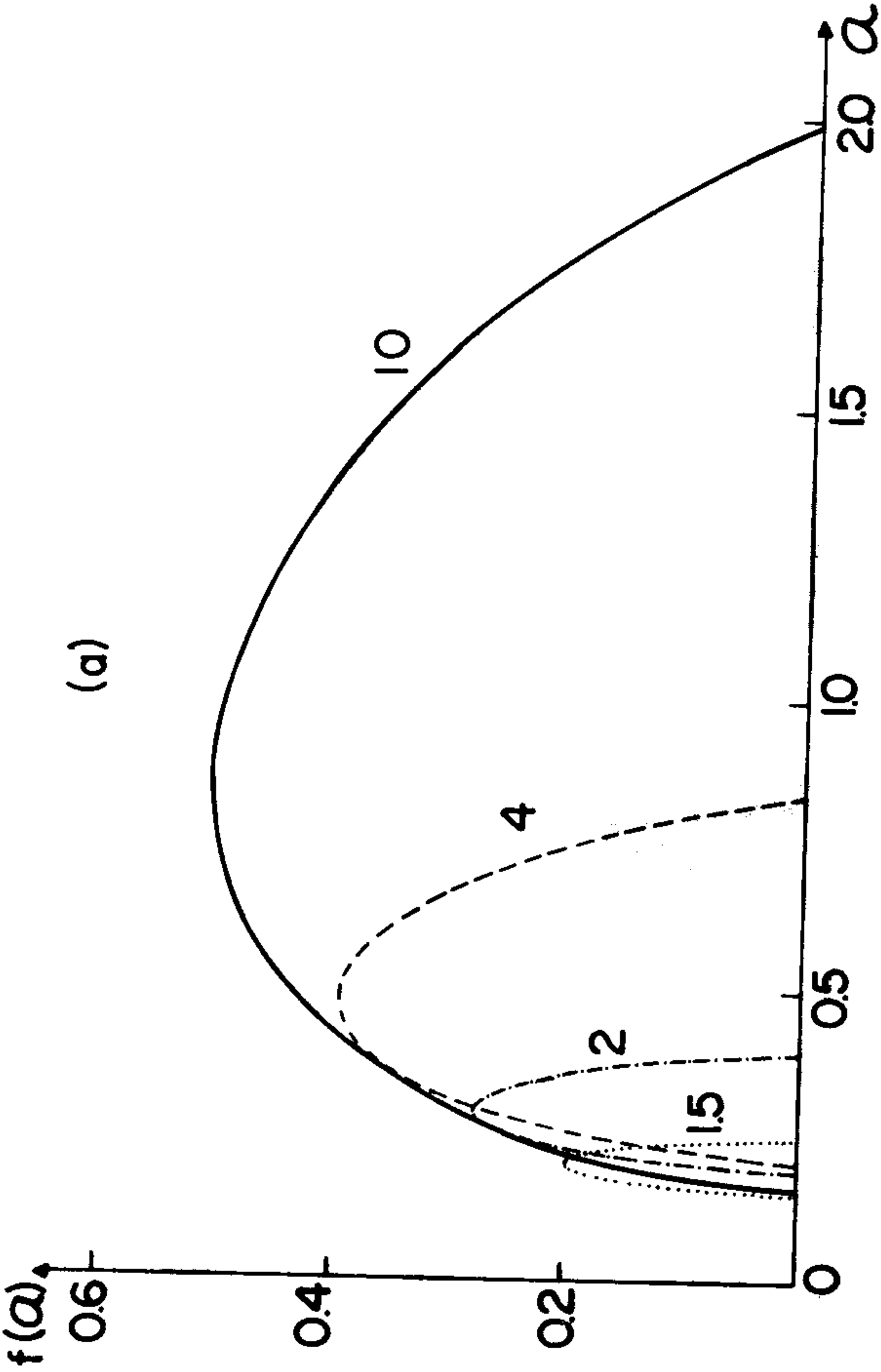

Fig. 4(a)

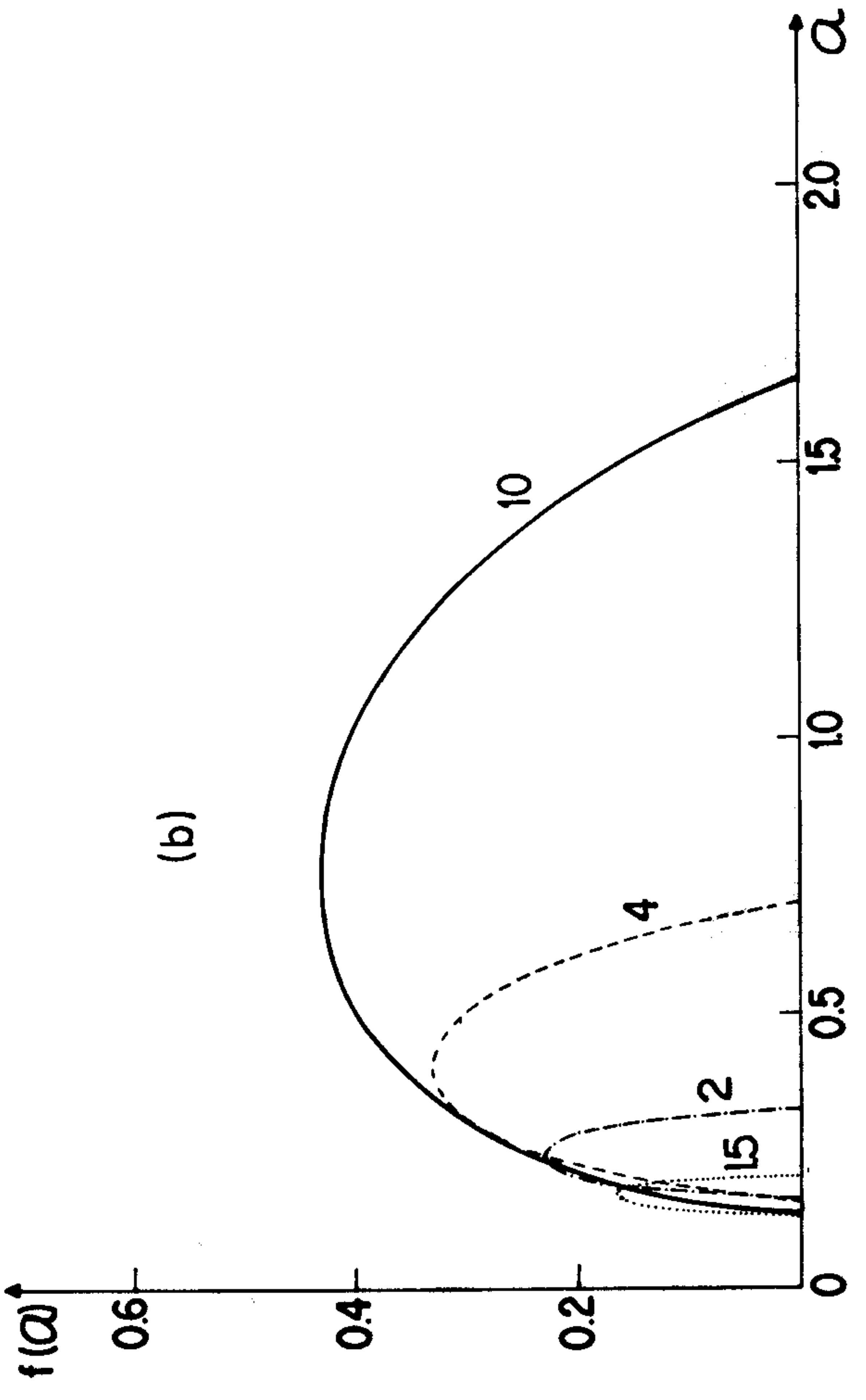

Fig. 4(b)

CAPTION FOR TABLE

Table I - Accumulation points $\tilde{\mu}_\infty$, and M-furcation rates δ and α for typical values of z and for the sequences $(RL)^{*n}$, $(RL^2)^{*n}$, $(RLR^2)^{*n}$, $(RL^2R)^{*n}$ and $(RL^3)^{*n}$. The results for $z=2,4,6$ and 8 are also calculated in Ref. 10., but the present numerical values are more accurate.

Table I

z		$(RL)^{*n}$	$(RL^2)^{*n}$	$(RLR^2)^{*n}$	$(RL^2R)^{*n}$	$(RL^3)^{*n}$
1.5	$\tilde{\mu}_\infty$	1.713540707	1.908140938	1.581073957	1.810146096	1.970391709
	δ	7.311×10^1	2.719×10^3	6.442×10^2	4.984×10^3	8.691×10^4
	α	3.010×10^1	3.234×10^2	1.292×10^2	5.143×10^2	3.174×10^3
2	$\tilde{\mu}_\infty$	1.786440255	1.942704354	1.631926654	1.862224022	1.985539530
	δ	5.524×10^1	9.816×10^2	2.555×10^2	1.287×10^3	1.693×10^4
	α	9.277	3.882×10^1	2.013×10^1	4.580×10^1	1.600×10^2
3	$\tilde{\mu}_\infty$	1.867865948	1.973452851	1.700204726	1.918298028	1.995250019
	δ	6.681×10^1	9.665×10^2	2.404×10^2	1.106×10^3	1.486×10^4
	α	4.364	1.063×10^1	6.720	1.125×10^1	2.645×10^1
4	$\tilde{\mu}_\infty$	1.909335470	1.985504660	1.743351015	1.945858588	1.997974021
	δ	8.578×10^1	1.275×10^3	2.919×10^2	1.418×10^3	2.099×10^4
	α	3.152	6.193	4.294	6.398	1.248×10^1
6	$\tilde{\mu}_\infty$	1.948866269	1.994205417	1.795920044	1.970972615	1.999432441
	δ	1.301×10^2	2.22×10^3	4.317×10^2	2.433×10^3	4.32×10^4
	α	2.281	3.659	2.790	3.727	6.007
8	$\tilde{\mu}_\infty$	1.966776434	1.997084404	1.827674871	1.981779236	1.999779411
	δ	1.789×10^2	3.49×10^3	5.89×10^2	3.79×10^3	7.87×10^4
	α	1.925	2.791	2.237	2.826	4.122
10	$\tilde{\mu}_\infty$	1.976500608	1.998317004	1.849408804	1.987431027	1.999896134
	δ	2.296×10^2	5.05×10^3	7.53×10^2	5.42×10^3	1.28×10^5
	α	1.729	2.355	1.949	2.375	3.326

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