CBPF-NF-036/95

Exact Solutions in Multidimensional Cosmology With Bulk Viscosity*

by

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Abstract

A multidimensional cosmological model describing the evolution of n Ricci-flat spaces is studied in the presence of a fluid with anisotropic pressure and bulk viscosity. The second equation of state is adopted in the form of a metric dependence of the bulk viscosity coefficients. For special sets of parameters exact solutions of Einstein equations are obtained in a Kasner-like form. Models with creation of matter, describing expansion of the external space and contraction of the internal ones, are singled out.

Key-words: Multidimensional Cosmology; Viscosity.

PACS number: 04.20.J, 04.60.+n, 03.65.Ge

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* To appear in Gravitation & Cosmology, Vol. 1 (1995), No. 2.

1. Introduction

We have recently studied different properties of multidimensional cosmology with a matter source of the multidimensional Einstein equations in the form of a perfect fluid [1-6]. But certainly models incorporating some viscosity effects may be more realistic. In 4-dimensional cosmology the viscous Universe was considered by a number of authors from different points of view. Without presenting a detailed review of the subject (an extensive review was given by Grøn [7]), we just mention some main trends in cosmology with viscous fluid as a source.

First, Misner [8] considered neutrino viscosity as a mechanism for reducing anisotropy in the Early Universe. Stewart [9] and Collins and Stewart [10] proved that it is possible only if initial anisotropies are small enough. Another series of papers was started by Weinberg [11] who studied entropy production in the viscous Universe. Both isotropization and entropy production during the lepton era in models of Bianchi types I and V were considered by Klimek [12]. Caderni and Fabbri [13] calculated coefficients of shear and bulk viscosity in the plasma and lepton eras within the Bianchi type I model. One more direction is connected with obtaining singularity-free "viscous" solutions. The first nonsingular solution was obtained by Murphy [14] within a flat Friedman-Robertson-Walker model with fluid possessing a bulk viscosity. Murphy supposed that the coefficient of bulk viscosity is proportional to the fluid density. However, Belinsky and Khalatnikov [15, 16] showed that this solution corresponds to a very peculiar choice of parameters and is unstable under anisotropy perturbations. Other nonsingular solutions with bulk viscosity were obtained by Novello and Aranjo [17], Romero [18], Oliveira and Salim [19].

In this paper we study a multidimensional cosmological model with a chain of Ricci-flat spaces for the source in the form of a fluid possessing pressure and bulk viscosity, both anisotropic. In Section 2 we describe the model and obtain the basic equations. To integrate them, we develop some vector formalism suggested in our previous papers. In Section 3 we integrate the equations of motion for two special sets of parameters in the first and second equations of state. Exact solutions are presented in a Kasner-like form and their properties are studied.

2. The model

Following our previous papers [1-6], we consider a multidimensional cosmological model with the metric

$$g = -\mathrm{e}^{2\gamma(t)} dt \otimes dt + \sum_{i=1}^{n} \exp[2x^i(t)] g^{(i)}, \quad n \geq 2, \tag{2.1}$$

defined on the D-dimensional manifold

$$M = R \times M_1 \times \ldots \times M_n, \tag{2.2}$$

where R is the time axis and M_i is an Einstein space of dimension N_i with a metric $g^{(i)}$. In this paper we consider only Ricci-flat spaces $M_1, \ldots, M_n$, i.e.

$$R_{n_i l_i}[g^{(i)}] = 0, \quad n_i, l_i = 1, \ldots, N_i. \tag{2.3}$$

It is easy to obtain in the usual way the following nonzero components of the Ricci-tensor for the metric (2.1) [3]:

$$R_0^0 = \mathrm{e}^{-2\gamma(t)} \Big(\sum_{i=1}^{n} N_i (\dot{x}^i)^2 + \ddot{\gamma}_0 - \dot{\gamma}\dot{\gamma}_0\Big), \tag{2.4}$$

$$R_{k_i}^{m_i} = \mathrm{e}^{-2\gamma(t)} (\ddot{x}^i + (\dot{\gamma}_0 - \dot{\gamma})\dot{x}^i) \delta_{k_i}^{m_i}, \tag{2.5}$$

where we have denoted $\gamma_0 = \sum_{i=1}^n N_i x^i$. The indices m_i and k_i range from $D - \sum_{j=i}^n N_j$ to $D - \sum_{j=i}^n N_j + N_i$ for $i = 1, \ldots, n$ ($D = 1 + \sum_{i=1}^n N_i = \dim M$).

We take the energy-momentum tensor for a viscous fluid in the standard form

$$T_B^A = \rho u^A u_B + (p - \zeta\theta) P_B^A, \tag{2.6}$$

where ρ and p are the fluid density and pressure, respectively, ζ is the bulk viscosity coefficient. The vector u^A is the D-dimensional velocity of the fluid and $P_B^A = \delta_B^A + u^A u_B$ is the projector to the $(D-1)$-dimensional space orthogonal to u^A. By θ we denote the scalar $\theta = u^A{}_{;A}$.

We impose the comoving observer condition for the D-dimensional velocity: $u^A = \delta_0^A \mathrm{e}^{-\gamma(t)}$. Then

$$(u^A u_B) = \mathrm{diag}(-1, 0, \ldots, 0), \tag{2.7}$$

$$(P_B^A) = \mathrm{diag}(0, 1, \ldots, 1), \tag{2.8}$$

$$\theta = \dot{\gamma}_0 \mathrm{e}^{-\gamma(t)}. \tag{2.9}$$

Let us remark that the function $\gamma(t)$ in (2.1) determines a time gauge for the comoving observer. We have the harmonic time gauge for $\gamma(t) = \gamma_0$ and the proper time gauge for $\gamma(t) = 0$. The harmonic time t and the proper time τ are connected by $d\tau = \exp[\gamma_0] dt$.

We admit that the pressure and the bulk viscosity term in (2.6) are anisotropic with respect to the whole space $M_1 \times \ldots \times M_n$. Such an admission leads to the following generalization of the expression (2.6):

$$(T_B^A) = \mathrm{diag}(-\rho, (p_1 - \theta\zeta_1)\delta_{k_1}^{m_1}, \ldots, (p_n - \theta\zeta_n)\delta_{k_n}^{m_n}) \tag{2.10}$$

where p_i and ζ_i are pressure and bulk viscosity coefficients in the spaces M_i. Furthermore, we suppose that a barotropic equations of state holds:

$$p_i = (1 - h_i)\rho(t), \tag{2.11}$$

where h_i =const for $i = 1, \ldots, n$.

The Einstein equations $R^A_B - \frac{1}{2}\delta^A_B R = \kappa^2 T^A_B$ (κ^2 is the gravitational constant) may be written as $R^A_B = \kappa^2 (T^A_B - \frac{T}{D-2}\delta^A_B)$. Further, we employ the equation $R^0_0 - \frac{1}{2}\delta^0_0 R = \kappa^2 T^0_0$ and the equations $R^{m_i}_{k_i} = \kappa^2 (T^{m_i}_{k_i} - \frac{T}{D-2}\delta^{m_i}_{k_i})$. Using (2.4), (2.5) and (2.10) we get

$$\sum_{i=1}^{n} N_i (\dot{x}^i)^2 - \dot{\gamma}_0^2 = -2\kappa^2 \mathrm{e}^{2\gamma}\rho, \tag{2.12}$$

$$\begin{aligned} \ddot{x}^i + (\dot{\gamma}_0 - \dot{\gamma})\dot{x}^i &= \kappa^2 \left\{ \left(-h_i + \frac{\sum_{k=1}^n N_k h_k}{D-2} \right) \rho \mathrm{e}^{2\gamma} \right. \\ &\quad \left. + \left(-\zeta_i + \frac{\sum_{k=1}^n N_k \zeta_k}{D-2} \right) \dot{\gamma}_0 \mathrm{e}^{\gamma} \right\}. \end{aligned} \tag{2.13}$$

To develop the integration procedure for the equations of motion (2.12) and (2.13) we introduce the n-dimensional real vector space R^n. By $e_1, \ldots, e_n$ we denote the canonical basis in R^n, i.e. $e_1 = (1, 0, \ldots, 0)$ etc.

Let $< .,. >$ be a symmetric bilinear form defined on R^n, such that

$$< e_i, e_j >= \delta_{ij} N_j - N_i N_j \equiv G_{ij}. \tag{2.14}$$

In our previous papers [1-6] this form was introduced as a minisuperspace metric for the cosmological models. It was shown that it is a nongenerate form with the pseudo-Euclidean signature $(-, +, \ldots, +)$. So, for vectors $a = a^1 e_1 + \ldots + a^n e_n$ and $b = b^1 e_1 + \ldots + b^n e_n$ we have

$$< a, b >= \sum_{i,j=1}^{n} G_{ij} a^i b^j. \tag{2.15}$$

The form $< a, b >$ may be also written as

$$< a, b >= \sum_{i=1}^{n} a_i b^i = \sum_{i=1}^{n} a^i b_i = \sum_{i,j=1}^{n} G^{ij} a_i b_j, \tag{2.16}$$

if we introduce the covariant components of vectors as

$$a_i = \sum_{j=1}^{n} G_{ij} a^j. \tag{2.17}$$

By $G^{ij} = \delta^{ij}/N_i + 1/(2 - D)$ we denote components of a matrix inverse to (G_{ij}).

We call a vector $y \in R^n$ time-like, space-like or isotropic if $< y, y >$ takes negative, positive or null values, respectively. The vectors y and z are called orthogonal if $< y, z >= 0$.

In our model the following vectors are used:

$$\begin{aligned} x &= x^1 e_1 + \ldots + x^n e_n, && (2.18) \\ u &= u^1 e_1 + \ldots + u^n e_n, && \\ u^i &= h_i - \frac{\sum_{k=1}^n N_k h_k}{D-2}, \quad u_i = N_i h_i && (2.19) \\ \xi &= \xi^1 e_1 + \ldots + \xi^n e_n, && \\ \xi^i &= \zeta_i - \frac{\sum_{k=1}^n N_k \zeta_k}{D-2}, \quad \xi_i = N_i \zeta_i. && (2.20) \end{aligned}$$

If $h_i = 1$ for $i = 1, \ldots, n$, we have dust in the whole space ($p_i = 0$, see (2.11)). The vector (2.19) corresponding to dust in the whole space is denoted by u_d. We note that

$$\begin{aligned} &(u_d)_i = N_i, \quad u^i_d = -1/(D-2), \\ &< u_d, u_d >= -(D-1)/(D-2), \quad < u_d, x > = \gamma_0. \end{aligned} \tag{2.21}$$

Thus, using (2.15), (2.18)-(2.20), we obtain the Einstein equations (2.12) and (2.13) in the form

$$< \dot{x}, \dot{x} >= -2\kappa^2 e^{2\gamma}\rho, \tag{2.22}$$

$$\ddot{x} + (< u_d, \dot{x} > -\dot{\gamma})\dot{x} = -\kappa^2(\rho e^{2\gamma}u + < u_d, \dot{x} > e^{\gamma}\xi). \tag{2.23}$$

Excluding the density ρ from (2.23) by (2.22), we get the equation

$$\ddot{x} + (<u_d, \dot{x}> - \dot{\gamma})\dot{x} = \frac{1}{2}<\dot{x}, \dot{x}>u - \kappa^2<u_d, \dot{x}>e^{\gamma}\xi. \tag{2.24}$$

To integrate Eq.(2.24) we need a second equation of state for the bulk viscosity coefficients ζ_i. To obtain an exact solution in a 4-dimensional flat Friedman-Robertson-Walker model with bulk viscosity, Murphy [14] used the second equation of state of the form $\zeta = \mathrm{const}\,\rho$. Belinsky and Khalatnikov [20] studied the qualitative behavior of this model with a more general equation: $\zeta = \alpha\rho^{\nu}$, where $\alpha, \nu = \mathrm{const}$. It is easy to show that, for this model on the manifold $R \times M_1^3$ for $\gamma(t) = 0$, the set of equations (2.22), (2.23) may be written as

$$3H^2 = \kappa^2\rho, \tag{2.25}$$

$$\dot{H} = \frac{\alpha}{2}3^{\nu+1}H^{2\nu+1} + \frac{3}{2}(h-2)H^2, \tag{2.26}$$

where H is the Hubble parameter of the 3-dimensional Ricci-flat manifold M_1^3, i.e. $H = \dot{x}^1$. The set of equations (2.25), (2.26) coincides with the one obtained by Belinsky and Khalatnikov [20]. It is easy to see that Eq.(2.26) for H is always integrable by quadratures. In the simplest case with $\nu = 1$ we get the exact solution obtained by Murphy [14]. Other solutions for special values of ν and h and a solution for arbitrary ν and h were also obtained (see [7] for details).

For a multidimensional cosmological model with the manifold $M = R \times M_1 \times \ldots \times M_n$ the set of equations (2.22), (2.23) is more complicated. We obviously have the set of nonlinear differential equations (2.24) for the scale factors e^{x^i} of the spaces $M_1, \ldots, M_n$. If we adopt Belinsky and Khalatnikov's condition $\zeta \sim \rho^{\nu}$, then rather complicated equations arise. In particular, for $\nu = 1$ Appel and Ricatti equations appear. Chakraborty and Nandy [21], within a 5-dimensional model with the manifold $R \times M_1^3 \times S_2^1$, avoided this difficulty by imposing an additional constraint for the scale factors: $\exp[x^2] = \mu\exp[\omega x^1]$, $\mu, \nu = \mathrm{const}$.

Here, with no loss of generality, we consider the integration of Eqs.(2.24) for another second equation of state. We suppose that the bulk viscosity coefficient ζ_i corresponding to the space M_i is proportional to $e^{-\gamma_0}$, i.e.,

$$\zeta_i \sim [\text{scale factor of } M_1]^{-\dim M_1} \times \ldots \cdot [\text{scale factor of } M_n]^{-\dim M_n}. \tag{2.27}$$

Physically, the assumption (2.27) means that the expansion of the spaces $M_1, \ldots, M_n$ is accompanied by a decreasing bulk viscosity effect.

Let us notice that the metric dependence of the bulk viscosity coefficient was also considered by other authors. Lukacs [22] integrated a homogeneous and isotropic 4-dimensional model with viscous dust with the second equation of state $\zeta = \mathrm{const}\ [scale\ factor]^{-1}$. The curvature-dependent bulk viscosity was studied in a multidimensional cosmology by Wolf [23]. Recently Motta and Tomimura [24] studied a 4-dimensional inhomogeneous cosmology with some metric dependence of the bulk viscosity coefficient.

3. Integrable models

We first study a model with essential anisotropic pressure and viscosity. Here we assume that the nonzero vectors u and ξ in (2.24) corresponding to the pressure and the bulk viscosity are parallel. Combining this assumption with the second equation of state (2.27), we get

$$\xi = \frac{\zeta_0}{\kappa^2}e^{-\gamma_0}u \quad \text{or} \quad \zeta_i = \frac{\zeta_0}{\kappa^2}e^{-\gamma_0}h_i, \tag{3.1}$$

where ζ_0 is a positive constant. We also suppose that the orthogonality condition

$$< u_d, u >= -\sum_{i=1}^{n}\frac{1}{D-2}N_i h_i = 0 \tag{3.2}$$

holds, which means a certain restriction on the eqs. of state (some h_i are negative). Then the set of equations (2.24) for the harmonic time gauge ($\gamma = \gamma_0$) looks as follows:

$$\ddot{x} = \Big(\frac{1}{2} < \dot{x}, \dot{x} > -\zeta_0 < u_d, \dot{x} >\Big)u. \tag{3.3}$$

To integrate (3.3), we use the decomposition

$$\begin{aligned} x &= < u_d, x > \frac{u_d}{< u_d, u_d >} \\ &+ <u,x> \frac{u}{<u,u>} + \sum_{i=3}^{n} <e'_i, x> e'_i. \end{aligned} \tag{3.4}$$

in the orthogonal basis $u_d, u, e'_3, \ldots, e'_n$ (if $n = 2$, we have only u_d and u). The orthogonality conditions together with (3.2) read

$$\begin{aligned} < u_d, e'_i > &= < u, e'_i >= 0, \\ < e'_i, e'_j > &= \delta_{ij}, \quad i,j = 3, \ldots, n. \end{aligned} \tag{3.5}$$

Under the condition (3.2) the vector u is spacelike ($<u,u> > 0$) since u_d is timelike. Substituting the expression for x (3.4) into (3.3), we get

$$< u_d, \ddot{x} > = 0, \tag{3.6}$$

$$< e'_i, \ddot{x} > = 0, \quad i = 3, \ldots, n, \tag{3.7}$$

$$\begin{aligned} < u, \ddot{x} > &= < u, u > \left[\frac{1}{2} \left(\frac{< u_d, \dot{x} >^2}{< u_d, u_d >} + \frac{< u, \dot{x} >^2}{< u, u >} \right.\right. \\ &+ \left.\left. \sum_{i=3}^{n} < e'_i, \dot{x} >^2 \right) - \zeta_0 < u_d, \dot{x} > \right]. \end{aligned} \tag{3.8}$$

Integration of (3.6)-(3.8) results in

$$< u_d, x > = p^1 t + q^1, \tag{3.9}$$

$$< e'_i, x > = p^i t + q^i, \quad i = 3, \ldots, n, \tag{3.10}$$

$$< u, x > = - \ln[C f^2], \tag{3.11}$$

where $p^1, p^3, \ldots, p^n, q^1, q^3, \ldots, q^n, C$ are arbitrary constants ($C > 0$). The function f in (3.11) is $(t - t_0)$ or 1 for $A = 0$, $\cos[\sqrt{A}(t-t_0)/2]$ for $A > 0$ and $\cosh[\sqrt{-A}(t-t_0)/2]$ or $\sinh[\sqrt{-A}(t-t_0)/2]$ for $A < 0$. For constant A we have

$$A = <u,u> \left[\frac{(p^1)^2}{<u_d, u_d>} + \sum_{i=3}^{n} (p^i)^2 - 2\zeta_0 p^1 \right]. \tag{3.12}$$

To present the scale factors in a Kasner-like form, we introduce the Kasner-like parameters α^i and β^i such that

$$\alpha = p^3 e'_3 + \ldots + p^n e'_n \equiv \alpha^1 e_1 + \ldots + \alpha^n e_n, \tag{3.13}$$

$$\beta = q^3 e'_3 + \ldots + q^n e'_n \equiv \beta^1 e_1 + \ldots + \beta^n e_n. \tag{3.14}$$

Combining (3.4), (3.9)-(3.11) and (3.13)-(3.14), we obtain the exact solution

$$\begin{aligned} e^{x^i} &= (Cf^2)^{-u^i/<u,u>} \exp\Big[(\alpha^i + \frac{p^1}{D-1}) t \\ &+ \beta_i + \frac{q^1}{D-1} \Big], \; i = 1, \ldots, n \end{aligned} \tag{3.15}$$

$$\begin{aligned} \rho &= \frac{\exp[-2(p^1 t + q^1)]}{2\kappa^2 < u,u >} \\ &\times \left[< u,u > \left(\frac{D-1}{D-2} (p^1)^2 - < \alpha, \alpha > \right) - F^2 \right], \end{aligned} \tag{3.16}$$

where the possible variants are

$$\begin{aligned} f &= \sinh[\sqrt{-A}(t-t_0)/2], \\ F &= -\sqrt{-A} \coth[\sqrt{-A}(t-t_0)/2], \; A < 0, \end{aligned} \tag{3.17}$$

$$\begin{aligned} f &= \cosh[\sqrt{-A}(t-t_0)/2], \\ F &= -\sqrt{-A} \tanh[\sqrt{-A}(t-t_0)/2], \; A < 0, \end{aligned} \tag{3.18}$$

$$\begin{aligned} f &= \cos[\sqrt{A}(t-t_0)/2], \\ F &= \sqrt{A} \tan[\sqrt{A}(t-t_0)/2], \; A > 0, \end{aligned} \tag{3.19}$$

$$f = 1, \quad F = 0, \quad A = 0, \tag{3.20}$$

$$f = t - t_0, \quad F = -2/(t-t_0), \quad A = 0. \tag{3.21}$$

For the Kasner-like parameters and the constants we have:

$$
\begin{aligned}
< \alpha, u_d > &= \sum_{i=1}^{n} \alpha^i N_i = 0, \\
< \alpha, u > &= \sum_{i=1}^{n} \alpha^i N_i h_i = 0, \qquad (3.22)\\
< \beta, u_d > &= \sum_{i=1}^{n} \beta^i N_i = 0, \\
< \beta, u > &= \sum_{i=1}^{n} \beta^i N_i h_i = 0, \qquad (3.23)\\
u^i &= h_i + \frac{1}{2-D}\sum_{i=1}^{n} N_i h_i = h_i, \\
< u, u > &= \sum_{i=1}^{n} (h_i)^2 N_i > 0, \qquad (3.24)\\
< \alpha, \alpha > &= \sum_{i=1}^{n} (\alpha^i)^2 N_i,
\end{aligned}
$$

$$
A = <u,u> \left[-\frac{D-2}{D-1}(p^1)^2 + <\alpha,\alpha> - 2\zeta_0 p^1 \right]. \qquad (3.25)
$$

Let us consider the properties of this model. We will say that a solution is physical if the weak energy condition $\rho(t) \geq 0$ holds for any t. Only solutions with $f = 1$ and $f = \cosh[\sqrt{-A}(t-t_0)/2]$ satisfy this condition. Here we study the solution with $f = 1$. It is easy to check that this condition holds for $p^1 < 0$. For the proper time τ introduced by $d\tau = \exp[<u_d, x>]dt$, this exact solution looks as follows:

$$
\mathrm{e}^{x^i(\tau)} = \mathrm{e}^{\tilde{\beta}^i} \left[(-p^1)(\tau_0 - \tau) \right]^{\gamma^i}, \quad \tau < \tau_0, \qquad (3.26)
$$

$$
\rho(\tau) = -\frac{\zeta_0}{\kappa^2 p^1} \frac{1}{(\tau_0 - \tau)^2}, \quad p^1 < 0, \qquad (3.27)
$$

where p^1, τ_0 are arbitrary constants and $\tilde{\beta}^i$ obey the relations (3.23). By γ^i we denote

$$
\gamma^i = \frac{1}{D-1} + \frac{\alpha^i}{p^1}, \qquad i = 1, \ldots, n. \qquad (3.28)
$$

This solution is singular at the final point of evolution $\tau = \tau_0$ because $\rho(\tau) \to +\infty$ as $\tau \to \tau_0 - 0$. We also notice that $\rho(\tau) \to 0$ as $\tau \to -\infty$, so this solution can be interpreted as that describing creation of matter in the Universe.

The behavior of the scale factors (3.26) near the singular point is of Kasner type. For the Kasner-like parameters γ^i we obtain the relations

$$
\sum_{i=1}^{n} \gamma^i N_i = 1, \qquad \sum_{i=1}^{n} \gamma^i N_i h_i = 0, \qquad (3.29)
$$

$$
\sum_{i=1}^{n} (\gamma^i)^2 N_i = 1 + 2\frac{\zeta_0}{p^1}. \qquad (3.30)
$$

It is clear that such a Kasner-like solution always provides the contraction for one part of spaces M_i and expansion for another part. So, we may interpret the expanding space as the external one and the contracting spaces as the internal ones. We notice that for $n \geq 3$ this solution cannot be stationary but special solutions with stationary internal spaces always exist.

Now we will study another model with pressure and viscosity both isotropic, i.e.,

$$
p_i = (1-h)\rho \qquad \text{or} \qquad u = h u_d, \qquad (3.31)
$$

$$
\zeta_i = \frac{\zeta_0}{\kappa^2}\mathrm{e}^{-\gamma_0} \ \text{or} \ \xi = \frac{\zeta_0}{\kappa^2}\mathrm{e}^{-\gamma_0} u_d, \quad i = 1, \ldots, n, \qquad (3.32)
$$

where ζ_0 and h are constants. Here we assume that

$$
0 < h < 2 \quad \text{and} \quad \zeta_0 > 0. \qquad (3.33)
$$

Then the set of equations (2.24) in the harmonic time gauge ($\gamma = \gamma_0$) looks as follows:

$$\ddot{x} = \frac{h}{2} < \dot{x}, \dot{x} > u_d - \zeta_0 < u_d, \dot{x} > u_d \tag{3.34}$$

(recall that $\gamma_0 =< u_d, x >$.) To integrate (3.34), we use the following decomposition of the vector x:

$$x = <u_d, x> \frac{u_d}{<u_d, u_d>} + \sum_{i=2}^{n} <e'_i, x> e'_i. \tag{3.35}$$

The vectors $u_d, e'_2, \ldots, e'_n$ form an orthogonal frame in R^n, i.e.

$$<u_d, e'_i> = 0, \quad <e'_i, e'_j> = \delta_{ij}, \quad i, j = 2, \ldots, n. \tag{3.36}$$

We notice that in this frame any vector e_i cannot be timelike because the vector u_d is timelike. The set of equations (3.34) may be written as

$$\begin{aligned} < u_d, \ddot{x} > &= < u_d, u_d > \Bigg[\frac{h}{2} \Bigg(\frac{< u_d, \dot{x} >^2}{< u_d, u_d >} \\ &+ \sum_{i=2}^{n} <e'_i, \dot{x}>^2 \Bigg) - \zeta_0 <u_d, \dot{x}> \Bigg], \end{aligned} \tag{3.37}$$

$$< e'_i, \ddot{x} > = 0, \quad i = 2, \ldots, n. \tag{3.38}$$

Integration of (3.37) and (3.38) results in

$$< u_d, x > = -\frac{1}{h} \ln[C f^2] + \frac{\zeta_0}{h} < u_d, u_d > t, \tag{3.39}$$

$$< e'_i, x > = p^i t + q^i, \quad i = 2, \ldots, n, \tag{3.40}$$

where p^i, q^i, t_0 and C are arbitrary constants ($C > 0$). The symbol f denotes $\sinh[Ah(t - t_0)/2]$ or $\cosh[Ah(t - t_0)/2]$ where

$$A = -\frac{\zeta_0}{h} < u_d, u_d > \sqrt{1 - \frac{h^2 \sum_{i=2}^{n} (p^i)^2}{\zeta_0^2 < u_d, u_d >}}. \tag{3.41}$$

A special solution arises for $f = \exp[\pm Ah(t - t_0)/2]$. In this case $C = \exp[\zeta_0 < u_d, u_d > t_0]$.

To present the scale factors $\exp[x^i]$ in a Kasner-like form, we introduce the vectors $\alpha, \beta \in R^n$

$$\alpha = p^2 e'_2 + \ldots + p^n e'_n \equiv \alpha^1 e_1 + \ldots + \alpha^n e_n, \tag{3.42}$$

$$\beta = q^2 e'_2 + \ldots + q^n e'_n \equiv \beta^1 e_1 + \ldots + \beta^n e_n. \tag{3.43}$$

Recall that the vectors $e_1, \ldots, e_n$ form a canonical frame in R^n. The coordinates α^i and β^i are the Kasner-like parameters. Using (3.35),(3.39),(3.40) and (3.42)-(3.43), we obtain the exact solution in the Kasner-like form

$$\mathrm{e}^{x^i} = (C f^2)^{-1/[h(D-1)]} \exp\left[(\alpha^i - \frac{\zeta_0}{h(D-2)}) t + \beta_i \right]. \tag{3.44}$$

The Kasner-like parameters obey the relations

$$< \alpha, u_d > = \sum_{i=1}^{n} \alpha^i N_i = 0,$$

$$< \beta, u_d > = \sum_{i=1}^{n} \beta_i N_i = 0, \tag{3.45}$$

$$< \alpha, \alpha > = \sum_{i=1}^{n} (\alpha^i)^2 N_i = \sum_{j=2}^{n} (p^j)^2. \tag{3.46}$$

Using (2.22), we obtain the density

$$\begin{aligned} \rho &= \frac{a^2 + < \alpha, \alpha >}{2\kappa^2} (C f^2)^{2/h} \exp\left[\frac{2a^2 h}{\zeta_0} t \right] \\ &\times \left(F + \frac{a - \sqrt{< \alpha, \alpha >}}{\sqrt{a^2 + < \alpha, \alpha >}} \right) \left(F + \frac{a + \sqrt{< \alpha, \alpha >}}{\sqrt{a^2 + < \alpha, \alpha >}} \right). \end{aligned} \tag{3.47}$$

For the functions f and F in (3.44) and (3.47) we have the following variants

$$\begin{aligned} f &= \sinh[Ah(t-t_0)/2], \\ F &= \coth[Ah(t-t_0)/2], \quad C>0, \end{aligned} \tag{3.48}$$

$$\begin{aligned} f &= \cosh[Ah(t-t_0)/2], \\ F &= \tanh[Ah(t-t_0)/2], \quad C>0, \end{aligned} \tag{3.49}$$

$$\begin{aligned} f &= \exp[Ah(t-t_0)/2], \\ F &= 1, \quad C=\exp[-\frac{\zeta_0(D-1)}{D-2}t_0], \end{aligned} \tag{3.50}$$

$$\begin{aligned} f &= \exp[-Ah(t-t_0)/2], \\ F &= -1, \quad C=\exp[-\frac{\zeta_0(D-1)}{D-2}t_0]. \end{aligned} \tag{3.51}$$

The constants A and a are such that

$$\begin{aligned} a &= \frac{\zeta_0}{h}\sqrt{\frac{D-1}{D-2}}, \\ A &= \frac{D-1}{D-2}\sqrt{\frac{\zeta_0^2}{h^2}+\frac{D-2}{D-1}<\alpha,\alpha>} \end{aligned} \tag{3.52}$$

It is easy to check that the solution for $f=\exp[Ah(t-t_0)/2]$ determines the density without a range of negative values. This solution can be written in terms of the proper time τ as follows:

$$e^{x^i(\tau)} = e^{\tilde\beta^i}\left(\frac{\tau_0-\tau}{T_0}\right)^{1/(D-1)-T_0\alpha^i}, \quad \tau<\tau_0, \tag{3.53}$$

$$\rho(\tau) = \frac{\zeta_0 T_0}{\kappa^2 h}\frac{1}{(\tau_0-\tau)^2}, \tag{3.54}$$

where τ_0 is an arbitrary constant and the parameters $\tilde\beta^i$ obey the relations (3.45). For the constant T_0 we have

$$\frac{1}{T_0}=\frac{D-1}{D-2}\left[\frac{\zeta_0}{h}+\left(\frac{\zeta_0^2}{h^2}+\frac{D-2}{D-1}<\alpha,\alpha>\right)^{1/2}\right]. \tag{3.55}$$

As in the previous case, this solution is singular at the final point of evolution $\tau=\tau_0$ and describes matter creation in the Universe.

It is also worth noticing that this solution describes contraction of at least one space of $M_1,\ldots,M_n$. Indeed, due to the relations (3.45) at least one of the Kasner-like parameters is nonpositive, so the corresponding scale factor monotonically decreases on the interval $(-\infty,\tau_0)$. This process can be interpreted as contraction of the internal space to the Planck scale (10^{-33} cm). Moreover, it can be shown that for some set of Kasner-like parameters the solution describes expansion of one part of spaces and contraction of the other part.

Let us consider this property for a simplest model on the manifold $R\times R^3\times T^d$, where R^3 is a 3-dimensional flat external space and T^d is an internal space having the shape of d-dimensional torus. The exact solution (3.53) gives

$$e^{x^1(\tau)} = e^{\tilde\beta^1}\left(\frac{\tau_0-\tau}{T_0}\right)^{1/(d+3)-T_0\alpha^1}, \tag{3.56}$$

$$e^{x^2(\tau)} = \exp[-\frac{3}{d}\tilde\beta^1]\left(\frac{\tau_0-\tau}{T_0}\right)^{1/(d+3)+\frac{3}{d}T_0\alpha^1}, \tag{3.57}$$

where

$$\frac{1}{T_0}=\frac{d+3}{d+2}\left(\frac{\zeta_0}{h}+\sqrt{\frac{\zeta_0^2}{h^2}+3\frac{d+2}{d}(\alpha^1)^2}\right), \tag{3.58}$$

τ_0, $\tilde\beta^1$ and α^1 are arbitrary constants. If $\alpha^1>0$, then the internal space monotonically contracts. It is easy to show that under the condition

$$\frac{(d+3)(d-1)}{d}\alpha > 2\frac{\zeta_0}{h} \tag{3.59}$$

we obtain the monotonic expansion of the external space on the interval $(-\infty,\tau_0)$. This condition can be satisfied for $d\geq 2$.

For exact solutions with other functions f the condition $\rho(t)\geq 0$ for any t is not satisfied. However, for the solution with $f=\sinh[Ah(t-t_0)/2]$ the density is positive for any $t\in(t_0,+\infty)$ and this interval corresponds to the proper time interval $(-\infty,\tau_0)$. So, we consider this solution to be also admissible from the physical point of view. It follows from (3.44) that this solution and that with $f=\exp[Ah(t-t_0)/2]$ have an identical behavior near the singular point $\tau=\tau_0$. For $2/h>1$ we have $\rho(\tau)\to 0$ as $\tau\to-\infty$, then this solution can be interpreted as that describing creation of matter.

Acknowledgement

V.R.G. and V.N.M. are grateful to the Russian Ministry of Science for partial support of this work. V.N.M. also thanks CNPq for the grant and possibility to work at CBPF in February-June, 1995.

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