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# Quantum Dynamics of Inhomogeneous Kaluza-Klein Cosmological Models near the Cosmological Singularity

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Abstract

Quantum dynamics of inhomogeneous Kaluza-Klein cosmological models in the vicinity of a cosmological singularity is considered. We use the Kasner-like parametrization to divide dynamical variables in two parts. The first part contains scale functions of Kasner vectors and behaves near the singularity like ordinary scalar fields. The second part containing residual variables has behavior like a set of vectors fields and in leading order turns out to be negligible as that of the ordinary matter sources having equations of state $\epsilon > p$. Within that approximation we solve the Wheelr-DeWitt equation and define the probability interpretation by means of singling out a positive-frequency sector in the space of solutions. In virtue of an ambiguity of such a procedure it is argued the need of some kind a third quantization for gravity. We suggest a scheem for third quantization and show that properties of inhomogeneities can be generated mostly by vacuum polarization effects in a third quantized theory.

**Key-words:** Inhomogeneous multidimensional cosmology; Quantum model; Third quantization.

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# 1 Introduction.

One of the most difficult problems of modern theoretical physics is the problem of the cosmological singularity. Singularities follow from the classical theory and, as is widely accepted, need quantum gravity to provide its exhaustive description. We do not have any reasonable theory of such a kind yet save, presumably, the superstring theory [1]. And as is known, the last one adds some new features to the existing Einstein gravity. In particular, the superstring theory predicts the dimension of the universe exceeds that of we use to experience at a macroscopic level. In the present Universe additional dimensions are supposed to be compactified to the Planckian size, and display themselves as a set of ordinary matter fields. However, close to the singularity one should expect that all dimensions to play an equal role, and have to be regarded on an equal footing. This enables us to consider more general than Einstein's one multidimensional theories of gravity [2] in order to study the nature and properties of singularitites.

From the classical point of view properties of general inhomogeneous cosmological Kaluza-Klein models near the singularity were recently considered in Ref. [3] (for more early investigations of the problem see also Refs.[4, 5]). It was shown that the properties of metric functions near the singularity may be well-described in the framework of asymptotic models. In this paper we are considering a quantum description of just those models and investigate their behavior near the singularity from the quantum point of view. The main result of this paper is that in the case of $n \leq 9$ ($n$ is the number of spatial dimensions) estimates for mean values of scale functions turns out to be of the same order as in the classical theory. For mean scale factor we get $< a_i >=< g^{Q_i} >\sim cg^{Q_{\min}}$ as $g \to 0$, where $g$ is the metric determinant which near the singularity may serve as a time variable, $Q_{\min} = -\frac{n-3}{n+1}$ is the minimal admissible value of the anisotropy parameters $Q_i$ and $c$ is a slowly varying with $g$ function, including quantum corrections, and differing from the classical one. When considering dimensions exceeding $n = 9$ the situation changes drastically. The potential does not restrict the configuration space and, therefore, we have no states which would be localized on the space of $Q_i$. If we get ready a localized state (a wave packet) the width of the packet spreads eventually more and more out and simultaneously the center of the wave packet runs to the infinity of the configuration space. In classical theory this signals us that the oscillatory mode becomes unstable and transforms into a Kasner-like behavior. Therefore, different mean values will depend upon the initial state crucially.

The paper is organized as follows. In Sec.2 we use generalized Kasner variables introduced first in Ref. [6] and adapted to the multidimensional models in Ref. [3] to divide basic variables into two parts. Near the singularity the first part has a behavior like a set of coupled scalar fields while residual variables behave as a set of vector fields and can be neglected in a leading order (in the same manner as it happens for the matter having an equation of state $\epsilon > p$, where $p$ and $\epsilon$ are an energy density and pressure respectively) [7, 3]. The asymptotic model is derived in Sec.3. In Sec.4 we consider the quantization of the model. The Wheeler-DeWitt equation turns out to be dependent upon the first group variables only. We solve this equation in a lattice approximation of the coordinate manifold. The probability interpretation is introduced by making use of an explicit selection of a positive frequency sector on the space of solutions to the Wheeler-DeWitt equation [8]. Such procedure implies an ambiguity and, therefore, the same ambiguity will be inherently presented in the obtained quantum gravity. In order to overcome this difficulty we, in Sec.5,6, discuss the possibility of the third quantization. We note that the third quantization seems to be the natural scheme providing a description of different possible topologies of the universe [9, 10]. We use the scheme proposed in [11] to show that in the course of the evolution the presence of matter, e.g. of an ordinary scalar field, can result in an increasing of quantum topology fluctuations and, therefore, properties of inhomogeneities of the metric may completely be determined by vacuum fluctuations in the third quantized theory. We conclude this paper with some estimates and speculations in Sec.7.

# 2 Generalized Kasner Solution, Generalized Kasner Variables

Aiming to obtain a quantum description of inhomogeneous Kaluza-Klein models we start with the canonical formulation of multidimensional gravity. In this formulation basic variables are the spatial Riemann metric components $g_{\alpha\beta}$ and the matter source which will be taken in the form of a scalar field $\phi$ and its conjugate momenta $\Pi^{\alpha\beta} = \sqrt{g}(K^{\alpha\beta} - g^{\alpha\beta}K)$ and $\Pi_\phi$. These variables are functions specified on the $n-$manifold $S$ ($\alpha = 1, ..., n$) and $K^{\alpha\beta}$ is the extrinsic curvature of $S$. For the sake of simplicity we shall consider $S$ to be compact i.e. $\partial S = 0$ (one may consider $S$ to be the $n$-dimensional sphere though this will not have any significance for our investigation). The action has the following form in Planck units (see for example [12])

$$I = \int_S (\Pi^{ij}\frac{\partial g_{ij}}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - NH^0 - N_\alpha H^\alpha) d^n x dt, \tag{2.1}$$

where

$$H^0 = \frac{1}{\sqrt{g}}\left\{\Pi^\alpha_\beta \Pi^\beta_\alpha - \frac{1}{n-1}(\Pi^\alpha_\alpha)^2 + \frac{1}{2}\Pi^2_\phi + g(W(\phi) - R)\right\}, \tag{2.2}$$

$$H^\alpha = -2\Pi^{\alpha\beta}_{|\beta} + g^{\alpha\beta}\partial_\beta\phi\Pi_\phi, \tag{2.3}$$

here

$$W(\phi) = \frac{1}{2}\left\{g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right\}. \tag{2.4}$$

It turns out to be convenient to use the so-called generalized Kasner-like parametrization of the dynamical variables [6, 3]. The metric components and their conjugate momenta are represented as follows

$$g_{\alpha\beta} = \sum_a \exp\left\{q^a\right\} l^a_\alpha l^a_\beta, \qquad \Pi^\alpha_\beta = \sum_a p_a L^\alpha_a l^a_\beta, \tag{2.5}$$

where $L^\alpha_a l^b_\alpha = \delta^b_a$ ($a, b = 0, ..., (n-1)$), and the vectors $l^a_\alpha$ contain only $n(n-1)$ arbitrary functions of spatial coordinates. Further parametrization may be taken in the form [3]

$$l^a_\alpha = U^a_b S^b_\alpha, \qquad U^a_b \in SO(n), \qquad S^a_\alpha = \delta^a_\alpha + R^a_\alpha \tag{2.6}$$

where $R^a_\alpha$ denotes a triangle matrix ($R^a_\alpha = 0$ as $a < \alpha$). Substituting Eq.(2.5), (2.6) into (2.1) we find the following expression for the action functional

$$I = \int_S (p_a\frac{\partial q^a}{\partial t} + T^\alpha_a\frac{\partial R^a_\alpha}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - NH^0 - N_\alpha H^\alpha) d^n x dt, \tag{2.7}$$

where $T^\alpha_a = 2\sum_b p_b L^\alpha_b U^b_a$ and the Hamiltonian constraint takes the form

$$H^0 = \frac{1}{\sqrt{g}}\left\{\sum p^2_a - \frac{1}{n-1}(\sum p_a)^2 + \frac{1}{2}\Pi^2_\phi + V\right\}. \tag{2.8}$$

In the case of $n = 3$ the functions $R^a_\alpha$ are connected purely with transformations of a coordinate system and may be removed by resolving momentum constraints $H^\alpha = 0$ [6]. However, in the multidimensional case the functions $R^a_\alpha$ contain $\frac{n(n-3)}{2}$ dynamical functions as well.

# 3 Asymptotic model in the case of arbitrary small times

As it was shown, [6, 13, 3] (see also [12]), in the vicinity of a singularity the potential term in (2.8) can be modeled by potential walls. To this end we represent the potential in the following form

$$V = \sum_{A=1}^{k} \lambda_A g^{\sigma_A}, \tag{3.1}$$

where $\lambda_A$ is a set of functions of all dynamical variables and of their derivatives and $\sigma_A$ is given by the expression

$$\sigma_{abc} = 1 + Q_a - Q_b - Q_c, \qquad b \neq c, \tag{3.2}$$

where $Q_a$ are the anisotropy parameters $Q_a = \frac{q^a}{\sum q}$. Then assuming the finiteness of the functions $\lambda_A$ and considering the limit $g \to 0$ we find that the potential $V$ may be modeled by potential walls

$$g^{\sigma_A} \to \theta_\infty[\sigma_A(Q)] = \begin{cases} +\infty\ , & \sigma_A < 0, \\ 0\ , & \sigma_A > 0 \end{cases} \tag{3.3}$$

Thus, if we put the expressions (3.3) into (2.8) we find that the Hamiltonian constraint does not depend on the variables $R^a_\alpha$ and its conjugate momenta $T^\alpha_a$. Of course this is an approximation and the real potential reserves a dependence of this group of variables and, therefore, one should consider the model (3.3), (2.8) as a first step in an approximation procedure. Then the rest of dynamical variables as well as an ordinary matter sources with the state equation satisfying the inequality $\epsilon > p$ may be accounted in subsequent steps of the approximation procedure [7].

Now we can remove the rest of dynamical functions $T^\alpha_a$, $R^a_\alpha$ from the action (2.7) by putting $N^\alpha = 0$. Then we get the reduced dynamical system

$$I = \int_S \left\{ p_a \frac{\partial q^a}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - \lambda \left\{ \sum p^2 - \frac{1}{n-1}(\sum p)^2 + \frac{1}{2}\Pi^2_\phi + U(Q) \right\} \right\} d^n x dt, \tag{3.4}$$

where $\lambda$ is expressed via the lapse function as $\lambda = \frac{N}{\sqrt{g}}$.

The configuration space $M$ of the system (3.4) (called also superspace) can be represented in the form of the direct product $M = \prod_{x\in S} M_x$. Moreover, every local space $M_x$ is the ordinary $n+1$-dimensional pseudo-Euclidean space. Indeed, one can choose on $M$ a new harmonic set of variables related to the old ones as follows

$$q^a = A^a_j z^j + z^0, \qquad z^n = \sqrt{\frac{2}{n(n-1)}}\phi, \tag{3.5}$$

where $j = 1, ..., n-1$, $a = 0, ...n-1$ and the matrix $A^a_j$ is a constant [3] which obeys the conditions

$$\sum_a A^a_j = 0, \qquad \sum_a A^a_j A^a_k = n(n-1)\delta_{jk}\ , \tag{3.6}$$

and can be expressed in the explicit form as

$$A^a_j = \sqrt{\frac{n(n-1)}{j(j+1)}}(\theta^a_j - j\delta^a_j), \qquad \theta^a_j = \begin{cases} 1\ , & j > a\ , \\ 0\ , & j \le a\ . \end{cases} \ . \tag{3.7}$$

Then the action (3.4) takes the form formally coincided with the action for a continueous set of relativistic particles

$$I = \int_S \left\{ P_r \frac{\partial z^r}{\partial t} - \lambda^{'}(P^2_i + U - P^2_0) \right\} d^n x dt, \tag{3.8}$$

where $r = 0, ..., n$, $i = 1, ..., n$, $\lambda^{'} = \frac{\lambda}{n(n-1)}$ and the kinetic term, that determines a metric on $M_x$, turns out to be coincided with that of the ordinary flat $n+1$-dimensional pseudo-Euclidean spacetime manifold.

# 4 Quantization and the probability interpretation

As it was mentioned above the action (3.8) resembles the action for a continueous set of relativistic particles. Therefore, quantization of such a system may be carried out in the complete analogy with that of relativistic particles [14]. The zero-energy Hamiltonian constraint leads to the set of the Wheeler-DeWitt equations [15]

$$(-\Delta_x + U_x + \xi P_x)\Psi = 0, \qquad x \in S, \tag{4.1}$$

where $\Psi$ is the wave function of the universe, $\Delta_x$ denotes a Laplace operator on $M_x$ : $\Delta_x = \frac{1}{\sqrt{-G}}\partial_A \sqrt{-G} G^{AB} \partial_B$, $G_{AB}$ is the metric on $M_x$ determined by the interval

$$\delta\Gamma(x)^2 = \frac{1}{4\lambda'}((\delta z^i(x))^2 - (\delta z^0(x))^2), \tag{4.2}$$

$P_x$ is the curvature scalar of $M_x$. The value of $\xi$ should be chosen as $\xi = \frac{n-1}{4n}$ to provide a conformal invariance of Eq.(4.1) which reflects the arbitrariness in the choice of the lapse function $\lambda$ . Indeed, the transformation

$$G_{AB} \to \widetilde{G}_{AB} = e^{-2\Omega} G_{AB}, \qquad \Psi \to \widetilde{\Psi} = e^{\frac{n-1}{2}\Omega}\Psi$$

transforms the Eq.(4.1) into

$$(-\widetilde{\Delta}_x + e^{2\Omega} U_x + \frac{n-1}{4n}\widetilde{P}_x)\widetilde{\Psi} = 0.$$

and the theory becomes independent on a particular choice of $\lambda$.

To solve the equation (4.1) we shall consider a lattice approximation. To this end we shall suppose the existence of a sufficiently small minimal scale of inhomogeneity for all fields $l_{min}$ , so that the coordinates $x$ will take discrete values only. The continueous limit one obtains tending $l_{min}$ to zero, though, from the other side, one may think of the lattice model as of a background model and treat the scales less than $l_{min}$ as small perturbations.

The system of equations (4.1) turns out to be uncoupled, for each from these equations contains a set of functions which are specified at a distinct point $x$ of $S$ . We shall call such sets as $x-$sets. Therefore, the space $H$ of solutions to this system takes the form of the tensor product of spaces $H_x$ ($H = \prod_{x \in S} H_x$) as that of $M$, where $H_x$ is the space of solutions to a distinct $x-$ equation ( 4.1). Accordingly, all $x-$sets of degrees of freedom may independently be considered. Therefore, at first it will be convenient to work out the probability interpretation and all the technique on the example of one local $x-$set of degrees of freedom and after that to generalize it to the case of all degrees of freedom.

## 4.1 The space of solutions to the WDW equation for a distinct $x$-set of degrees of freedom.

Every local $x$-equation (4.1) admits the conserved current $J_A(\Psi, \Psi) = i[\Psi^* \nabla_A \Psi - \Psi \nabla_A \Psi^*]$ which may be used to determine the inner product in the space $H_x$

$$< \varphi \mid \chi >= i \int_{\Sigma_x} J_A(\varphi, \chi) d\Sigma_x^A, \tag{4.3}$$

where $\Sigma_x$ is an arbitrary space-like surface on $M_x$ and $\nabla_A$ denotes a covariant derivative on $x$-metric (4.2).

To construct a complete set of solutions to the local Eq.(4.1) it turns out to be convenient by making use of the so-called Misner-Chitre like variables [6, 3] ($\vec{y} = y^j$, $j = 1, ..., n-1$)

$$z^0 = -e^{-\tau}\frac{1+y^2}{1-y^2}, \qquad \vec{z} = -2e^{-\tau}\frac{\vec{y}}{1-y^2}, \qquad y = \mid \vec{y} \mid \leq 1. \tag{4.4}$$

In these variables the anisotropy parameters become independent of the timelike variable $\tau$

$$Q_a(y) = \frac{1}{n}\left\{1 + \frac{2A_j^a y^j}{1+y^2}\right\} \tag{4.5}$$

and that of the potential $U(Q)$ in Eq.(4.1). The metric (4.2) in the new variables takes the form

$$\delta\Gamma(x)^2 = \frac{e^{-2\tau}}{4\lambda'}\left(\frac{4(\delta y^j)^2}{(1-y^2)^2} + e^{2\tau}(\delta z^n)^2 - (\delta\tau)^2\right). \tag{4.6}$$

For the sake of simplicity we shall use the gauge $4\lambda' e^{2\tau} = 1$ in what follows.

The part of the configuration space $M_x$ related to the variables $\vec{y}$ is a realization of the $(n-1)$-dimensional Lobachevsky space and the potential $U$ cuts a part $K$ of it [4, 5, 3]

$$\sigma_{abc} = 1 + Q_a - Q_b - Q_c \geq 0, \qquad a \neq b \neq c \tag{4.7}$$

which in the case $n \leq 9$ has a finite volume. We shall suppose that there is a set of solutions to the eigenvalue problem for the Laplace - Beltrami operator

$$(\Delta_y + k_J^2 + \frac{(n-2)^2}{4})\varphi_J(z) = 0, \qquad \varphi_J\mid_{\partial K} = 0, \tag{4.8}$$

where the Laplace operator $\Delta_y$ is constructed via the metric $dl^2 = h_{ij}dy^i dy^j = \frac{4(dy)^2}{(1-y^2)^2}$ and $J$ collects all indices numbering the eigenfunctions $\varphi_J$. In the case of $n < 10$ the region $K$ has a finite volume and $J$ takes discrete values ($J = 0, 1, 2, \ldots$), while for $n \geq 10$ the volume of $K$ is infinite and the spectrum of the Laplace - Beltrami operator becomes a continueous one. The functions $\varphi_j$ obey the orthogonality and normalization relations

$$(\varphi_I, \varphi_J) = \int_K \varphi_I^*(y)\,\varphi_J(y)d\mu(y) = \delta_{IJ}, \tag{4.9}$$

where $d\mu(y) = \frac{1}{c}\sqrt{h}d^{n-1}y = \frac{2^{n-1}}{c}\frac{d^{n-1}y}{(1-y^2)^{n-1}}$ , and $c$ is the volume of $K$. The completeness conditions are

$$\sum_I \varphi_I^*(y)\,\varphi_I(y') = \frac{\delta(y-y')}{\sqrt{h}}.$$

Then a complete orthonormal set $\{u_p, u_p^*\}$ of solutions to $x-$equation (4.1) is constituted by functions of the form

$$u_p = \exp(-\frac{1}{2}\tau)\chi_p(\tau)\Phi_p(y,z) \quad , \quad \Phi_p(y,z) = (2\pi)^{-1/2}\varphi_J(y)\exp(i\epsilon z^n) \tag{4.10}$$

where $p = (J, \epsilon)$. Functions $\chi_p(\tau)$ satisfy the equation following from (4.1):

$$\frac{d^2\chi_p}{d\tau^2} + \omega_p^2(\tau)\chi_p = 0, \qquad \omega_p^2(\tau) = k_J^2 + \epsilon^2 e^{-2\tau} \tag{4.11}$$

with the normalization condition $\chi_p^*\frac{d\chi_p}{d\tau} - \chi_p\frac{d\chi_p^*}{d\tau} = -i$, and are expressed via the Bessel functions. The initial conditions to Eq.(4.11) at a moment $\tau_0$ are to be taken in the form $\chi_p(\tau_0) = \frac{1}{\sqrt{\omega_p(\tau_0)}}$ , $\chi_p'(\tau_0) = -i\omega_p(\tau_0)\,\chi_p(\tau_0)$ .

The set of solutions (4.10) is orthonormal in the sense of the scalar product (4.3), i.e. they satisfy the relations

$$< u_p \mid u_q >= - < u_p^* \mid u_q^* >= \delta_{pq}, \qquad < u_p \mid u_q^* >= 0. \tag{4.12}$$

Thus, an arbitrary solution $f$ to the local Wheeler-DeWitt equation (4.1) can be represented in the form

$$f = \sum_p A_p^+ u_p + A_p^- u_p^*, \tag{4.13}$$

where $A_p^\pm$ are arbitrary constants which are to be specified by initial conditions.

## 4.2 Probability interpretation and the case of all degrees of freedom

Since the norm determined by the scalar product (4.3) turns out to be sign-indefinite we face up with the difficulty of probability interpretation. The simplest way to define a positive-definite inner product is to separate a submanifold $H_x^+$ on the space $H_x$ which is of "positive frequency". If we suppose $A_p^- = 0$ in (4.13), then the normalization condition for $f$ takes the form

$$< f \mid f >= \sum_p \mid A_p^+ \mid^2 = 1, \tag{4.14}$$

and meets no difficulties. Thus, the subspace of physical states $H_x^+$ becomes the ordinary Hilbert space and we can adopt the standard probability interpretation [14].

Now the generalization to the case of all degrees of freedom may be carried out straightforwardly. The positive frequency sector $H^+$ in the total space of solutions $H$ we determine as the direct product of positive frequency local submanifolds $H^+ = \prod_{x \in S} H_x^+$ . Thus, the wave function takes the form

$$\Psi = \sum_{[p(x)]} F_{p(x)} U_{p(x)}, \qquad U_{p(x)} = \prod_{x \in S} u_{p(x)} \tag{4.15}$$

with the scalar product induced by (4.12)

$$\langle \chi | \psi \rangle = \sum_{[p(x)]} B^*_{p(x)} A_{p(x)} \ , \tag{4.16}$$

where $\chi = \sum B_{p(x)} U_{p(x)}$ and $\psi = \sum A_{p(x)} U_{p(x)}$ are arbitrary vectors from $H^+$ .

Dispite that Eq. (4.15) and (4.16) give already well defined probability interpretation it is necessary to mention that the procedure of the choice of $H_x^+$ in the $H_x$ is not uniquely defined. We can use a Bogoliubov transformation to construct a new set of modes

$$v_{p,x} = \sum_q \left\{ \alpha\left(x\right)_{pq} u_q + \beta\left(x\right)_{pq} u_q^* \right\} \tag{4.17}$$

where we add the label $x$ to point out the possible dependence on $x \in S$ and while $\beta\left(x\right)_{pq} \neq 0$ different sets of modes (4.17) define different submanifolds $H_x^+$. The situation will be worse still when considering the total space $H$. Therefore, the probability interpretation turns out to be crucially dependent upon the particular choice of the physical sector $H^+$ in $H$ . Here we face with the main inherent difficulty of quantum cosmology which, apparently, cannot be solved in the framework of the ordinary "one-particle" quantum gravity. To overcome this difficulty it is necessary to use the procedure of second (or "third") quantization of the wave function of the Universe [9, 16, 10, 17].

# 5 Third quantization

In addition to provide a probability interpretation third quantization has another goal. This theory allows describe processes connected with topology changes. The simplest processes of such a kind was widely discussed earlier in connection with wormholes and baby universes [9] and in the context of a description of a quantum creation of the Universe from nothing [16, 17]. In the present section we use a new approach pointed out in Refs. [10, 11] which generalizes the third quantization and allows to describe arbitrary topologies of the universe. That generalization follows from the fact that the system of WDW equations (4.1) is uncoupled in the leading order. Therefore, one may secondly quantize every $x$-set of degrees of freedom independently from each other. In quantum gravity this corresponds to the situation when the number of points of the physically observable space, specified at a particular point of the basic coordinate manifold $S$, turns out to be a variable and topology of the physical space may be different from that of $S$ [10, 11] (below we shall follow Ref. [11]) .

Let us consider a distinct $x-$set of the degrees of freedom. While we do not account for interactions between these sets we can describe quantum states of each set by a local wave functions $\Psi_x$ . When the third quantization is imposed the wave functions $\Psi_x$ become field operators and can be expanded in the form (4.13) (for simplicity we consider $\Psi_x$ to be a real scalar function):

$$\Psi_x = \sum C(p,x) u(p,x) + C^+(p,x) u^*(p,x), \tag{5.1}$$

where $u(p,x)$ is the set of modes (4.12) and the label $x$ we add to point out the possible dependence on spatial coordinates. Now we consider the operators $C(p,x)$ and $C^+(p,x)$ to satisfy the standard commutation relations

$$[C(p,x), C^+(q,x')] = \delta_{p,q} \delta(x,x'). \tag{5.2}$$

The field operators $\Psi_x$ act on a Hilbert space of states which has the well known structure in the Fock representation. The vacuum state is defined by the relations $C(x,p) \mid 0 >= 0$ (for all $x \in S$), $< 0|0 >= 1$.

Acting by the creation operators $C^+(p,x)$ on the vacuum state we can construct states describing the Universe with arbitrary spatial topologies. In particular, the states of the type (4.15) describing the Universe whose spatial topology coinsides with the topology of $S$ take the structure

$$|f>=\sum_{[p(x)]} F_{p(x)}|1_{p(x)}>, \qquad |1_{p(x)}>=\frac{1}{Z_1}\prod_{x\in S} C^+(x,p(x))|0>, \tag{5.3}$$

where $Z$ is a normalization constant and the wave function (4.15) describing the simple-topology Universe can be found as

$$<0|\Psi|f>=<0|\prod_{x\in S}\Psi_x|f>=\sum_{[p(x)]} F_{p(x)}U_{p(x)}. \tag{5.4}$$

The states describing the Universe with $n$ disconnected spatial components have the structure

$$|n>=|1_{p_1(x)},...,1_{p_n(x)}>=\frac{1}{Z_n}\prod_{i=1}^{n}\prod_{x\in S} C^+(x,p_i(x))|0> \tag{5.5}$$

(we recall that in the model under consideration in virtue of the existence of $l_{min}$ the coordinates $x$ take discrete values). Besides these states describing simplest topologies the approach considered allows to construct nontrivial topologies as well. This is due to the fact that the tensor product in (5.3), (5.5) may be defined either over the whole coordinate manifold $S$ or over part of it $D\subset S$. In this manner, taking sufficiently small pieces $D_i$ of the coordinate manifold $S$ we can glue arbitrarily complex physical spaces. In order to construct the states of such a kind it is convenient to introduce a set of operators as follows

$$a(D,p(D))=\prod_{x\in D} C(x,p(x)), \qquad a^+(D,p(D))=\prod_{x\in D} C^+(x,p(x)). \tag{5.6}$$

These operators have a clear interpretation, e.g. the operator $a^+(D,p(D))$ creates the whole region $D\in S$ having the quantum numbers $p(D)$. Thus, in the general case states of the Universe will be described by vectors of the type

$$|\Phi>=c_0|0>+\sum_I c_I a_I^+|0>+\sum_{I,J} c_{IJ}a_I^+a_J^+|0>+.... \tag{5.7}$$

Now consider an interpretation of the scheme suggested in [11]. Ordinary measurements are usually performed only on a part $K$ of the coordinate manifold $S$. There are two possibilities. The first one is that an observer measures all of the quantum state of the region $K$ and, the second, more probable one is when the observer measures only a part of the state. In the second case the observer considers $K$ as if it were a part of the ordinary flat space. Therefore, the part of the quantum state which will be measured, appears to be in a mixed state. This means the loss of quantum coherence widely discussed in Refs.[9]. In order to describe measurements of the second type we define the following density matrix for the region $K$

$$\rho^{nm}(K)=\frac{1}{N(K)}<\Phi|a^+(K,n(K))a(K,m(K))|\Phi>, \tag{5.8}$$

where $|\Phi>$ is an arbitrary state vector of the (5.7) type and $N(K)$ is a normalization function which measures the difference between the real spatial topology and the coordinate manifold $S$. If we consider the smallest region $K$ which contains only one point $x$ of the space $S$ the normalization function $N(x)$ in (5.8) will play the role of a "density" of the physical space. For the states (5.3), (5.5) we have $N(x)=1$ and $N(x)=n$ respectively. Thus, if $A(K)$ is any observable we find $<A>=\frac{1}{N}Tr(A\rho)$.

# 6 Topology fluctuations and quantum creation of the Universe from nothing

Since the WDW Eq.(4.1) has an explicit "time"-dependent form one could expect the existence of quantum polarization effects (topology fluctuation or the so-called spacetime foam [18, 19]). These effects can be calculated either by singling out the asymptotic *in* and *out* regions on the configuration space $M$ for which we can determine positive-frequency solutions to Eq.(4.1) (see for example [17]), or by using the diagonalization of the Hamiltonian technique [20] by means of calculating depending on time Bogoliubov's coefficients. Let us consider solutions (4.10) of the arbitrary local $x$-equation (4.1). The function $\chi_p$ can be decomposed in positive and negative frequency parts

$$\chi_p = \frac{1}{\sqrt{2\omega_p}}(\alpha_p e^{-i\theta_p} + \beta_p e^{i\theta_p}), \qquad \frac{d\chi_p}{d\tau} = -i\sqrt{\frac{\omega_p}{2}}(\alpha_p e^{-i\theta_p} - \beta_p e^{i\theta_p}), \tag{6.1}$$

where $\theta_p = \int_{\tau_0}^{\tau} \omega_p d\tau$ . The functions $\alpha_p$ and $\beta_p$ satisfy identity $|\alpha_p|^2 - |\beta_p|^2 = 1$ and define the depending on time Bogoliubov coefficients [20]. The depending on time creation and annihilation operators take the form

$$b_\tau(x,p) = \alpha_p(\tau)\,C(x,p) + \beta_p^*(\tau)\,C^+(x,p) \quad , \quad b_\tau^+(x,p) = \alpha_p^*(\tau)\,C^+(x,p) + \beta_p(\tau)\,C(x,p) \quad . \tag{6.2}$$

In terms of these operators the super-Hamiltonian of the field $\Psi_x$ (the Hamiltonian density) becomes diagonal

$$E_x = \int\limits_{\Sigma^\tau} \Theta_{\tau A}\, d\Sigma_x^A = \frac{1}{2}\sum_p \omega_p(\tau)\left(b_\tau^+(x,p)\,b_\tau(x,p) + b_\tau(x,p)\,b_\tau^+(x,p)\right) \ , \tag{6.3}$$

where $\Theta_{AB} = \nabla_A\Psi_x\nabla_B\Psi_x - \frac{1}{2}G_{AB}\left(\nabla_C\Psi_x\nabla^C\Psi_x - (U+\xi P)\Psi_x^2\right)$ and $d\Sigma_x^\tau = \sqrt{G^n}d^{n-1}ydz$, $G^n$ is the metric on $\Sigma_x^\tau$ induced by (4.6) . The ground state of the Hamiltonian is deternined by the conditions $b_\tau(x,p)\,|0_\tau\rangle = 0$ for all $x$ and $p$ and is also depending on time. The excitations of (6.3) are interpreted as points of physical space having the coordinate $x \in S$ .

Now we determine two asymptotic regions as *in* ($\tau \to -\infty$) and *out* ( $\tau_0 \to +\infty$). In these regions the functions $\alpha_p$ and $\beta_p$ take constant values. Substituting the initial conditions $\alpha_p = 1$, $\beta_p = 0$ as $\tau_0 \to -\infty$ in (4.11), (6.1) we find that in the *out* region the Bogoliubov coefficients are

$$\alpha_p = (\exp(\pi k_J)/2sh(\pi k_J))^{\frac{1}{2}} \, , \qquad \beta_p = (\exp(-\pi k_J)/2sh(\pi k_J))^{\frac{1}{2}} \, . \tag{6.4}$$

Then, for example, if the initial state of the "superspace"-Hamiltonian (6.3) is the ground state $|0_{in}\rangle$, in the *out* region the density matrix (5.8) takes form

$$\rho^{pq}(K) = \prod_{x\in K} \rho^{p(x)q(x)}(x), \tag{6.5}$$

where $\rho(x)$ is the one-point density matrix

$$\rho^{pq}(x) = \frac{1}{N(x)}|\beta_p|^2\delta(p,q) = \frac{1}{N(x)}\frac{1}{e^{2\pi k_J}-1}\delta(p,q) \ , \tag{6.6}$$

with $N(x)$ being the normalization function $N(x) = \sum_p \frac{1}{e^{2\pi k_J}-1}$ . This one-point density matrix does not depend on spatial coordinates and has the Plankian form with the temperature $T = \frac{1}{2\pi}$ and therefore, the density matrix (6.5) describes a Universe which in average turns out to be homogeneous.

# 7 Estimates and concluding remarks

In this manner the Universe appears to be homogeneous just after topology fluctuations are accounted for. If, on the contrary, one does not consider topology fluctuations, properties of inhomogeneities of the metric depend crucially upon the choice of initial data. Despite this, when $n \le 9$ , near the singularity the behavior of lengths in time shows universal features. This occurs, in the first place, due to the fact that the main contrubution to the mean scales $\langle g^{Q_a}\rangle$ is given just by those regions of the configuration space in which the anisotropy parameters $Q_a$ take the minimal values. They are the points $Q_a^* = -\frac{n-3}{n+1}$ lying on the boundary $\partial K$ (see, for more detail Ref [3]). Since at the boundary the eigenfunctions $\varphi_J = 0$

, in the neighborhood of $\partial K$ we have $\varphi_J \approx k_J(Q - Q^*)$ and the probability density can be estimated as $P(Q) \sim (Q - Q^*)^n$(we recall that in classical theory we had $P_{cl}(Q) \sim (Q - Q^*)^{n-2}$ and the need to average out the scale function appeared as a result of a stochastic behavor of the metric functions in space and time). Thus, in the same way as in Ref [3] for $n > 3$ in the limit $g \to 0$ we find for moments of the scale functions ($M > 0$)

$$\left\langle g^{MQ_a} \right\rangle = C_a(M, \tau) \frac{g_*^{MQ^*}}{(M \ln 1/g_*)^{n+1}} ,$$

in the case $n > 3$ and for $n = 3$ the esimate

$$\left\langle g^{MQ_a} \right\rangle = C_a(M, \tau) (M \ln 1/g_*)^{-5/2}$$

where $g_* = g(\tau, Q^*)$ and $C_a$ is a slowlly varying in time function which includes information of initial quantum state. Thus, one can see that in quantum theory the average lengths are also increasing.

In the case of $n > 9$ the volume of $K$ is infinite and the eigenfunction (4.8) proves to be non-normalizable and, therefore, we have no states which would be localized on $K$. If we get ready a localized state (a wave packet) the width of the packet spreads eventually more and more out and simultaneously the center of the wave packet runs to the infinity of the configuration space. In classical theory this signals us that the oscillatory mode becomes unstable and transforms into a Kasner-like behavior. Therefore, different mean values depend upon the initial state crucially.

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