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The Caldirola-Kanai Theory from "Brownian" Path Integral Quantum Mechanics

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Abstract

We deduce the classsical one-dimensional Caldirola-Kanai action from quantum mechanics, at the W.K.B. limit, formulated as an eletronic flux interacting with an enviroment in the usual Langevin-Brownian phenomenological path integral framework.

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It is an interesting problem in quantum mechanics to introduce interactions leading to damping in the equations of motion, at the classical limit of the associated quantum systems ([1],[4]).

In this brief report, we propose a phenomenological framework for the above mentioned problem by considering instanton configurations associated to a path integral ([2]) describing the W.K.B. leading limit of a quantum particle wave function interacting with a white noise reservoir with a friction interaction of a Brownian type.

Let us start by writing the one dimensional particle Schroedinger wave equation in a convenient way to implement the hydrodynamical analogy with quantum mechanics:

$$\Psi(x,t) \;=\; \rho^2(x,t)e^{\frac{i}{\hbar}S(x,t)} \tag{1}$$

$$\frac{\partial S(x,t)}{\partial t} + \frac{1}{2m}|\frac{d}{dx}S|^2(x,t) + \frac{\hbar^2}{2m}\frac{\frac{d^2}{dx^2}\rho(x,t)}{\rho(x,t)} + V(x,t) \;=\; 0 \tag{2}$$

$$\frac{\partial(\rho^2(x,t))}{\partial t} + \frac{d}{dx}(\rho^2(x,t)\frac{d}{dx}S(x,t)) \;=\; 0 \tag{3}$$

In order to write the above Schroedinger equation as a hydrodynamical equation suitable to introduce dissipation as usualy implemented in classical fluid fluxes, we consider a W.K.B. pure phase approximation for Eqs.(1)-(3) and relevant for probing the classical limit of the system

$$\Psi(x,t) \sim \exp(\frac{i}{\hbar}S(x,t)) \tag{4}$$

By substituting Eq.(4) into Eq.(2), we get as our basic phenomenological dynamical equation, the inviscid Burger equation for the eletronic flow defined by the wave function phase gradient $v(x,t) \;=\; \frac{d}{dx}S(x,t)$, namely:

$$\frac{\partial}{\partial t}v(x,t) + \frac{1}{2m}(v\frac{d}{dx})v \;=\; -\frac{d}{dx}V(x,t) \tag{5}$$

At this point it is worth to remark that the quantum mechanical probability conservation law, Eq.(3), will be neglected at the zeroth W.K.B. limit, Eq.(4), together with the quantum

potential $\frac{\hbar^2}{2m}\frac{\frac{d^2}{dx^2}\rho(x,t)}{\rho(x,t)}$ which has the meaning of describing a complicated pressure term in the quantum hydrodynamical interpretation ([4]).

In order to modelling, phenomenologically, the friction with a thermal reservoir, we follow the usual Langevin-Brownian procedure by introducing the viscosity stress tensor on the right-hand side of Eq.(5), as a damping term of the form $-\nu v(x,t)$. The reservoir fluctuation interaction will be modelled by random impurities producing a pure Gaussian stochastic force stirring the eletronic flux $v(x,t)$ ([5]).

As a consequence of the above made model assumptions, we are lead to consider the following Beltrami-Burger equation, in the range $0 \leq t \leq \infty$, describing our damped quantum eletronic system in the phase approximation, Eq.(4),

$$\frac{\partial v(x,t)}{\partial t} + \frac{1}{2m}(v\frac{d}{dx})v(x,t) \;=\; -\nu v(x,t) + F(x,t) - (\frac{d}{dx}V)(x,t) \tag{6}$$

Here the phenomenological damping term, ν, is temperature dependent and the stochastic force posesses the non-trivial two-point correlation form below, similar to phenomenological studies of Brownian motion ([5]),

$$\langle F(x,t)F(x',t')\rangle \;=\; \delta(x-x')\delta(t-t') \tag{7}$$

Proceeding as in Refs ([2], [3]), the space time characteristic functional of the random "forced" (Brownian) eletronic flux, Eq.(6), is given by the following Wyld-Rosen path integral

$$\begin{aligned} Z[J(x,t)] \;=\; & \frac{1}{Z[0]}\int \mathcal{D}^F[\Pi(x,t)]\mathcal{D}^F[v(x,t)] \times \\ & e^{i\int_{-\infty}^{+\infty}dx\int_0^{\infty}dt\Pi(x,t)(\frac{\partial v}{\partial t}+\frac{1}{2m}(v\frac{d}{dx})v+\nu v+\frac{d}{dx}V)(x,t)} \times \\ & e^{[-\frac{1}{2}\int_{-\infty}^{+\infty}d^3x\int_0^{\infty}dt\Pi^2(x,t)]}e^{[i\int J(x,t)v(x,t)dxdt]} \end{aligned} \tag{8}$$

Following Ref.[2], we consider the classical (instanton) equations of motion associated with the path integral, Eq.(8), with $J(x,t) \equiv 0$,

$$\frac{\partial v_{ins}(x,t)}{\partial t} + \frac{1}{2m}(v_{ins}(x,t)\frac{d}{dx})v_{ins}(x,t) + \frac{d}{dx}V(x) + \nu v_{ins}(x,t) \;=\; \Pi_{ins}(x,t) \tag{9}$$

$$-\frac{\partial \Pi_{ins}(x,t)}{\partial t} - \frac{1}{2m}(v_{ins}(x,t)\frac{d}{dx})\Pi_{ins}(x,t) + \nu \Pi_{ins}(x,t) \;=\; 0 \tag{10}$$

A straightforward calculation leads to the following solutions for $t > 0$:

$$\Pi_{ins}(x,t) \;=\; 0 \tag{11}$$

$$v_{ins}(x,t) \;=\; e^{-\nu t}\hat{v}_{ins}(x,t) \tag{12}$$

with $\hat{v}_{ins}(x,t)$ satisfying the equation

$$\frac{\partial \hat{v}_{ins}(x,t)}{\partial t} + \frac{1}{2me^{\nu t}}(\hat{v}_{ins}\frac{d}{dx})\hat{v}_{ins}(x,t) + e^{\nu t}(\frac{d}{dx}V)(x) \;=\; 0. \tag{13}$$

Now, proceeding backward of the steps taken at the begining of this brief report, we can write the effective Schroedinger equation for the wave function associated with the phase-flux instanton, Eq.(12),

$$i\hbar\frac{\partial \Psi_{ins}(x,t)}{\partial t} \;=\; (-\frac{\hbar^2}{2me^{\nu t}}\frac{d^2}{dx^2} + e^{\nu t}V(x))\Psi_{ins}(x,t) \tag{14}$$

It has a time dependent mass term and a linear forcing potencial as a consequence or our proposed Brownian-Ohmic friction enviroment interaction. It is worth to remark that effective Schroedinger equation, Eq.(14), may be formally associated with the well known Caldirola-Kanai Lagrangean([6]):

$$\mathcal{L}_{classical}(x,t) \;=\; \frac{1}{\hbar}\int_0^t d\sigma e^{\nu\sigma}[\frac{1}{2}m(\frac{dx}{d\sigma})^2 - V(x(\sigma))] \tag{15}$$

As a consequence of the above exposed results, we get our main result,i.e, the damped classical equation of motion for the quantum particle interacting with our Brownian reservoir model, at the W.K.B.($\hbar \to 0$) limit.

$$\frac{d^2x_{cl}(t)}{dt^2} + \nu\frac{dx_{cl}(t)}{dt} \;=\; -\frac{d}{dx}V(x_{cl}(t). \tag{16}$$

Finally, and for completness of our previous path integral study, let us briefly discuss the Fokker-Planck approach ([7]) for our Beltrami-Burger stochastic equation, Eq.(6), where for simplicity we have taken $V(x) \equiv 0$.

The Fokker-Planck probability distribution function in a Fourier domain is given by the equal time occupation random variable, i.e,

$$P((x_1,...,x_n);(p_1,...,p_n);t) \;=\; \langle \exp(i\sum_{k=1}^{N} p_k v(x_k;t))\rangle \tag{17}$$

where the average $\langle ...\rangle$ is defined by the stochastic proccess of the eletronic fluid velocity induced by the Gaussian stochastic force with general spatial correlation

$$\langle F(x,t)F(x',t')\rangle \;=\; K(x-x')\delta(t-t') \tag{18}$$

The Fokker-Planck equation is easily deduced by following closelly Ref.[6], i.e,

$$-i\frac{\partial}{\partial t}P((x_1,...,x_n);(p_1,...,p_n);t) \;=$$
$$\sum_{\ell=1}^{N}[p_\ell\frac{\partial}{\partial p_\ell}(\frac{1}{p_\ell}\frac{\partial}{\partial x_\ell}) - \nu p_\ell\frac{\partial}{\partial p_\ell}] + \sum_{\ell=1,\ell'=1}^{N}(K(x_\ell - x_{\ell'})p_\ell p_{\ell'})P((x_1,...,x_n);(p_1,...,p_n);t) \tag{19}$$

The main remark related to this Fokker-Planck approach is that Eq .(9) is a closed partial differential equation. Let us map Eq.(9) into a deterministic many-particle Schroedinger equation. In order to i8mplement such a mapping, we introduce mixed coordinates, i.e,

$$p_j + x_j \;=\; u_j \quad ; \quad p_j - x_j \;=\; v_j \tag{20}$$

The Fokker-Planck Eq.(19) takes, thus, the following closed form(after analytic continuation $v \to iv$).

$$-i\frac{\partial}{\partial t}P((u_1,...,u_n);(iv_1,...,iv_n);t) \;=$$
$$[\sum_{\ell=1}^{n}\frac{1}{4}[\frac{\partial^2}{\partial u_\ell^2} + \frac{\partial^2}{\partial v_\ell^2} - (\frac{2}{u_\ell + iv_\ell})(\frac{\partial}{\partial u_\ell} + i\frac{\partial}{\partial v_\ell}) - \frac{\nu}{2}(u_\ell + iv_\ell)(\frac{\partial}{\partial u_\ell} - i\frac{\partial}{\partial v_\ell})]$$
$$+\frac{1}{4}\sum_{\ell=1;\ell'=1}^{n}(u_\ell + iv_\ell)(u'_\ell + iv_{\ell'})K(u_\ell - u_{\ell'} + i(v_{\ell'} - v_\ell))]P((u_1,...,u_n);(iv_1,...,iv_n);t) \tag{21}$$

At this point we remark that it is straightforward to implement a numerical, "low viscosity" ($\nu \ll 1$), perturbative calculations by considering a slowly varying (even function) correlation function of the form (see Ref.[2]):

$$K(x) \sim K(0) - \frac{\ell_0}{2}x^2; \quad |x| \ll (\frac{K(0)}{\ell_0})^1/2 \equiv L$$
$$K(x) \sim 0; \quad |x| \gg L \tag{22}$$

Numerical studies of Eq.(21) are in progress and will appear elsewhere.

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