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Random Environments: a Path Integral Study

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Abstract

We employ a path integral techniques to write quantum and classical propagators of a particle interacting with a classical gas and a non-equilibrium scalar phonon field respectively and moddeling random (friction) enviroments.

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I. INTRODUCTION

The interaction of a classical particle with friction, originated from shocks with a random enviroment, has been very successfully described, at a phenomenological level, by means of the famous Langevin-Einstein equation with a damping term([1]).

In this paper we intend to analyse the quantum version of the above classical friction phenomenological interaction framework by means of path integral techniques borrowed from our previous work ([2] and [5]).

The scope of the paper is as follows: In section II we deduce the quantum propagator of a particle interacting with a random enviroment modelled by a classical gas, from an exact microscopic point of view. In sections III and IV we present the same analysis, but now with the particle interacting with a scalar phonon field with random initial conditions, as a random enviroment. Finally in the appendix we present the path integral solution of a classical Brownian charged particle in the presence of a magnetic field.

II. THE QUANTUM PROPAGATOR OF A PARTICLE INTERACTING WITH A CLASSICAL GAS

Let us start our study by considering the grand canonical partition functional of a classical gas, in a volume $\Omega \subset R^3$, at temperature $T = (k\beta)^{-1}$ and with a two body interaction potential V(r)

$$Z(z,\beta,\Omega) \;=\; \sum_{N=0}^{\infty} \frac{z^N}{N!}(\frac{2\pi m}{\beta\hbar^2})^{3N/2} Z_N(\beta,\Omega) \tag{1}$$

with

$$Z_N(z,\beta,\Omega) \;=\; \frac{1}{\hbar^{3N}N!}\int_\Omega \prod_{\ell=1}^{N} d^3\vec{r}_\ell \prod_{\ell=1}^{N} d^3\vec{P}_\ell \times e^{(-\beta[\sum_{\ell=1}^{N}\frac{(\vec{P}_\ell)^2}{2m}+\sum_{\ell<j}^{N} V(|\vec{r_\ell}-\vec{r_j}|)])} \tag{2}$$

By following our previous study (see [2] and references therein), we can re-write Eq.(1) as a functional integral for a field theory of a disorder field $\Phi(\vec{r})$ for this classical gas

$$Z(z,\beta,\Omega) \;=\; \langle \exp(\alpha \int d^3 r e^{i\gamma\Phi(\vec{r})})\rangle_\Phi, \tag{3}$$

where the disorder field average is defined by the following Gaussian field path-integral

$$\langle ... \rangle_\Phi \;=\; \frac{1}{<1>} \int \mathcal{D}^F[\Phi(\vec{r})] \times e^{-\frac{1}{2}\int d^3\vec{r}\Phi(\vec{r})(\mathcal{L}\Phi(\vec{r}))} \tag{4}$$

with the constant coupling identifications

$$\begin{aligned} \gamma^2 \;&=\; \beta \;=\; \frac{1}{kT} \\ \alpha \;&=\; z(\frac{2\pi m}{\beta\hbar^2})^{3/2} \exp\frac{1}{2}\beta V(0). \end{aligned} \tag{5}$$

Here, the kinetic term, of the disorder gas field in Eq.(4) has the two-body potential, as the Green function, i.e,

$$\mathcal{L}V(\vec{r}-\vec{r'}) \;=\; \delta^{(3)}(\vec{r}-\vec{r'}) \tag{6}$$

The interaction, i.e, sucessive shocks of a quantum particle of mass M with the particle gas, is easily implemented in the field theory of disordered gas by considering the Feynman path integral propagator interacting with the disordered gas field by the usual expression with a coupling constant g (see [2]):

$$\begin{aligned} &\langle G((\vec{r},t);(\vec{r'},t'))\rangle_\Phi \quad \equiv \quad \bar{G}((\vec{r},t);(\vec{r'},t')) \;= \\ &\int \mathcal{D}^F[\vec{r}(\sigma)] \times e^{\frac{i}{\hbar}\int_t^{t'}\frac{1}{2}m(\frac{d\vec{r}}{d\sigma})^2} \times \langle e^{\frac{i}{\hbar}\int_t^{t'} d\sigma g\Phi(\vec{r}(\sigma)) \;+\; \alpha\int d^3\vec{r} e^{i\gamma\Phi(\vec{r})}}\rangle_\Phi \end{aligned} \tag{7}$$

It is a straightforward α perturbative calculation to obtain a closed expression for our proposed "damped" quantum propagator

$$\begin{aligned} &\bar{G}_{dam}((\vec{r},t);(\vec{r'},t')) \;= \\ &\sum_{N=0}^{\infty}\frac{\alpha^N}{N!}\int_\Omega d^3r_1...d^3r_N \times (\int_{\vec{r}(t)=\vec{r};\vec{r}(t')=\vec{r'}} \mathcal{D}^F[\vec{r}(\sigma)] e^{\frac{i}{\hbar}\int_{t'}^{t} d\sigma\frac{1}{2}m(\frac{d\vec{r}}{d\sigma})^2} e^{-\frac{\gamma^2}{2}\sum_{k,j=1}^{N} V(|\vec{r_k}-\vec{r_j}|)} \\ &e^{-\frac{\gamma g}{\hbar}\sum_{k=1}^{N}\int_{t'}^{t} d\sigma V(\vec{r_k}-\vec{r}(\sigma))} e^{-\frac{g^2}{2\hbar^2}\int_{t'}^{t} d\sigma\int_{t'}^{t} d\sigma' V(\vec{r}(\sigma)-\vec{r}(\sigma'))}) \end{aligned} \tag{8}$$

In the regime of high-temperature, $\gamma \to 0$, the leading contribution is exactly given by the non local Feynman path integral ($\hbar = 1$)

$$\bar{G}_{dam}((\vec{r},t);(\vec{r}',t')) \sim \int_{\vec{r}(t)=\vec{r};\vec{r}(t')=\vec{r}'} \mathcal{D}^F[\vec{r}(\sigma)] e^{i\int_{t'}^{t} d\sigma \frac{1}{2} m (\frac{d\vec{r}}{d\sigma})^2(\sigma)} e^{-\frac{g^2}{2}\int_{t'}^{t} d\sigma \int_{t'}^{t} d\sigma' V(\vec{r}(\sigma)-\vec{r}(\sigma'))} \tag{9}$$

and this is our main result of this section.

In order to obtain closed expressions in the one dimensional case for Eq.(9), let us approximate the non local interaction by the two-time quadratic action near a equilibrium point of the particle gas potential ($V(0) = V'(0) = 0$), namelly

$$\frac{g^2}{2}\int_{t'}^{t} d\sigma \int_{t'}^{t} d\sigma' V(\vec{r}(\sigma)-\vec{r}(\sigma')) \sim \frac{g^2}{2}\frac{V"(0)}{2}\int_{0}^{t} d\sigma \int_{0}^{t} d\sigma' (r(\sigma)-r(\sigma'))^2. \tag{10}$$

In this case the path integral Eq.(9)-(10), was exactly evaluated and the result, for finite time t, is (see [3]):

$$\begin{aligned}\bar{G}_{dam}((r,t);(r',0)) &= (\frac{m}{2\pi i t})^{1/2}(\frac{g(V"(0))^{1/2}\frac{t^{3/2}}{2}}{\sin(\frac{t^{3/2}}{2}g(V"(0))^{1/2})}) \\ \exp[(\frac{img}{4}&t^{1/2}(V"(0))^{1/2}\cot(\frac{g(V"(0))^{1/2}t^{3/2}}{2}))(r-r')^2]\end{aligned} \tag{11}$$

It worth to remark that the above quantum propagator still displays the divergence of the quantum probability $|\bar{G}_{dam}((r,t);(r',0))|^2$ in the equilibrium limit $t \to \infty$ as similar phenomena obtained in pure phenomenological studies([4]).

Finally we point out that the above damped quantum propagator, Eq.(9), satisfies the following Schroedinger equation in the presence of a random potential([2]).

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}G((\vec{r},t);(\vec{r}',t')) &= (-\frac{\hbar^2}{2m}\triangle + \Phi(\vec{r}))G((\vec{r},t);(\vec{r}',t')) \\ \lim_{t\to t'} G((\vec{r},t);(\vec{r}',t')) &= \delta^{(3)}(\vec{r}-\vec{r}') \end{aligned} \tag{12}$$

where the random potential satisfies the Gaussian statistics

$$\langle \Phi(\vec{r})\Phi(\vec{r}')\rangle_{\Phi} = g\delta^{(3)}(\vec{r}-\vec{r}'). \tag{13}$$

III. THE PATH INTEGRAL FOR ONE DIMENSIONAL MOTION OF A PARTICLE INTERACTING WITH PHONON FIELD AS A RANDOM ENVIROMENT

In this section we deduce the phenomenological Langevin equation ([1]) as an effective macroscopic equation obtained from a microscopic theory of a particle interacting weakly with a (non-equilibrium) friction enviroment, modelled by a scalar phonon field with a random initial conditions.

Let us, thus, start this section by considering the following classical one dimensional phonon field $\phi(x,t)$ with Lagrangean

$$\mathcal{L}_\phi = \int_0^\infty dt \int_{-\infty}^{+\infty} dx \frac{1}{2}[(\frac{\partial\phi}{\partial t})^2 - \beta(\frac{\partial\phi}{\partial x})^2], \tag{14}$$

where the phonon field time evolution is in the range $[0,\infty)$ and β denotes a microscopic constant related to the cristal (phonon) tension. We shall consider, together with the Eq.(14), the following initial conditions:

$$\begin{aligned} \phi(x,0) &= g(x) \\ \dot{\phi}(x,0) &= f(x), \end{aligned} \tag{15}$$

where $g(x)$ and $f(x)$ are fixed functions to be specified later.

At this point, we take a classical one dimensional particle of mass M in the presence of a harmonic potential with the following Lagrangean:

$$\mathcal{L}_Q = \int_0^\infty (\frac{1}{2}M\dot{Q}^2 - \frac{M}{2}\omega^2 Q^2)dt. \tag{16}$$

We thus propose the following interaction Lagrangean of the classical particle and the classical phonon field,

$$\mathcal{L}_{int} = i\int_0^\infty \int_{-\infty}^{+\infty} dx \phi(x,t)\delta(x - gQ(t)), \tag{17}$$

where g is a dimensionless coupling constant which couples the particle position Q(t) with the phonon field at the cristal position x. In order to proceed with our study, let us introduce the plane wave expansion for phonon field,

$$\phi(x,t) \;=\; \int_{-\infty}^{+\infty} dk \phi_k(t) e^{ikx} \tag{18}$$

with,

$$\phi_{-k}(t) \;=\; \phi_k^*(t). \tag{19}$$

At this point, it worth to replace the full interaction, Eq.(17), by the first order equivalent interaction Lagrangean in the leading limit of weak coupling, i.e, $g \to 0^+$

$$i\int_0^\infty dt \int_{-L}^{+L} dx(\phi(x,t)\delta(x-gQ(t)) \sim i\lim_{L\to\infty}\int_0^\infty dt\int_{-L}^{+L} dx(\phi(0,t)+\frac{\partial\phi}{\partial x}(0,t)x+...)\delta(x-gQ(t))$$
$$\sim -g\int_{-\infty}^{+\infty} dk k\phi_k(t)Q(t). \tag{20}$$

The complete Lagrangean for the system is, thus, given by the equation below:

$$\mathcal{L}_\phi + \mathcal{L}_Q + \mathcal{L}_{int} \;=\;$$
$$\int_0^\infty dt \int_{|k|<\Lambda} dk[\frac{1}{2}\dot{\phi}_k^2(t) - \frac{1}{2}\beta k^2\phi_k^2(t)] \;+$$
$$\int_0^\infty dt(\frac{1}{2}M\dot{Q}^2(t) - \frac{M}{2}\omega^2 Q^2(t)) - g\int_0^\infty dt \int_{|k|<\Lambda} dk\; k\phi_k(t)Q(t), \tag{21}$$

where

$$\phi(x,0) \;=\; \int_{|k|<\Lambda} dk\; g_k e^{ikx}$$
$$\dot{\phi}(x,0) \;=\; \int_{|k|<\Lambda} dk\; f_k e^{ikx}. \tag{22}$$

Here we have introduced a momentum cut-off $|k| < \Lambda$ which may be removed at the end of our calculations and leading to a formal renormalization of the macroscopic parameters of the effective equation of the particle motion, as we are going to show at the end of this section.

The classical motion equations for each harmonic oscilator is easily obtained from Eq.(21) and are as follows:

$$M\ddot{Q}(t) \;=\; -M\omega_0^2 Q(t) - gk\phi_k(t)$$
$$\ddot{\phi}(t) \;=\; -\beta k^2\phi_k(t) - gkQ(t). \tag{23}$$

By considering the above equations of motion in the frequency domain, by means of a Laplace transform, we get the following algebraic equations in the frequency domain in place of Eq.(23), i.e,

$$M(s^2\tilde{Q}(s) - sQ(0) - \dot{Q}(0)) \;=\; -M\omega_0\tilde{Q}(s) - gk\tilde{\phi}_k(s) \tag{24}$$

$$s^2\tilde{\phi}_k(s) - sg_k - f_k \;=\; -\beta k^2\phi_k(s) - gk\tilde{Q}(s). \tag{25}$$

The use of identity

$$\frac{1}{s^2+\beta k^2} \;=\; (1 - \frac{s^2}{s^2+\beta k^2})\frac{1}{\beta k^2} \tag{26}$$

allow us to get the following equation for the particle of mass M:

$$\begin{aligned}\int_{|k|<\Lambda} dk\ (Ms^2\tilde{Q}(s) - sQ(0) - \dot{Q}(0)) \;=&\; -\int_{|k|<\Lambda} dk\ M\omega^2\tilde{Q}(s) \\ &-g\int_{|k|<\Lambda} dk\ k[\frac{sg_k + f_k - gk\tilde{Q}(s)}{s^2+\beta k^2}].\end{aligned} \tag{27}$$

Now we note that the $\tilde{Q}(s)$ term, coming from the last equation in the right-hand side of the Eq.(27) is explicitly given by:

$$\frac{g^2}{\beta}\int_{|k|<\Lambda} dK\ \tilde{Q}(s) - \frac{g^2 2\pi}{\sqrt{\beta}}s\tilde{Q}(s). \tag{28}$$

The time domain equation associated with Eq.(27) (and taking into account Eq.(28)) leads, straightforwardly, to the following damped equation

$$M\ddot{Q}(t) \;=\; (-M\omega_0^2 + \frac{g^2}{\beta})Q(t) - \nu^{(\Lambda)}\dot{Q}(t) + F(t) \tag{29}$$

where we have a finite shift in the frequency of the harmonic oscilator and also a renormalized viscosity explicitly expressed in terms of our microscopic parameters as

$$\nu^{(\Lambda)} \;=\; \frac{2\pi g^2}{\sqrt{\beta}}\frac{1}{(\int_{|k|<\Lambda} dk)}. \tag{30}$$

Note that by choosing random Gaussian ultra-local initial conditions for our phonon field enviroment, i.e,

$$\phi(x,0) = g(x) = 0 \tag{31}$$

$$\dot{\phi}(x,0) = f(x) \tag{32}$$

with

$$\langle f(x)f(x')\rangle = \gamma\delta(x-x'), \tag{33}$$

we obtain that the external non-vanishing force in Eq.(29) satisfies the random initial condition in time

$$\langle F(t)F(t')\rangle = \gamma\delta(t-t') \tag{34}$$

which is the famous Langevin equation.

Now we will present a path integral representation for the above random equation, Eq.(29).

IV. THE PATH INTEGRAL FOR THE BROWNIAN MOTION

In this section we consider the Brownian motion of a one dimensional harmonic hoscilator with the Langevin equation in the interval $[0,\infty]$ (see Eq.(29))

$$\frac{d^2x}{dt^2} + \nu\frac{dx}{dt} + \omega^2 x = f(t) \tag{35}$$

where $f(t)$ is the one-dimensional phenomenological random force with Gaussian statistics

$$\begin{aligned}\langle f(t)\rangle &= 0\\ \langle f(t)f(t')\rangle &= \gamma\delta(t-t').\end{aligned} \tag{36}$$

Physical quantities in this system are functionals of the one-dimensional particle position $x(t,[f])$ and must be averaged over the random force. The whole averaging information is contained in the probabilistic characteristic functional

$$Z[J(t)] = \langle \exp - \int_0^\infty dt J(t)x(t,[f])\rangle. \tag{37}$$

In order to write the above average in terms of a path integral over trajectories $x(t), 0 \leq t \leq \infty$, we follow the same procedure of section II which leads us to the following result:

$$\begin{aligned} Z[J(t)] &= \int \mathcal{D}^F[x(t)]\ det[\frac{d^2}{dt^2} + \nu \frac{d}{dt} + \omega^2] \times \\ & e^{-\frac{1}{2\gamma}\int_0^\infty dt(\frac{d^2x}{dt^2}+\nu\frac{dx}{dt}+\omega^2 x)^2} . e^{-\int_0^\infty dt J(t)x(t)}. \end{aligned} \tag{38}$$

It is instrutive to remark the non-triviality of the functional determinant in Eq.(38), a result that is opposite to that associated to first-order in time kinetic operator of the Ref.[5].

In order to proceed its evaluation, we use the following similarity relationship between the kinetic operator for the damped equation of motion and that for the non-damped harmonic oscilator, i.e,

$$e^{-\alpha(t)}(\frac{d^2}{dt^2} + \Omega^2)e^{+\alpha(t)} = \frac{d^2}{dt^2} + \nu\frac{d}{dt} + \omega^2, \tag{39}$$

where one must choose $\alpha(t)$ and $\Omega(t)$ satisfying the following relationship

$$2\frac{d\alpha}{dt} = \nu \tag{40}$$

$$\omega^2 = (\frac{d\alpha}{dt})^2 + \frac{d^2\alpha}{dt^2} + \Omega^2 \tag{41}$$

or exactly

$$\alpha(t) = \frac{\nu}{2}t \tag{42}$$

$$\omega^2 = \frac{\nu^2}{4} + \Omega^2. \tag{43}$$

As a consequence of the above similarity relationship, we have the validity of the results written below:

$$\begin{aligned} det[\frac{d^2}{dt^2} + \nu\frac{d}{dt} + \omega^2] &= det[e^{-\frac{\nu}{2}t}(\frac{d^2}{dt^2} + \Omega^2)e^{\frac{\nu}{2}t}] \\ \equiv det[\frac{d^2}{dt^2} + (-\frac{\nu^2}{4} + \omega^2)] &= \frac{1}{\sqrt{2\pi\gamma(t_2-t_1)}}\sqrt{\frac{(-\frac{\nu^2}{4}+\omega^2)^{1/2}(t_2-t_1)}{\sin[(-\frac{\nu^2}{4}+\omega^2)^{1/2}(t_2-t_1)]}}. \end{aligned} \tag{44}$$

The Gaussian fourth-order action in Eq.(12) has the following Green function for its kinetic operator ([1]):

$$\bar{G}(t_1,t_2) \;=\; \{(\frac{d^2}{dt^2}+\nu\frac{d}{dt}+\omega^2)^{-1}.[(\frac{d^2}{dt^2}+\nu\frac{d}{dt}+\omega^2)^{-1}]^*\}(t_1,t_2) \tag{45}$$

where the symbol * denotes the operation of taking the (formal) adjoint of the operator under consideration ([5]).

Again, as a consequence of the similarity relationship, Eq.(39), we can easily write the above Green functions in terms of the well-known causal Euclidian Green function for the harmonic oscilator, i.e,

$$(e^{-\frac{\nu}{2}t}(\frac{d^2}{dt^2}+\Omega^2)e^{\frac{\nu}{2}t})^{-1}(t,t') \;=\; \theta(t-t')e^{-\frac{\nu}{2}t}\frac{\sin(-\frac{\nu^2}{4}+\omega^2)^{1/2}(t-t')}{\sqrt{\omega^2-\frac{\nu^2}{4}}}e^{-\frac{\nu}{2}t'} \tag{46}$$

and

$$\begin{aligned}[(\frac{d^2}{dt^2}+\nu\frac{d}{dt}+\omega^2)^*]^{-1}(t,t') \;&=\; (\frac{d^2}{dt^2}-\nu\frac{d}{dt}+\omega^2)^{-1}(t,t')\\ &= e^{+\frac{\nu}{2}t}\theta(t-t')\frac{\sin(-\frac{\nu^2}{4}+\omega^2)^{1/2}(t-t')}{\sqrt{\omega^2-\frac{\nu^2}{4}}}e^{+\frac{\nu}{2}t'}.\end{aligned} \tag{47}$$

Now a simple evaluation leads, finally, to the following expression for the fourth-order Green function:

$$\begin{aligned}\bar{G}(t_1,t_2) \;&=\; \int_0^\infty dt'\; e^{-\frac{\nu}{2}t_1}\frac{\sin(-\frac{\nu^2}{4}+\omega^2)^{1/2}(t-t')}{\sqrt{\omega^2-\frac{\nu^2}{4}}}e^{-\frac{\nu}{2}t'}\times\\ &\theta(t_1-t')\theta(t'-t_2)e^{\frac{\nu}{2}t'}\frac{\sin(-\frac{\nu^2}{4}+\omega^2)^{1/2}(t-t')}{\sqrt{\omega^2-\frac{\nu^2}{4}}}e^{\frac{\nu}{2}t_2}\\ &= \frac{e^{-\frac{\nu}{2}(t_1-t_2)}}{\frac{\nu^2}{4}+\omega^2}\int_{t_1}^{t_2} dt'\sin[(-\frac{\nu^2}{4}+\omega^2)^{1/2}(t_1-t')]\sin[(-\frac{\nu^2}{4}+\omega^2)^{1/2}(t'-t_2)].\end{aligned} \tag{48}$$

The generating functional, Eq.(37), thus takes the closed form

$$Z[J(t)] = e^{-\frac{1}{2\gamma}\int_0^\infty\; dt\int_0^\infty\; dt'\;\; J(t)\bar{G}(t,t')J(t'')}. \tag{49}$$

As we have shown, the path integral techniques of Refs.[2] and [5] provides a quick, mathematically and conceptually, simple way to analyse random forced physical systems.

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APPENDIX A: A PATH INTEGRAL SOLUTION FOR THE BROWNIAN MOTION OF A CHARGED PARTICLE UNDER A MAGNETIC FIELD.

Let us start this appendix by considering the Langevin equation in a three dimensional infinite medium ($t > 0$), for the Brownian velocity of a charged particle in the presence of a external magnetic field, i.e,

$$\frac{d\vec{v}(t)}{dt} - \frac{e}{c}(\vec{v}X\vec{B})(t) \;=\; -\nu\vec{v}(t)\vec{f}(t), \tag{A1}$$

where $\vec{B} \;=\; B\hat{k}$ is the magnetic field in the z-direction; $\vec{v}(t) \;=\; v_1(t)\hat{\imath} + v_2(t)\hat{\jmath} + v_3(t)\hat{k}$ denotes the velocity vector of the Brownian particle, ν is the Brownian damping viscosity and $\vec{f}(t)$ is the phenomenological random force satisfying the Gaussian statistics ($\gamma > 0$)

$$\begin{aligned} \langle\vec{f}(t)\rangle \;&=\; 0 \\ \langle f_i(t)f_j(t')\rangle \;&=\; \gamma\delta(t-t')\delta_{ij}. \end{aligned} \tag{A2}$$

The probability of observing that the particle has a prescribed velocity $\vec{v}_{in}$ at a time t_1 and another prescribed velocity $\vec{v}_{out}$ at a time t_2, with $t_2 > t_1$, denoted from here on by $P[\vec{v}(t_1) = \vec{v}_{in}; \vec{v}(t_2) = \vec{v}_{out}]$, is given by the following path integral average of the probabilistic velocity occupation time ([2])

$$\begin{aligned} &P[\vec{v}(t_1) = \vec{v}_{in}; \vec{v}(t_2) = \vec{v}_{out}] \;= \\ &\int \mathcal{D}[\vec{f}(t)]e^{-\frac{1}{2}\int_0^\infty dt\vec{f}(t).\vec{f}(t)}\delta^{(3)}(\vec{v}(t_1,[\vec{f}]) - \vec{v}_{in})\delta^{(3)}(\vec{v}(t_2,[\vec{f}]) - \vec{v}_{out}), \end{aligned} \tag{A3}$$

where we have used the notation $\vec{v}(t,[\vec{f}])$ to emphasize that the particle vector velocity is a functional of the Brownian random force $\vec{f}(t)$ whose precise functional form satisfy Eq.(A1).

At this point, we follow previous studies ([1]) to re-write Eq.(A3) as a Feynmam-Wiener path integral over trajectories $\vec{v}(t); t_1 \le t \le t_2$ ([2]), i.e,

$$\begin{aligned} P[\vec{v}(t_1) = \vec{v}_{in}; \vec{v}(t_2) = \vec{v} = \vec{v}_{out}] \;&=\; \int \mathcal{D}^F[\vec{f}(t)]e^{-\frac{1}{2\gamma}\int_0^\infty dt(\vec{f}.\vec{f})(t)} \times \\ \int_{\vec{v}(t_1)=\vec{v}_{in};\vec{v}(t_2)=\vec{v}_{out}} \mathcal{D}^F[\vec{v}(t)]\delta^F[\frac{d\vec{v}(t)}{dt} &- \frac{e}{c}(\vec{v}X\vec{B})(t) + \nu\vec{v}(t) - \vec{f}(t)]det_F[(\frac{d}{dt}+\nu)\delta_{i\ell} - \frac{e}{c}\epsilon^{i\ell 3}B]. \end{aligned} \tag{A4}$$

We now observe that the functional determinant in Eq.(A4) is unity as a straightforward consequence that the Green function of the operator $\frac{d}{dt}$ is the step function when operating in the domain $C^\infty[t_1, t_2]$.

By using the Lagrange-multiplier representation for the delta functional inside the average Eq.(A4), namely:

$$\delta^{(F)}[\frac{d\vec{v}}{dt} - \frac{e}{c}\vec{v}X\vec{B} + \nu\vec{v}(t) - \vec{f}] = \int_{\vec{\lambda}(t_1)=\vec{\lambda}(t_2)=0} \mathcal{D}^F[\vec{\lambda}(t)] e^{i\int_{t_1}^{t_2} dt \vec{\lambda}(t)(\frac{d\vec{v}}{dt} - \frac{e}{c}\vec{v}X\vec{B} + \nu\vec{v}(t) - \vec{f})}, \quad \text{(A5)}$$

and evaluating the resulting $\vec{\lambda}(t)$ and $\vec{f}(t)$ Gaussian functional integrals, we arrive at the exact path integral representation for the two-velocity probability for the Brownian (<u>classical</u>)particle Eq.(1):

$$P[\vec{v}(t_1) = \vec{v}_{in}; \vec{v}(t_2) = \vec{v}_{out}] = \int_{\vec{v}(t_1)=\vec{v}_{in};\vec{v}(t_2)=\vec{v}_{out}} \mathcal{D}^F[\vec{v}(t)] e^{-\frac{1}{2\gamma}\int_{t_1}^{t_2} dt([\frac{d\vec{v}}{dt} - \frac{e}{c}\vec{v}X\vec{B} + \nu\vec{v})^2}, \quad \text{(A6)}$$

which, by its turn, leads to exactly soluble path integrals of a 2D Euclidian quantum harmonic oscilator under the presence of a external magnetic field with cyclotron frequency, $\omega_c = eB/c$, and a 1D Euclidian harmonic oscilator into the z direction with frequency ν^2:

$$\begin{aligned} P[\vec{v}(t_1) = \vec{v}_{in}; \vec{v}(t_2) = \vec{v}_{out}] &= \\ \int_{v_1(t_1)=v_1^{in};v_1(t_2)=v_1^{out}} \mathcal{D}^F[v_1(t)] &\int_{v_2(t_1)=v_2^{in};v_2(t_2)=v_2^{out}} \mathcal{D}^F[v_2(t)] \quad \times \\ e^{-\frac{1}{2\gamma}\int_{t_1}^{t_2}(\dot{v}_1^2+\dot{v}_2^2+(\omega_c^2+\nu^2)(v_1^2+v_2^2)-\omega_c(\dot{v}_1 v_2 - v_1\dot{v}_2))} &\quad \times \\ \int_{v_3(t_1)=v_3^{in};v_3(t_2)=v_3^{out}} \mathcal{D}^F[v_3(t)] &e^{-\frac{1}{2\gamma}\int_{t_1}^{t_2} dt\ (\dot{v}_3^2+\nu^2 v_3^2)}. \end{aligned} \quad \text{(A7)}$$

In the special (pedagogical) case of $\omega_c^2 + \nu^2 = \frac{\omega_c^2}{4}; \gamma = 1$ *and* $t_1 = 0$, one can evaluate easily the first propagator with the result (see [6]):

$$\begin{aligned} P[\vec{v}(0) = \vec{v}_{in}; \vec{v}(t) = \vec{v}_{out}] &= \\ \frac{1}{4\pi t} e^{-\frac{1}{4t}(v_1^{out} - \sqrt{(v_1^{in})^2+(v_2^{in})^2}\cos[\omega_c t + \arctan(\frac{v_2^{in}}{v_1^{in}})])^2} &\quad \times \end{aligned}$$

$$e^{-\frac{1}{4t}(v_2^{out}-\sqrt{(v_1^{in})^2+(v_2^{in})^2}\sin[\omega_c t+\arctan(\frac{v_2^{in}}{v_1^{in}})])^2} \times$$
$$(\pi\sqrt{\frac{4}{\nu}}\sin\sqrt{4\nu}t)^{-1/2} \times$$
$$e^{-\sqrt{\frac{\nu}{4}}\frac{((v_3^{in})^2+(v_3^{out})^2)\cosh\sqrt{4\nu}t-2v_3^{in}v_3^{out}}{\sin\sqrt{4\nu}t}}. \tag{A8}$$

In the general physical case of generics B and ν, even if these external parameters are time-dependent, one can write the associated Euclidian Schroedinger (diffusion) equation and solve exactly such (initial conditions) Feynman propagators by making use of the space-time transformations of Refs. [4]-[7].