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Thermo Field Dynamics of Deformed Systems

by

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Abstract

We discuss the formalism of Thermo Field Dynamics for deformed systems. We apply this method to the study of the statistics of q-bosonic oscillators.

Key-words: Thermal physics; Quantum algebras.

Quantum Groups and Algebras [1–4] have attracted a lot of attention in the last few years. They can be interpreted as non-trivial generalizations of Lie groups and algebras which are recovered in the limit $q \to 1$, where q is a deformation parameter, or a set of parameters, introduced in the deformed theories. These mathematical structures have found their application in several areas of physics [4–6] such as: inverse scattering method, vertex models, anisotropic spin chains hamiltonians, knot theory, conformal field theory, heuristic phenomenology of deformed molecules and nuclei, non-commutative approach to quantum gravity and anyon physics.

On the other hand Thermo Field Dynamics (TFD) [7] is a formalism whereby the usual field theory defined in real space-time can be generalized to the case with finite temperature. In this formalism the Feynman diagram method can be easily formulated by means of the real-time causal Green's function [8], which are expressed in terms of "temperature-dependent vacuum" expectation values, and all the operator relations of $T = 0$ field theory are preserved. Thermo Field Dynamics has been extensively developed and has been applied to problems in condensed matter physics as well as in high-energy physics [8–9].

In this letter we discuss the initial steps in the development of TFD of deformed systems and as a specific, and simple, example we study the statistics of bosonic q-oscillators (or deformed Heisenberg algebras) exploiting the algebraic structure which emerges from the TFD method. The study of bosonic q-oscillators is not a new subject [10] and recently their connection with Quantum Algebras and Groups have been established [11–12].

We start by considering the average of a quantity $\hat{O}$ over the canonical ensemble at temperature T

$$< \hat{O} > \equiv Z^{-1}(\beta) Tr[\hat{O} e^{-\beta H}] \tag{1}$$

where

$$Z(\beta) = Tr[e^{-\beta H}] \tag{2}$$

with $\beta = (k_B T)^{-1}$ and k_B the Botzmann constant.

In the formalism of TFD [7–9] one constructs a temperature dependent vacuum, $|0(\beta)>$, in which the statistical average of $\hat{O}$ defined in (1) coincides with the vacuum expectation value using the new vacuum $|0(\beta)>$, i.e.

$$< \hat{O} > = Z^{-1}(\beta) Tr[e^{-\beta H} \hat{O}] = < 0(\beta)|\hat{O}|0(\beta) > \ . \tag{3}$$

Let $\{|n>\}$ be the orthonormal basis of the state vector space $\mathcal{H}$ consisting of eigenstates of the Hamiltonian H

$$\begin{aligned} H|n> &= E_n|n> \\ <m|n> &= \delta_{m,n} \ . \end{aligned} \tag{4}$$

In order to construct such a state $|0(\beta)>$ one introduces a fictitious system (tilde system) characterized by the Hamiltonian $\tilde{H}$ and the state vector space $\tilde{\mathcal{H}}$ spanned by $\{|n>\}$ obeying

$$
\begin{aligned}
\tilde{H}|n> &= E_n|n> \\
<n|m> &= \delta_{n,m} \ .
\end{aligned}
\tag{5}
$$

The state vector $|0(\beta)>$ belongs to the tensor product space $\mathcal{H} \otimes \tilde{\mathcal{H}}$ and is given by:

$$
|0(\beta)>= Z^{-1/2}(\beta)\sum_n e^{-\beta E_n/2}|n> \otimes |n> \equiv Z^{-1/2}(\beta)\sum_n e^{-\beta E_n/2}|n,\tilde{n}> \ . \tag{6}
$$

If one uses (6) in (3) one has

$$
\begin{aligned}
<0(\beta)|\hat{O}|0(\beta)> &= Z^{-1}(\beta)\sum_{n,m} e^{-\beta E_n/2}e^{-\beta E_m/2} <\tilde{n},n|\hat{O}|m,\tilde{m}> \\
&= Z^{-1}(\beta)\sum_n e^{-\beta E_n} <n|\hat{O}|n>=<\hat{O}>
\end{aligned}
\tag{7}
$$

which is the result claimed in (3). We shall see later that the tilde system has a sensible physical interpretation.

We are now going to extend this formalism to the case of statistical averages of systems made up with q-oscillators. One calls bosonic q-oscillators the associative algebra generated by the elements a, a^+ and N satisfying the relations [11–13]

$$
\begin{aligned}
&[N,a^+] = a^+ \quad , \quad [N,a] = -a \\
&aa^+ - qa^+a = q^{-N}
\end{aligned}
\tag{8}
$$

where $q \in \mathbf{C}$; a, a^+ and N are the annihilation, creation and number operators respectively. There are different forms of the above algebra (8); if for instance one defines the operators

$$
A = q^{N/2}a \quad , \quad A^+ = a^+q^{N/2} \tag{9}
$$

they satisfy the algebra

$$
AA^+ - q^2A^+A = 1 \ . \tag{10}
$$

It is possible to construct the representation of the relations (8) or (10) in the Fock space $\mathcal{F}$ generated by the normalized eigenstates $|n>$ of the number operator N as

$$
\begin{aligned}
\mathcal{A}|0> &= 0 \quad N|n>=n|n> \qquad n=0,1,2,... \\
|n> &= \frac{1}{\sqrt{[n]_{\mathcal{A}}!}}(\mathcal{A}^+)^n|0>
\end{aligned}
\tag{11}
$$

where $\mathcal{A} = a$, $[n]_a = (q^n - q^{-n})/(q-q^{-1})$ and $[n]_a! = [n]_a \cdots [1]_a$ for the relations (8) and $\mathcal{A} = A$, $[n]_A = (q^{2n}-1)/(q^2-1)$ and $[n]_A! = [n]_A \cdots [1]_A$ in the case of relations (10).

In the Fock space $\mathcal{F}$ it is possible to express the deformed oscillators, A or a, in terms of the standard bosonic ones b, b^+; for instance in the case of a, a^+ one has [13]

$$a = \left(\frac{[N+1]}{N+1}\right)^{1/2} b \quad , \quad a^+ = b^+ \left(\frac{[N+1]}{N+1}\right)^{1/2} \tag{12}$$

and it can easily be shown in $\mathcal{F}$ that

$$\mathcal{A}\mathcal{A}^+ = [N+1]_{\mathcal{A}} \quad , \quad \mathcal{A}^+\mathcal{A} = [N]_{\mathcal{A}} \, . \tag{13}$$

Let us consider now an ensemble of q-bosons with Hamiltonian given by

$$H = wN \tag{14}$$

with eigenvalues $wn \, (n = 0, 1, 2, \cdots)$ on $\mathcal{F}$. We introduce the Hamiltonian of the tilde system as

$$\tilde{H} = w\tilde{N} \tag{15}$$

where the tilde q-operators we are considering satisfy the following relations

$$\begin{array}{l} [\tilde{N}, \tilde{a}^+] = \tilde{a}^+ \quad , \quad [\tilde{N}, \tilde{a}] = -\tilde{a} \\ \tilde{a}\tilde{a}^+ - q\tilde{a}^+\tilde{a} = q^{-\tilde{N}} \end{array} \tag{16}$$

or

$$\begin{array}{l} [\tilde{N}, \tilde{A}^+] = \tilde{A}^+ \quad , \quad [\tilde{N}, \tilde{A}] = -\tilde{A} \\ \tilde{A}\tilde{A}^+ - q^2\tilde{A}^+\tilde{A} = 1 \end{array} \tag{17}$$

and $[\tilde{\mathcal{A}}, \tilde{\mathcal{A}}] = [\mathcal{A}, \tilde{\mathcal{A}}^+] = 0$.
The temperature dependent vacuum $|0(\beta)>$ in the case of the relations (8) and (16) is thus given by (with an analogous formula for (10) and (17))

$$\begin{aligned} |0(\beta)> &= Z^{-1/2}(\beta) \sum_n e^{-\beta n\omega/2} \frac{1}{[n]_a!} (a^+)^n (\tilde{a}^+)^n |0> \\ &= (1 - e^{-\beta\omega})^{1/2} \exp_{q_a}(e^{-\beta\omega/2} a^+ \tilde{a}^+) |0> \end{aligned} \tag{18}$$

with $\exp_{q_{\mathcal{A}}} x = \sum_{n=0}^{\infty} \frac{1}{[n]_{\mathcal{A}}!} x^n$ the q-exponential [14], and $|0> = |0> \otimes |0>$. The form of the above expression (18) is dictated by the fact that the partition function of q-bosons [15] corresponding to the Hamiltonian (14) coincides with the usual one for harmonic oscillators. We can easily see that the non-deformed case is recovered in the $q \to 1$ limit

We denote

$$
\begin{aligned}
u_\beta &= (1-e^{-\beta\omega})^{-1/2} \\
v_\beta &= (e^{\beta\omega}-1)^{-1/2} \\
G_B &= -i\theta_\beta\left[\left(\frac{\tilde{N}+1}{[\tilde{N}+1]_a}\right)^{1/2}\left(\frac{N+1}{[N+1]_a}\right)^{1/2}\tilde{a}a\right. \\
&- \left.\left(\frac{\tilde{N}}{[\tilde{N}]_a}\right)^{1/2}\left(\frac{N}{[N]_a}\right)^{1/2}\tilde{a}^+a^+\right]
\end{aligned}
\tag{19}
$$

where $\cosh\theta_\beta = u_\beta$. With these definitions we can rewrite the temperature dependent vacuum $|0(\beta)>$, (18), as

$$
|0(\beta)>=\exp(-iG_B)|0>\equiv B|0> \ . \tag{20}
$$

Let us now define the temperature dependent operators; they are given by:

$$
\begin{aligned}
a_\beta &\equiv \exp(-iG_B)a\exp(iG_B) \\
\tilde{a}_\beta &\equiv \exp(-iG_B)\tilde{a}\exp(iG_B)\ .
\end{aligned}
\tag{21}
$$

It is interesting to observe that this transformation preserves the q-Heisenberg algebra (8) or (10)), i.e.

$$
a_\beta a_\beta^+ - q a_\beta^+ a_\beta = q^{-N_\beta}\ . \tag{22}
$$

This is easily seen by the use relation (12), thus the transformation B is a kind of q-Bogoliubov transformation. Obviously

$$
a_\beta|0(\beta)>=\tilde{a}_\beta|0(\beta)>=0 \tag{23}
$$

and the Fock space can be constructed by applying the B-transformation (20) on (11), i.e.

$$
|n>\frac{1}{\sqrt{[n]_{\mathcal{A}}!}}(\mathcal{A}_\beta^+)^n|0(\beta)> \tag{24}
$$

for $n = 0, 1, \cdots$.

Let us now compute the average of a^+a which, as we are going to see, depends on the deformation considered. In the TFD approach this average is given by:

$$
<a^+a>=<0(\beta)|a^+a|0(\beta)>=<0(\beta)|[N]_a|0(\beta)> \ . \tag{25}
$$

In order to perform this calculation we go to the basis of the non-deformed bosonic operators. In the non-deformed case one has [7]

$$
\begin{aligned}
b_\beta &= \exp(-iG_B)b\exp(iG_B) = u_\beta b - v_\beta\tilde{b}^+ \\
\tilde{b}_\beta &= \exp(-iG_B)b\exp(iG_B) = u_\beta\tilde{b} - v_\beta b^+
\end{aligned}
\tag{26}
$$

with the inverse given by

$$\begin{aligned} b &= u_\beta b_\beta + v_\beta \tilde{b}^+ \\ \tilde{b} &= u_\beta \tilde{b}_\beta + v_\beta b_\beta^+ \end{aligned} \tag{27}$$

where

$$G_B = -i\theta_B(\tilde{b}b - \tilde{b}^+ b^+) \; . \tag{28}$$

We recall that the generators of $SU(1,1)$ algebra

$$[J_0, J_\pm] = \pm J_\pm \quad , \quad [J_+, J_-] = -2J_0 \tag{29}$$

can be realized à la Schwinger [17] as

$$J_+ = b_\beta^+ \tilde{b}_\beta^+ \quad , \quad J_- = \tilde{b}_\beta b_\beta \quad , \quad J_0 = \frac{1}{2}(N_\beta + \tilde{N}_\beta + 1) \; . \tag{30}$$

Thus using (27) and (30) we have the following expression for the number operator

$$N = 2v_\beta^2 J_0 + u_\beta v_\beta (J_+ + J_-) + N_\beta \tag{31}$$

which can be rewritten as

$$N = (u_\beta^2 + v_\beta^2) J_0 + u_\beta v_\beta (J_+ + J_-) + \frac{1}{2} C \tag{32}$$

with

$$C = N_\beta - \tilde{N}_\beta - 1 \; . \tag{33}$$

Note that C commutes with the first two terms of the right-hand side of expression (32).

Using now (32) and expressing q^m as $\exp\lambda$, the relevant terms in the calculation of (25) have the form

$$< 0(\beta)|e^{\lambda N}|0(\beta) > = < 0(\beta)|e^{\lambda[(u_\beta^2+v_\beta^2)J_0 + u_\beta v_\beta (J_+ + J_-) + \frac{1}{2}C]}|0(\beta) > \; . \tag{34}$$

This last expression can be computed by means of the Baker-Campbell-Hausdorff (BCH) formula, which can be derived for the $SU(1,1)$ algebra [16]. The BCH formula for the case we are considering is given by:

$$e^{\lambda[(u_\beta^2+v_\beta^2)J_0 + u_\beta v_\beta (J_+ + J_-)]} = e^{\rho J_+} e^{\gamma J_0} e^{\rho J_-} \tag{35}$$

where

$$\begin{aligned} \rho &= \frac{2v_\beta \sinh(\lambda/2)}{\cosh(\lambda/2) - (u_\beta^2 + v_\beta^2)\sinh(\lambda/2)} \\ \gamma &= -2\ln[\cosh(\lambda/2) - (u_\beta^2 + v_\beta^2)\sinh(\lambda/2)] \; . \end{aligned} \tag{36}$$

The above procedure amounts to normal ordering eq. (34). Using this we obtain

$$< 0(\beta)|q^{mN}|0(\beta) >= \frac{2q^{-m/2}}{q^{m/2} + q^{-m/2} - (u_\beta^2 + v_\beta^2)(q^{m/2} - q^{-m/2})}, \qquad (37)$$

and finally using (25) and (37) we can easily see that

$$< 0(\beta)|a^+ a|0(\beta) >= \frac{e^{\beta\omega} - 1}{e^{2\beta\omega} - (q + q^{-1})e^{\beta\omega} + 1} \qquad (38)$$

for the bosonic q-algebra given in (8), and for that one given in (10) we have

$$< 0(\beta)|A^+ A|(\beta) >= \frac{1}{e^{\beta\omega} - q^2} \qquad (39)$$

which agrees with the results given in ref. [15].

We have thus seen how to implement the formalism of Thermo Field Dynamics to the case of deformed bosonic systems. We are now going to show that the physical interpretation of the tilde system, (16) and (17), is the same as in the non-deformed case. To this end we give the explicit expression of the temperature dependent operators in terms of those at $T = 0$; they are given by:

$$\begin{aligned}
a_\beta &= u_\beta \left(\frac{[N_\beta + 1]_a}{N_\beta + 1}\right)^{1/2} \left(\frac{N + 1}{[N + 1]_a}\right)^{1/2} a \\
&- v_\beta \left(\frac{[N_\beta + 1]_a}{N_\beta + 1}\right)^{1/2} \left(\frac{\tilde{N}}{[\tilde{N}]_a}\right)^{1/2} \tilde{a}^+ \\
\tilde{a}_\beta &= u_\beta \left(\frac{[\tilde{N}_\beta + 1]_a}{\tilde{N}_\beta + 1}\right)^{1/2} \left(\frac{\tilde{N} + 1}{[\tilde{N} + 1]_a}\right)^{1/2} \tilde{a} \\
&- v_\beta \left(\frac{[\tilde{N}_\beta + 1]_a}{\tilde{N}_\beta + 1}\right)^{1/2} \left(\frac{\tilde{N}}{[\tilde{N}]_a}\right)^{1/2} \tilde{a}^+ \, .
\end{aligned} \qquad (40)$$

For their inverse we have

$$\begin{aligned}
a &= u_\beta \left(\frac{[N + 1]_a}{N + 1}\right)^{1/2} \left(\frac{N_\beta + 1}{[N_\beta + 1]_a}\right)^{1/2} a_\beta \\
&+ v_\beta \left(\frac{[N + 1]_a}{N + 1}\right)^{1/2} \left(\frac{\tilde{N}_\beta}{[\tilde{N}_\beta]_a}\right)^{1/2} \tilde{a}_\beta^+ \\
\tilde{a} &= u_\beta \left(\frac{[\tilde{N} + 1]_a}{\tilde{N} + 1}\right)^{1/2} \left(\frac{\tilde{N}_\beta + 1}{[\tilde{N}_\beta + 1]_a}\right)^{1/2} \tilde{a}_\beta \\
&+ v_\beta \left(\frac{[\tilde{N} + 1]_a}{\tilde{N} + 1}\right)^{1/2} \left(\frac{N_\beta}{[N_\beta]_a}\right)^{1/2} \tilde{a}_\beta^+ \, .
\end{aligned} \qquad (41)$$

Using conveniently (40-41) on the temperature dependent vacuum state we see that

$$a_\beta^+|0(\beta)>=\frac{1}{u_\beta}\left(\frac{N}{[N]_a}\right)^{1/2} a^+|0(\beta)>=\frac{1}{v_\beta}\left(\frac{\tilde{N}+1}{[\tilde{N}+1]_a}\right)^{1/2}\tilde{a}|0(\beta)> \qquad (42)$$

which shows that the one particle state is built up from the thermal equilibrium state $|0(\beta)>$ by adding one particle or by eliminating one particle with tilde. Thus, analogously to the non-deformed case [7] we may interpret that the particle with a tilde is a hole of the physical particle.

In summary, we have shown how to implement the formalism of Thermo Field Dynamics in the study of the statistical properties of q-oscillators. To this end it was important to introduce a temperature dependent vacuum, by means of a q-Bogoliubov transformation acting on a $T = 0$ vacuum. With this temperature dependent vacuum we computed thermal averages as expectation values instead of traces over the Fock space. As the formalism of TFD is a real time one and preserves the operator relations of the $T = 0$ theories, we believe that it would be useful to study anyonic statistical models and statistical models with quantum symmetries.

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