Superspace Gauge Fixing of Topological Yang-Mills Theories

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Abstract

We revisit the construction of topological Yang-Mills theories of the Witten type with arbitrary space-time dimension and number of "shift supersymmetry" generators, using a superspace formalism. The super-BF structure of these theories is exploited in order to determine their actions uniquely, up to the ambiguities due to the fixing of the Yang-Mills and BF gauge invariance. UV finiteness to all orders of perturbation theory is proved in a gauge of the Landau type

Key-words: Quantum field; Topological theory; Gauge theory.

1 Introduction

Observables in topological theories possess a global character, such as the knot invariants of Chern-Simons theory, the Wilson loops, etc. The problem of finding all theses invariants is a problem of equivariant cohomology, as proposed by Witten in 1988 [1] for Yang-Mills topological theory in four-dimensional space-time. Equivariant cohomology is the cohomology of a BRST-like operator – the "shift supersymmetry operator", associated to a local shift transformation of the connection field – in a space of gauge invariant field polynomials. A superspace formulation of Witten's model was proposed by Horne [2] and developed later on, in particular by Blau and Thompson [3, 4], who extended it to the cases of more than one supersymmetry generator and in different space-time dimensions. In various cases these topological theories are seen to arise from super-Yang-Mills theories through some twist of group representations [1, 3, 4], possibly accompanied by dimensional reduction. The reader may see [5] for the systematic construction of topological theories from super-Yang-Mills ones using this technique. Our proposal is to systematize the superspace construction of actions in the most general setting involving an arbitrary number N_T of topological supersymmetry generators in any space-time dimension D. Our construction will be direct, not passing through the twist procedure. The question of the existence, in each case, of a corresponding super-Yang-Mills theory will not be touched.

The theory intends to describe gauge field configurations with null curvature – or also selfdual curvature, in the 4-dimensional case. The null or selfdual curvature condition is implemented through a Lagrange multiplier field B which has the same hierarchy of zero-modes as the B field of a BF type theory [6]. The point of view usually adopted in the literature [1, 2, 3, 4] is that of considering the supersymmetry generator(s) as BRST operator(s) associated to the local shift invariance, and fixing the latter with suitable Lagrange multiplier fields. In the present paper, in order to avoid certain ambiguities which may arise in the usual scheme, we shall consider the theory as a rigid supersymmetric theory with two gauge invariances, namely the usual Yang-Mills gauge invariance and the gauge invariance of the B field, like the one encountered in BF-theories. Both invariances are supergauge invariances, their parameters being superspace functions. We shall see that this is enough to define the theory in an unambiguous way, apart from the freedom in the choice of a gauge fixing procedure. Moreover, in the case of a gauge fixing of the Landau type, we shall show, using supergraph techniques, that perturbative radiative corrections are completely absent. The theory thus turns out to be obviously ultraviolet finite.

A very important point is the systematic characterization of all observables of a topological theory. This has been fully done in [7] for the $N_T = 1$ theories. Partial results exist in the literature. In particular, a set of observables has been given in [8] for the case of $N_T = 2$ in a 4-dimensional Kähler manifold. In the present paper we shall only show a rather general set of observables, however without determining if it represents the most general one.

The plan of the paper is the following. After reminding the principal features of the original Witten-Donaldson's topological Yang-Mills theory [1] in section 2 and of superspace

formalism for topological theories in section 3, we shall show the construction of the action as a super-BF one, with the appropriate gauge fixing, in section 4. Examples of observables are given in section 5, and the ultraviolert problem is dealt with in section 6. A discussion of the results is done in the concluding section. Some of our conventions and notations are given in two appendices.

2 Shift Supersymmetry

We are going to review here, for illustrative purpose, how "shift supersymmetry" may describe the gauge fixing of gauge field configurations with null curvature, or alternatively with selfdual curvature. We shall concentrate on the original Donaldson-Witten model [1, 9], with one supersymmetry generator in four dimensional space-time.

2.1 Transformation rules and invariant actions

We recall that this model implies, beyond the gauge connection a_μ associated to some gauge group G, a fermion 1-form ψ_μ and a 0-form ϕ, with "shift" supersymmetry defined by the infinitesimal transformations

$$\tilde{Q}a = \psi \ , \quad \tilde{Q}\psi = -D(a)\phi \equiv -\left(d\phi + [a, \phi]\right) \ , \quad \tilde{Q}\phi = 0 \ . \tag{2.1}$$

All fields here and in the rest of the paper are valued in the Lie algebra of the gauge group – assumed to be a compact Lie group. Details on the notation are given in appendix ??.

The usual Yang-Mills gauge transformations read, written as BRST transformations with ghost c:

$$\mathcal{S}a = -D(a)c \ , \quad \mathcal{S}\psi = -[c, \psi] \ , \quad \mathcal{S}\phi = -[c, \phi] \ , \quad \mathcal{S}c = -c^2 \ . \tag{2.2}$$

Whereas the BRST operator $\mathcal{S}$ is nilpotent, the fermionic generator $\tilde{Q}$ is nilpotent modulo a ϕ-dependent gauge transformation:

$$\tilde{Q}^2 a = -D(a)\phi \ , \quad \tilde{Q}^2\psi = -[\psi, \phi] \ , \quad \tilde{Q}^2\phi = 0 \ . \tag{2.3}$$

This means that $\tilde{Q}$ is nilpotent when restricted to gauge invariant quantities. Following Witten, we may thus interpret the shift supersymmetry invariance as a BRST-like invariance in the space of gauge invariant field functionals. The 1-form ψ represents the ghost of local shift invariance and ϕ is its ghost of ghost. Then the following counting of degrees of freedom holds: counting 4 degrees of freedom for a, -4 for the ghost ψ and 1 for the ghost of ghost ϕ, we arrive to a total of 1 degree of freedom, which corresponds to the scalar mode of the field a – which in turn is eliminated thanks to the usual Yang-Mills gauge invariance. The final number zero of local degrees of freedom is of course characteristic of a topological theory.

In view of the absence of local degrees of freedom, the theory may be defined through an action which will be purely of a gauge fixing type. This fixing of the local shift supersymmetry

may be done introducing Lagrange multipliers fields $^0b_2$, $^1\lambda_1$, $^0\lambda_0$ and $^{-1}\eta_0$, together with the corresponding "antighost" fields $^{-1}\bar{b}_2$, $^0\bar{\psi}_1$, $^{-1}\bar{\psi}_0$ and $^{-2}\bar{\phi}_0$, where the lower right index denotes the form degree and the upper left one the degree of supersymmetry or "SUSY-number". The latter number corresponds to a ghost number in the interpretation of shift supersymmetry as a BRST transformation. Each "antighost" transforms under $\tilde{Q}$ into its corresponding Lagrange multiplier:

$$\tilde{Q}\ ^{-1}\bar{b}_2 = \ ^0b_2\ , \qquad\qquad \tilde{Q}\ ^{p-1}\bar{\psi}_p = \ ^p\lambda_p \quad (p=1,2)\ , \qquad \tilde{Q}\ ^{-2}\bar{\phi}_0 = \ ^{-1}\eta_0\ ,$$
$$\tilde{Q}\ ^0b_2 = -[\ ^{-1}\bar{b}_2, \phi]\ , \qquad \tilde{Q}\ ^p\lambda_p = -[\ ^{p-1}\bar{\psi}_p, \phi]\ , \qquad\qquad \tilde{Q}\ ^{-1}\eta_0 = -[\ ^{-2}\bar{\phi}_0, \phi]\ ,$$

(2.4)

the transformation rules of the Lagrange multipliers assuring the nilpotency of $\tilde{Q}$ modulo a ϕ-dependent gauge transformation. If one intends to study the instanton gauge field configurations, i.e. those with selfdual curvature $F = P_+ F$, where P_+ is defined by (A.6), the associated "antighost" and Lagrange mutiplier have to be chosen as anti-selfdual: $P_+\ ^{-1}\bar{\psi}_0 = 0$, $P_+\ ^0\lambda_0 = 0$.

A gauge invariant and $\tilde{Q}$- invariant action may be taken as (following [4])

$$\tilde{Q}\ \mathrm{Tr} \int \Big(\ ^{-1}\bar{b}_2\ F(a) + \ ^{-2}\bar{\phi}_0\ D(a) * \psi + \ ^0\bar{\psi}_1\ D(a) *^{-1}\bar{b}_2 + \ ^{-1}\bar{\psi}_0\ D(a) *^0 \bar{\psi}_1\Big)$$
$$= \mathrm{Tr} \int \Big(\ ^0b_2 F(a) + \ ^{-1}\eta_0\ D(a) * \psi + \ ^1\lambda_1\ D(a) *^{-1}\bar{b}_2 + \ ^0\lambda_0\ D(a) *^0 \bar{\psi}_1$$
$$+\ ^{-1}\bar{b}_2\ D(a)\psi + \ ^{-2}\bar{\phi}_0\ D(a) * D(a)\phi + \ ^0\bar{\psi}_1\ D(a) *^0 b_2 + \ ^{-1}\bar{\psi}_0\ D(a) *^1 \lambda_1$$
$$+\ ^{-2}\bar{\phi}_0[\psi, *\psi] - \ ^0\bar{\psi}_1[\psi, *^{-1}\bar{b}_2] - \ ^{-1}\bar{\psi}_0[\psi, *^0\bar{\psi}_1]\Big)\ ,$$

(2.5)

where $F(a) = da + a^2$ is the curvature of the Yang-Mills connection a and $*$ is the Hodge duality operator (see appendix **??**). One sees that the Lagrange multiplier $^0b_2$ implements the zero-curvature condition $F(a) = 0$ – or the selfduality condition, as in the original Witten's paper. $^{-1}\eta_0$ is the Lagrange multiplier fixing the zero-mode of ψ. Morover, $^1\lambda_1$ fixes the zero-mode of $^{-1}\bar{b}_2$, $^0\lambda_0$ that of $^0\bar{\psi}_1$. Finally, $^0\bar{\psi}_1$ fixes the zero-mode of $^0b_2$ and $^{-1}\bar{\psi}_0$ that of $^1\lambda_1$.

This action corresponds to a generalized "Landau gauge" fixing. However, it is still possible to add one invariant term quadratic in the Lagrange multipliers without spoiling gauge invariance, supersymmetry and SUSY-number conservation. It reads

$$\tilde{Q}\ \mathrm{Tr} \int \frac{\xi}{2}\ ^{-2}\bar{\psi}_2 * \ ^0b_2 = \mathrm{Tr} \int \frac{\xi}{2} \Big(\ ^0b_2 * \ ^0b_2 + \ ^{-2}\bar{\psi}_2 * [\ ^{-2}\bar{\psi}_2, \phi]\Big)\ . \qquad (2.6)$$

In this case, which corresponds to a generalized Feynman gauge, the Lagrange mmultiplier $^0b_2$ becomes an auxiliary fields, whose elimination through its equation of motion gives rise, for $\xi = 1$, to the original action of Witten [1], which describes selfdual configurations when choosing the Lagrange multiplier $^0b_2$ and its corresponding "antighost" as anti-selfdual 2-forms.

2.2 Observables

According to Witten, the algebra of observables of the theory is generated by sets of gauge invariant forms $w_p^{(n)}$ ($0 \leq p \leq 4$, n integer) obeying "descent equations"

$$\tilde{Q} w_p^{(n)} + d w_{p-1}^{(n)} = 0 \quad (4 \geq p \geq 1) , \quad \tilde{Q} w_0^{(n)} = 0 , \tag{2.7}$$

and are uniquely fixed up to total derivatives by

$$w_0^{(n)} = C^{(n)}(\phi) ,$$

where $C^{(n)}(\phi)$ is an invariant corresponding to a Casimr operator $\mathcal{C}^{(n)}$ of the gauge group. Each p-form w_p^n being then integrated on some p-dimensional submanifold M_p, represents an equivariant cohomology class and defines a basis element of the algebra of observables.

3 N_T-Extended Supersymmetry

Our purpose in this section is to review and develop a superspace formalism describing topological theories such as Wittens's theory described in section 2 and generalizations of it for more than one supersymmetric generators and for any space-time dimension, starting from the formalism described in [2, 4, 7].

3.1 N_T Superspace formalism

N_T supersymmetry is generated by the fermionic charges Q_I, $I = 1, \ldots, N_T$ obeying the Abelian superalgebra [2]

$$[Q_I, Q_J] = 0 , \tag{3.1}$$

commuting with the space-time symmetry generators and the gauge group generators. The gauge group is some compact Lie group.

A representation of supersymmetry is provided by superspace, a supermanifold with D bosonic and N_T fermionic dimensions[3]. The respective coordinates are denoted by (x^μ, $\mu = 0, \ldots, D-1$), and (θ^I, $I = 1, \ldots, N_T$). A superfield is by definition a superspace function $F(x, \theta)$ which transforms as

$$Q_I F(x, \theta) = \partial_I F(x, \theta) \equiv \frac{\partial}{\partial \theta^I} F(x, \theta) \tag{3.2}$$

under an infinitesimal supersymmetry transformation.

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[2] The bracket is here an anti-commutator.
[3] Notations and conventions on superspace are given in appendix ??.

An expansion in the coordinates θ^I of a generic superfield reads

$$F(x, \theta) = f(x) + \sum_{n=1}^{N} \frac{1}{n!} \theta^{I_1} ... \theta^{I_n} f_{I_1 ... I_n}(x) \tag{3.3}$$

where the space-time fields $f_{I_1 ... I_n}(x)$ are completely antisymmetric in the indices $I_1 ... I_n$. We recall that all fields (and superfields) are Lie algebra valued. We shall also deal with superforms. A p-superform may be written as

$$\hat{\Omega}_p = \sum_{k=0}^{p} \Omega_{p-k; \, I_1 ... I_k} d\theta^{I_1} \cdots d\theta^{I_k} \, , \tag{3.4}$$

where the coefficients $\Omega_{k-1; \, I_1 ... I_k}$ are (Lie algebra valued) superfields which are space-time forms of degree $(p-k)$. They are completely symmetric in their indices since, the coordinates θ being anti-commutative, the differentials $d\theta^I$ are commutative. The superspace exterior derivative is defined as

$$\hat{d} = d + d\theta^I \partial_I \, , \quad d = dx^\mu \partial_\mu \, , \tag{3.5}$$

and is nilpotent: $\hat{d}^{\,2} = 0$.

The basic superfield of the theory is the superconnection $\hat{A}$, a 1-superform:

$$\hat{A} = A + E_I d\theta^I \, , \tag{3.6}$$

with $A = A_\mu(x, \theta) dx^\mu$ a 1-form superfield and $E_I = E_I(x, \theta)$ a 0-form superfield. The superghost $C(x, \theta)$ is a 0-superform. We expand the components of the superconnection (3.6) as

$$A = a(x) + \sum_{n=1}^{N} \frac{1}{n!} \theta^{I_1} ... \theta^{I_n} a_{I_1 ... I_n}(x) \, , \tag{3.7}$$

where the 1-form a is the gauge connection, and the 1-forms $a_{I_1 ... I_n}$ its supersymmetric partners. The expansions of E_I and of the ghost superfield C read

$$\begin{aligned}
E_I &= e_I(x) + \sum_{n=1}^{N} \frac{1}{n!} \theta^{I_1} ... \theta^{I_n} e_{I, I_1 ... I_n}(x) \, , \\
C &= c(x) + \sum_{n=1}^{N} \frac{1}{n!} \theta^{I_1} ... \theta^{I_n} c_{I_1 ... I_n}(x) \, .
\end{aligned} \tag{3.8}$$

The infinitesimal supergauge transformations of the superconnection are expressed as the nilpotent BRST transformations

$$\mathcal{S}\hat{A} = -\hat{d}C - [C, \hat{A}] \, , \quad \mathcal{S}C = -C^2 \, , \quad \mathcal{S}^2 = 0 \, . \tag{3.9}$$

In terms of component superfields we have

$$\mathcal{S}A = -dC - [C, A] \, , \quad \mathcal{S}E_I = -\partial_I C - [C, E_I] \, , \quad \mathcal{S}C = -C^2 \, . \tag{3.10}$$

The supercurvature

$$\hat{F} = \hat{d}\hat{A} + \hat{A}^2 = F(A) + \Psi_I \, d\theta^I + \Phi_{IJ} \, d\theta^I d\theta^J \tag{3.11}$$

transforms covariantly:

$$\mathcal{S}\hat{F} = -[C, \hat{F}] \ ,$$

as well as its components

$$F(A) = dA + A^2 \ , \quad \Psi_I = \partial_I A + D(A)E_I \ , \quad \Phi_{IJ} = \frac{1}{2} \left(\partial_I E_J + \partial_J E_I + [E_I, E_J] \right) \ , \tag{3.12}$$

where the covariant derivative is defined by $D(A)(\cdot) = d(\cdot) + [A, (\cdot)]$

For further use and comparisons with the literature, let us give the explicit examples of $N_T = 1, 2$.

Example 1 – Case $N_T = 1$

The superconnection (3.6) and the expansions (3.7 - 3.8) read

$$A(x, \theta) = a(x) + \theta\psi(x) \ , \quad E(x, \theta) = \chi(x) + \theta\phi(x) \ , \quad C(x, \theta) = c(x) + \theta c'(x) \ . \tag{3.13}$$

The BRST transformations of the component fields are

$$\mathcal{S}a = -D(a)c \ , \quad \mathcal{S}\psi = -[c, \psi] - D(a)c' \ , \quad \mathcal{S}\phi = -[c, \phi] - [\chi, c'] \ , \quad \mathcal{S}\chi = -[c, \chi] - c' \ ,$$
$$\mathcal{S}c = -c^2 \ , \qquad \mathcal{S}c' = -[c, c'] \tag{3.14}$$

As for the supersymmetry transformations defined by (3.2), we have:

$$Qa = \psi \ , \ Q\psi = 0 \ , \qquad Q\chi = \phi \ , \ Q\phi = 0 \ , \qquad Qc = c' \ , \ Qc' = 0 \ . \tag{3.15}$$

The supercurvature components (3.12) read

$$F(A) = F(a) - \theta D(a)\psi \ ,$$
$$\Psi = \psi + D(a)\chi - \theta \left(D(a)\phi - [\psi, \chi] \right) \ , \tag{3.16}$$
$$\Phi = \phi + \chi^2 + \theta[\phi, \chi] \ .$$

Example 2 – Case $N_T = 2$

The superconnection (3.6) and the expansions (3.7 - 3.8) now read (with $I = 1, 2$)

$$A(x, \theta) = a(x) + \theta^I \psi_I(x) + \tfrac{1}{2}\theta^2\alpha \ , \quad E_I(x, \theta) = \chi_I(x) + \theta^I \phi_{IJ}(x) + \tfrac{1}{2}\theta^2\eta_I \ ,$$
$$C(x, \theta) = c(x) + \theta^I c_I(x) + \tfrac{1}{2}\theta^2 c_F \ . \tag{3.17}$$

The BRST transformations of the component fields are

$$\mathcal{S}a = -D(a)c \ , \quad \mathcal{S}\psi_I = -[c,\psi_I] - D(a)c_I, \quad \mathcal{S}\alpha = -[c,\alpha] - D(a)c_F + \epsilon^{IJ}[c_I,\psi_J] \ ,$$

$$\mathcal{S}\chi_I = -[c,\chi_I] - c_I \ , \quad \mathcal{S}\phi_{IJ} = -[c,\phi_{IJ}] - \epsilon_{IJ}c_F + [\chi_I,c_J] \ ,$$

$$\mathcal{S}\eta_I = -[c,\eta_I] - [c_F,\chi_I] + \epsilon^{JK}[c_J,\phi_{IK}] \ ,$$

$$\mathcal{S}c = -c^2 \ , \quad \mathcal{S}c_I = -[c,c_I] \ , \quad \mathcal{S}c_F = -[c,c_F] + \tfrac{1}{2}\epsilon^{IJ}[c_I,c_J] \ .$$

$$(3.18)$$

The supersymmetry transformations read

$$Q_I a = \psi_I \ , \qquad Q_I \psi_J = -\epsilon_{IJ}\alpha \ , \qquad Q_I \alpha = 0 \ ,$$

$$Q_I \chi_J = \phi_{JI} \ , \qquad Q_I \phi_{Jk} = -\epsilon_{IK}\eta_J \ , \qquad Q_I \eta_J = 0 \ ,$$

$$Q_I c = c_I \ , \qquad Q_I c_I = -\epsilon_{IJ}c_F \ , \qquad Q_I c_F = 0 \ .$$

$$(3.19)$$

The supercurvature components (3.12) read now

$$F(A) = F(a) - \theta^I D(a)\psi_I + \tfrac{1}{2}\theta^2 (D(a)\alpha - \tfrac{1}{2}\epsilon^{IJ}[\psi_I,\psi_J]) \ ,$$

$$\Psi_I = \psi_I + D(a)\chi_I + \theta^J(\epsilon_{IJ}\alpha - D(a)\phi_{IJ} + [\psi_J,\chi_I])$$

$$+ \tfrac{1}{2}\theta^2 (D(a)\eta_I - \epsilon^{KJ}[\psi_K,\phi_{IJ}] + [\alpha,\chi_I]) \ ,$$

$$(3.20)$$

$$\Phi_{IJ} = \frac{1}{2}\left(\phi_{IJ} + \phi_{JI} + [\chi_I,\chi_J] + \theta^K(\epsilon_{IK}\eta_J + \epsilon_{JK}\eta_I + [\phi_{JK},\chi_I] + [\phi_{IK},\chi_J])\right.$$

$$\left. + \tfrac{1}{2}\theta^2([\chi_I,\eta_J] + [\eta_I,\chi_J] - \epsilon^{KL}[\phi_{IK},\phi_{JL}])\right) \ .$$

Counting the number of degrees of freedom:

The numbers of degrees of freedom, i.e. the numbers of component fields – remembering that a p-form has $D!/[p!(D-p!)]$ components – are shown in Table 1. If we were considering the

Fields:	$a_{I_1 \ldots I_n}(x)$	$A(x,\theta)$	$e_{I,I_1 \ldots I_n}(x)$	$E_I(x,\theta)$	$c_{I_1 \ldots I_n}(x)$	$C(x,\theta)$
Numbers of fields:	$D\binom{N_T}{n}$	$D\,2^{N_T}$	$N_T\binom{N_T}{n}$	$N_T\,2^{N_T}$	$\binom{N_T}{n}$	2^{N_T}

Table 1: Numbers of component fields. D = space-time dimension, N_T = number of supersymmetry generators.

present theory as a usual supersymmetric gauge theory, with (super)gauge invariance defined by the BRST transformations (3.10), the number of physical degrees of freedom would be given by the total number of components of the forms A and E_I minus the number of components of the superghost C. However, considering it as a topological theory we have to treat supersymmetry as a local invariance, too, all fields excepted the Yang-Mills connection a being ghosts or ghosts of ghosts, as in the example shown in section 2. The SUSY-number

s is thus a ghost number as well as the usual ghost number[4] g. Thus the effective ghost number is equal to $s + g$ and, in the counting of the physical degrees of freedom, we must therefore assign a sign $(-)^{s+g}$ to the number of degrees of freedom of a field, as shown in Table 2. One sees that there is a complete cancellation of the local degrees of freedom, as it

Fields:	$a_{I_1 \dots I_n}$	A	$e_{I,I_1 \dots I_n}$	E_I	$c_{I_1 \dots I_n}$	C
SUSY #:	n	0	$n+1$	1	n	0
Ghost #:	0	0	0	0	1	1
Degrees of freedom:	$(-1)^n D \binom{N_T}{n}$	0	$(-1)^{n+1} N_T \binom{N_T}{n}$	0	$(-1)^{n+1} \binom{N_T}{n}$	0

Table 2: Numbers of physical degrees of freedom. D = space-time dimension, N_T = number of supersymmetry generators.

should in a topological theory.

3.2   Wess-Zumino gauge

The contact with the formalism described in section 2 is made by choosing a special gauge fixing [4] of the Wess-Zumino type [10]. The BRST transformations of the component fields can be calculated from the superfield expressions (3.10). They are explicitly given, for $N_T = 1$ and 2, by (3.14,3.18). We shall only write explicitly the linear part – or Abelian approximation – of the transformations in the general case, which will be sufficient for our argument:

$$\mathcal{S}a = -dc + \cdots , \qquad \mathcal{S}a_{I_1 \dots I_N} = -dc_{I_1 \dots I_N} + \cdots ,$$

$$\mathcal{S}e_I = -c_I + \cdots , \qquad \mathcal{S}e_{I,I_1 \dots I_N} = -c_{I I_1 \dots I_N} + \cdots , \tag{3.21}$$

$$\mathcal{S}c = \cdots , \qquad\qquad \mathcal{S}c_{I_1 \dots I_N} = \cdots ,$$

where the dots represent nonlinear terms. These transformaions indicate that $e_I(x)$ and the completely antisymmetric part of the fields $e_{I,I_1 \dots I_n}(x)$ are pure gauge degrees of freedom. A possible gauge fixing is therefore of setting these fields to zero. This defines the Wess-Zumino (WZ) gauge:

$$e_I = 0 , \quad e_{[I,I_1 \dots I_n]} = 0 \ (1 \le n \le N_T) . \tag{3.22}$$

This fixes the gauge degreees of freedom corresponding to the ghosts $c_{I_1 \dots I_n} \ (1 \le n \le N_T)$. The remaining gauge degree of freedom parametrized by the ghost c, which is of the usual Yang-Mills type, can be fixed in a usual way.

The WZ gauge condition (3.22) is not stable under supersymmetry transformations, but one can redefine the generators Q_I into new generators $\tilde{Q}_I$, compatible with the WZ condition, resulting from a combination of Q_I and of a field dependent supergauge transformation.

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[4]s and g are defined by attributing $s = g = 0$ to the gauge connection $a(x)$, $s = 1$, $g = 0$ for the supersymmetry genrators Q_I – hence $s = -1$ to θ^I – and $s = 0$, $g = 1$ for the BRST generator $\mathcal{S}$.

Thus, let us combine an infinitesimal supersymmetry transformation of constant commuting parameters ϵ^I with a supergauge transformation δ_Λ of anticommuting parameters (fermionic superfield) $\Lambda(x,\theta)$:

$$\tilde{Q} = \epsilon^I Q_I + \delta_\Lambda \equiv \epsilon^I \tilde{Q}_I \; . \tag{3.23}$$

δ_Λ is in fact a BRST transformation (3.14), with C substituted by Λ. This will define the modified supersymmetry generator $\tilde{Q}_I$, provided we choose Λ in such a way to preserve the WZ gauge condition (3.23). It is convenient to rewrite the WZ condition in a superspace way:

$$\theta^I E_I(x,\theta) = 0 \; , \tag{3.24}$$

where E_I is the $d\theta$-part of the superconnection (3.6). We shall denote by $\tilde{E}_I$ the solution of this condition, and by $\tilde{e}_{[I,I_1...I_n]}$ $(0 \leq n \leq N_T)$ the components of its θ-expansion – which are therefore solutions of (3.22). The latters are tensors with mixed symmetry. Applying $\tilde{Q}$ to (3.24) we find, after some integration by part in θ:

$$\tilde{Q}(\theta^I E_I) = -\theta^I \left(\epsilon^J \partial_J E_I - \partial_I C - [C, E_I] \right) = -\epsilon^I E_I + \theta^I \partial_I \Lambda + \partial_J(\epsilon^J \theta^I E_I) + [\theta^I E_I, \Lambda] \; , \tag{3.25}$$

which shows that the WZ condition (3.24) is stable if, and only if, Λ obeys the equation

$$\theta^I \partial_I \Lambda = \epsilon^I E_I \; . \tag{3.26}$$

The solution reads

$$\Lambda = \epsilon^I \sum_{n=1}^{N_T} \frac{1}{n!n} \theta^{I_1} \cdots \theta^{I_n} \tilde{e}_{I,I_1...I_n}(x) \; , \tag{3.27}$$

where the functions $\tilde{e}_{I,I_1...I_n}(x)$ are the coefficients of the superfield expansion of $\tilde{E}_I$, solution of (3.24).

One can now check that the superalgebra now closes up to field dependent gauge transformations $\delta_{\tilde{e}_{IJ}}$:

$$[\tilde{Q}_I, \tilde{Q}_J] = -2\delta_{\tilde{e}_{IJ}} \; .$$

Physical degrees of freedom in the WZ gauge:

The numbers of component fields are now given in Table 3. Remember that the only remaining ghost is $c(x)$, since the $c_{I_1...I_n}$ for $n \geq 1$ correspond to the gauge degrees of freedom which have been fixed. In order to count the physical degrees of freedom we must again take

Fields:	$a_{I_1...I_n}(x)$	$A(x,\theta)$	$e_{I,I_1...I_n}(x)$	$E_I(x,\theta)$	$c(x)$
Numbers of fields:	$D\binom{N_T}{n}$	$D\,2^{N_T}$	$\binom{N_T+1}{n+1}n$	$(N_T-1)2^{N_T}+1$	1

Table 3: Numbers of component fields in the WZ gauge. D = space-time dimension, N_T = number of supersymmetry generators.

Fields:	$a_{I_1 \cdots I_n}(x)$	$A(x, \theta)$	$e_{I, I_1 \cdots I_n}(x)$	$E_I(x, \theta)$	$c(x)$
SUSY #:	n	0	$n+1$	1	0
Ghost #:	0	0	0	0	1
Degrees of freedom:	$(-1)^n D \binom{N_T}{n}$	0	$(-1)^{n+1} \binom{N_T+1}{n+1} n$	1	-1

Table 4: Numbers of physical degrees of freedom. D = space-time dimension, N_T = number of supersymmetry generators.

into account the sign $(-)^{s+g}$ characterizing the ghost nature of each field, thus obtaining the results shown in Table 4. There is again a complete cancellation of the local degrees of freedom, as it should.

Let us consider more explicitly the cases of $N_T = 0$ and 1.

Example 3 – Case $N_T = 1$

The WZ gauge condition reads $\chi = 0$, we have

$$\tilde{E}(x, \theta) = \theta \phi(x) \ ,$$

and the parameter Λ of the compensating supergauge transformation is given by

$$\Lambda = \epsilon \, \theta \phi \ .$$

In 4 dimensions we recover the Donaldson-Witten theory of section 2. In particular, the modified supersymmetry transformations are those given by (2.1). It is moreover easy to check the nilpotency of $\tilde{Q}$ modulo a ϕ-dependent gauge tansformation δ_ϕ:

$$\tilde{Q}^2 = \delta_\phi \ .$$

Example 4 – Case $N_T = 2$

In terms of the component fields defined by (3.17), the WZ gauge condition reads

$$\chi_I = 0 \ , \quad \phi_{IJ} - \phi_{JI} = 0 \ ,$$

so that

$$\tilde{E}_I = \theta^J \phi_{(IJ)} + \frac{1}{2} \theta^2 \eta_I \ , \quad with \quad \phi_{(IJ)} = \frac{1}{2}(\phi_{IJ} + \phi_{JI}) \ .$$

The parameter Λ of the compensating supergauge transformation is given by

$$\Lambda = \epsilon^I \left(\theta^J \phi_{(IJ)} + \frac{1}{4} \theta^2 \eta_I \right) \ ,$$

and the modified supersymmetry transformations are

$$\tilde{Q}_I a = \psi_I \ , \quad \tilde{Q}_I \psi_J = -D(a)\phi_{IJ} - \epsilon_{IJ}\alpha \ , \quad \tilde{Q}_I \alpha = \epsilon^{JK}[\phi_{IJ}, \psi_K] + D(a)\eta_I \ ,$$

$$\tilde{Q}_I \phi_{JK} = \frac{1}{2}\left(\epsilon_{IJ}\eta_K + \epsilon_{IK}\eta_J\right) \ , \quad \tilde{Q}_I \eta_J = \epsilon^{KL}[\phi_{IK}, \phi_{JL}] \ .$$

The superalgebra closes on the ϕ dependent gauge transformations $\delta_{\phi_{(IJ)}}$:

$$[\tilde{Q}_I, \tilde{Q}_J] = -2\delta_{\phi_{(IJ)}} \ .$$

4   Actions

4.1   Action for $N_T = 1$ in D-dimensions

4.1.1   The geometrical sector

We follow here [2, 3, 4, 11]. In such theories, the action is purely of gauge fixing type, the gauge condition being that of zero Yang-Mills curvature – or possibly of selfdual curvature, in four dimensions, as in the original Witten's paper [1]. The "gauge invariance" which has to be fixed is the local shift supersymmetry expressed by the nilpotent operator Q (or $\tilde{Q}$ in the WZ gauge). For this we have to introduce a Lagrange multiplier field[5] ${}^0 b^0_{D-2}$ and an associated "antighost" ${}^{-1}\bar{b}^0_{D-2}$. In the case of a selfduality condition in $D = 4$ dimensions, both ${}^0 b^0_{D-2}$ and ${}^{-1}\bar{b}^0_{D-2}$ are to be taken as anti-selfdual 2-forms. One has still to introduce the Lagrange multiplier ${}^{-2}\bar{\phi}^0_0$ and its associated "antighost" ${}^{-1}\eta^0_0$ in order to fix the zero mode of the 1-form field ${}^1\psi^0_1$. "Antighosts" and Lagrange multipliers transform as

$$\begin{aligned} Q \ {}^{-1}\bar{b}^0_{D-2} &= {}^0 b^0_{D-2} \ , &\quad Q \ {}^0 b^0_{D-2} &= 0 \ , \\ Q \ {}^{-2}\bar{\phi}^0_0 &= {}^{-1}\eta^0_0 \ , &\quad Q \ {}^{-1}\eta^0_0 &= 0 \ . \end{aligned} \tag{4.1}$$

The best way to write down an invariant action is to use the superspace formalism, introducing the two "Lagrange multiplier superfields"

$$^{-1}B^0_{D-2} = {}^{-1}\bar{b}^0_{D-2} + \theta \ {}^0 b^0_{D-2} \ , \quad {}^{-2}\overline{\Phi}^0_0 = {}^{-2}\bar{\phi}^0_0 + \theta \ {}^{-1}\eta^0_0 \ ,$$

corresponding to the transformation rules (4.1). One must impose the anti-selduality condition $P_+ \ {}^{-1}B^0_{D-2} = 0$ if one is interested in the instanton configurations (see (A.6) for the definition of the (anti-)selfduality projectors). An action which fixes local shift supersymmetry may be given by the following supergauge invariant and supersymmetric expression,

---

[5]Recall that the indices p and s in ${}^s\varphi^g_p$ respectively denote the form degree and the SUSY-number. The indice g denotes the ghost number associated to the BRST invariance defined by (3.9) – or (2.2), in the WZ gauge.

written as a superspace integral:

$$S_{\text{inv}} = \text{Tr} \int d^1\theta \ \left(\ ^{-1}B^0_{D-2} \ F(A) + \ ^{-2}\overline{\Phi}^0_0 \ D(A) * \Psi \right)$$

$$= \text{Tr} \int \left(\ ^0b^0_{D-2} \ F(a) + \ ^{-1}\eta^0_0 \ D(a) * (\psi + D(a)\chi) + \ (-1)^{D-1} \ ^{-1}\bar{b}^0_{D-2} \ D(a)\psi \right. \tag{4.2}$$

$$\left. + \ ^{-2}\bar{\phi}^0_0 \Big(- \ D(a) * (D(a)\phi + [\psi, \chi]) + [\psi, *(\psi + D(a)\chi)] \Big) \right),$$

where * is the Hodge duality symbol. In the second term of the first line, we have used the supercurvature component Ψ given in (3.16) instead of ψ for the sake of supergauge invariance. In the WZ gauge, $\chi = 0$, we have

$$S_{\text{inv}} = \text{Tr} \int \left(\ ^0b^0_{D-2} \ F(a) + \ ^{-1}\eta^0_0 \ D(a) * \psi + \ (-1)^{D-1} \ ^{-1}\bar{b}^0_{D-2} \ D(a)\psi \right.$$

$$\left. - \ ^{-2}\bar{\phi}^0_0 \ D(a) * D(a)\phi + [\psi, *\psi] \right). \tag{4.3}$$

Beyond the zero-mode of the connection superfield A due to super-Yang-Mills invariance (3.10), there still remains 0-modes for the $(D-2)$-form superfield $\ ^{-1}B^0_{D-2}$, due to an invariance under local transformations of the form

$$\delta \ ^{-1}B^0_{D-2} = D(A) \ ^{-1}\Sigma^0_{D-3} \ . \tag{4.4}$$

Before describing our way of fixing these zero-modes, let us briefly remind of the scheme introduced in [4].

4.1.2 The Blau-Thompson gauge fixing

The fixing of the zero-modes of $\ ^{-1}B^0_{D-2}$ by the authors of [3, 4, 11] is based on the Batalin-Vilkovisky procedure [12], adapted to the case where gauge invariance is the shift symmetry, with a corresponding system of ghosts for ghosts, antighosts and Lagrange multipliers. The result is rather cumbersome and redundant, but the authors of [4] succeeded to construct a reduced procedure with a minimum number of fields. The reduced procedure amounts to introduce a set of superfields, which we shall denote by

$$^0\overline{\Psi}^0_{D-3} \ , \quad ^{-1}\overline{\Psi}^0_{D-4} \ , \quad ^0\overline{\Psi}^0_{D-5} \ , \quad \cdots \ , \quad ^{-k}\overline{\Psi}^0_0 \ , \quad \text{with } k = \tfrac{1}{2}\left(1 + (-1)^D\right) \ , \tag{4.5}$$

and to add to the action (4.2) the terms

$$S_{\text{BT}} = \text{Tr} \int d^1\theta \ \left(\ ^0\overline{\Psi}^0_{D-3} \ D(A) * \ ^{-1}B^0_{D-2} \ + \ ^{-1}\overline{\Psi}^0_{D-4} \ D(A) * \ ^0\overline{\Psi}^0_{D-3} \right.$$

$$\left. + \ ^0\overline{\Psi}^0_{D-5} \ D(A) * \ ^{-1}\overline{\Psi}^0_{D-4} \ + \cdots + \ ^{-k}\overline{\Psi}^0_0 \ D(A) * \ ^{k-1}\overline{\Psi}^0_1 \right), \tag{4.6}$$

which by construction is a Q-variation. If supplemented by a gauge fixing action for the Yang-Mills supergauge invariance, the fixing of the zero-modes is complete, propagators are well defined and the quantum theory may be calculated. However, the latter is not defined unambiguously. This can be seen, at the perturbative level, from the possible occurrence of gauge invariant and supersymmetric counterterms different from the terms already present in the action. For instance, in $D = 4$ dimensions, possible such counterterms are given by superspace integrals of traces of expressions such as

$$^0\overline{\Psi}_1^0 D(A) \, ^{-1}B_2^0 \, , \qquad ^0\overline{\Psi}_1^0 \, ^{-2}\overline{\Phi}_0^0 * \Psi \, , \qquad ^{-1}\overline{\Psi}_1^0 \, ^{-2}\overline{\Phi}_0^0 * \Phi \, , \qquad ^{-1}B_2^0 \left(\partial_\theta \, ^{-1}B_2^0 + [E, \, ^{-1}B_2^0] \right) \, , \quad \text{etc.} \tag{4.7}$$

This fact may jeopardize the stability of the theory under radiative corrections.

Let us remind that there is an alternative way [2], which may be used in the instanton configuration case, in $D = 4$ dimensions. It consists in adding to the action (4.3), instead of the terms (4.6), a term quadratic in the Lagrange multiplier $^{-1}B_{D-2}^0$:

$$\frac{1}{2}\text{Tr} \int d^1\theta \left(\, ^{-1}B_2^0 \, * \left(\partial_\theta \, ^{-1}B_2^0 + [E, \, ^{-1}B_2^0] \right) \right) \, , \tag{4.8}$$

equal to

$$\frac{1}{2}\text{Tr} \int \left(\, ^0b_2^0 \, * \, ^0b_2^0 + \, ^{-1}\bar{b}_2^0[\, ^{-1}\bar{b}_2^0, \phi] \right) \, , \tag{4.9}$$

in the WZ gauge, and substituting the now auxiliary field $^0b_2^0$ by its equation of motion $^0b_2^0 = P_-F(a)$, where P_- is the anti-selfduality projector defined in (A.6). This leads to the term

$$S_\text{H} = \frac{1}{2}\text{Tr} \int \left(P_-F(a) \right)^2 \, ,$$

as pointed out in [2], thus leading to Witten's original action[6]. This alternative way is analogous to the way leading from a gauge fixing of the Landau type to one of the Feynman type in usual gauge theories. We note that the action $S_\text{inv} + S_\text{H}$ represents a complete gauge fixing, too, since the BF-type gauge invariance is explicitly broken. Moreover, it is stable under the radiative corrections, to the contrary of the action $S_\text{inv} + S_\text{BT}$. However, this alternative procedure appears unsuitable for generalization to higher dimension and higher supersymmetry.

On the other hand, the reduced Blau-Thompson procedure may be easily generalized to higher dimension and higher N_T shift supersymmetry: this has been done in [4] for $D = 3$ and 4, $N_T = 1$ and 2. However the same problem of unstability will persist.

4.1.3   The super-BF gauge fixing

Our proposal is to treat the theory as a supersymmetric theory with supergauge invariance, and to eliminate the zero-modes of the superfield $^{-1}B_{D-2}^0$ by explicitly using the supergauge

---

[6]This point is discussed in [13] together with an argument indicating the equivalence of both versions.

invariance of the type encountered in topological BF theories and fixing it accordingly to the Batalin-Vilkovisky (BV) prescription [12], as in BF theories. Implementing this new gauge invariance within the BRST algebra, we first introduce a ghost $^{-1}B^1_{D-3}$ as well as a series of ghosts for ghost $^{-1}B^g_{D-2-g}$, $g = 2, \ldots, D-2$, and the BRST transformation rules

$$\mathcal{S}\,{}^{-1}B^0_{D-2} = -[C,\,{}^{-1}B^0_{D-2}] - D(A)\,{}^{-1}B^0_{D-3}\;,$$

$$\mathcal{S}\,{}^{-1}B^g_{D-2-g} = -[C,\,{}^{-1}B^g_{D-2-g}] - D(A)\,{}^{-1}B^{g+1}_{D-3-g} \quad (g = 1, \ldots, D-3)\;, \qquad (4.10)$$

$$\mathcal{S}\,{}^{-1}B^{D-2}_0 = -[C,\,{}^{-1}B^{D-2}_0]\;,$$

where we have incorporated the super-Yang-Mills transformations with superghost C. We note that, if the space-time dimension D is greater or equal to 4, these transformations hold only on-shell, namely modulo terms linear in the curvature $F(A)$, the latter being an equation of motion as a consequence of the action (4.2). Indeed, $\mathcal{S}^2 = 0$ when applied to all the fields, except

$$\mathcal{S}^2\,{}^{-1}B^g_{D-2-g} = -\left[F(A),\,{}^{-1}B^{g+2}_{D-4-g}\right] \quad (g = 1, \ldots, D-3\;;\quad D \geq 4)\;. \qquad (4.11)$$

The transformations as written in (4.10) hold in the generic case describing the gauge field configurations of null curvature: $F(a) = 0$. If we are interested in the selfdual configurations in four-dimensional space-time, $P_- F(a) = 0$, the Lagrange multiplier superfield $^{-1}B^0_2$ has to be chosen as an anti-selfdual 2-form:

$$P_+\,{}^{-1}B^0_2 = 0,\;, \qquad (4.12)$$

and the BRST transformations (4.10) must be redefined accordingly:

$$\mathcal{S}\,{}^{-1}B^0_2 = -[C,\,{}^{-1}B^0_4] - P_-\left(D(A)\,{}^{-1}B^0_1\right)\;,$$

$$\mathcal{S}\,{}^{-1}B^1_1 = -[C,\,{}^{-1}B^1_1] - D(A)\,{}^{-1}B^2_0\;, \qquad (4.13)$$

$$\mathcal{S}\,{}^{-1}B^2_0 = -[C,\,{}^{-1}B^2_0]\;.$$

One readily verifies that on-shell nilpotency still holds, $F(A)$ in (4.11) being replaced by $P_-F(A)$, which is now the relevant equation of motion.

The fixing of the gauge invariance (4.4) is completed through the addition of antighost and Lagrange multiplier superfields $^s\overline{C}^{g-1}_p$ and $^s\Pi^g_p$. The ghosts B and antighosts BC form together a Batalin-Vilkovisky triangle, whose upper summit is the superfield $^{-1}B^0_{D-2}$ and bottom line is made of 0-forms:

$$^{-1}B^0_{D-2}$$

$$^0\overline{C}^{-1}_{D-3} \qquad\qquad\qquad {}^{-1}B^1_{D-3}$$

$$^{-1}\overline{C}^0_{D-4} \qquad\qquad {}^0\overline{C}^{-2}_{D-4} \qquad\qquad\qquad {}^{-1}B^2_{D-4}$$

$$^0\overline{C}^{-1}_{D-5} \qquad\qquad {}^{-1}\overline{C}^1_{D-5} \qquad\qquad {}^0\overline{C}^{-3}_{D-5} \qquad\qquad\qquad {}^{-1}B^3_{D-5}$$

$$\cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots$$

The Lagrange multipliers form a smaller triangle corresponding to the antighost subtriangle:

$$^0\Pi^0_{D-3}$$

$$^{-1}\Pi^1_{D-4} \qquad\qquad\qquad ^0\Pi^{-1}_{D-4}$$

$$^{-0}\Pi^0_{D-5} \qquad\qquad ^{-1}\Pi^2_{D-5} \qquad\qquad ^0\Pi^{-2}_{D-5}$$

$$\cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots$$

The set of BRST transformations given by (3.10) for the connection superfields A and E, by (4.10) for $^{-1}B^0_{D-2}$ and its ghosts, is completed by

$$\mathcal{S}\ {}^s\overline{C}^{g-1}_p = {}^s\Pi^g_p\ ,\quad \mathcal{S}\ {}^{-1}\Pi^g_p = 0\ , \tag{4.14}$$

for the antighost and Lagrange multipliers, and finally by

$$\mathcal{S}\ {}^{-2}\overline{\Phi}^0_0 = -[C,\ {}^{-2}\overline{\Phi}^0_0]\ , \tag{4.15}$$

the nilpotency property being preserved. Introducing still the antighost and Lagrange multiplier superfields $\overline{C}$ and Π for fixing super-Yang-Mills gauge invariance, we are ready to write down a complete action. Since the "BF gauge symmetry" algebra is closed only on-shell, one must use the complete Batalin-Vilkovisky setting, including the introduction of the antifields, demand that the action solves the master equation, thereby obtaining an action involving terms quadratic in the ghosts. This has been done in quite generality for the usual BF models [3, 6] and will not be repeated here. We shall only indicate the part, written as a superspace integral, of the action linear in the ghost fields, which may be obtained adding to the invariant action (4.2) a BRST variation:

$$S_{\text{(linear part in the ghosts)}} = S_{\text{inv}}$$

$$-\mathcal{S}\ \text{Tr} \int d^1\theta\ \left(\overline{C}\ d * A +\ {}^0\overline{C}^{-1}_{D-3}\ d *^{-1} B^0_{D-2} +\ {}^0\overline{C}^{-2}_{D-4}\ d *^{-1} B^1_{D-3}\right.$$

$$\left. +\ {}^{-1}\overline{C}^0_{D-4}\ d *^0 \overline{C}^{-1}_{D-3} + \cdots\right)$$

$$= \text{Tr} \int d^1\theta\Big(\ {}^{-1}B^0_{D-2}\ F(A) +\ {}^{-2}\overline{\Phi}^0_0\ D(A) * \Psi + \Pi\ d * A - \overline{C}\ d * \mathcal{S}A$$

$$+\ {}^0\Pi^0_{D-3}\ d *\ {}^{-1}B^0_{D-2} +\ {}^0\Pi^{-1}_{D-4}\ d *\ {}^{-1}B^1_{D-3} +\ {}^{-1}\Pi^1_{D-4}\ d *\ {}^0\overline{C}^{-1}_{D-3}$$

$$-\ {}^0\overline{C}^{-1}_{D-3}\ d * \mathcal{S}\ {}^{-1}B^0_{D-2} -\ {}^0\overline{C}^{-2}_{D-4}\ d * \mathcal{S}\ {}^{-1}B^1_{D-3} -\ {}^{-1}\overline{C}^0_{D-4}\ d * \mathcal{S}\ {}^0\overline{C}^{-1}_{D-3} + \cdots\Big)\ . \tag{4.16}$$

In fact, the dependence in the Lagrange multipliers is exact and completely fixed if one imposes, as it may be done in usual gauge theories [14], the Landau type "gauge conditions"

$$\frac{\delta S}{\delta \Pi} = d * A\ ,\quad \frac{\delta S}{\delta\ {}^0\Pi^0_{D-3}} = d *\ {}^{-1}B^0_{D-2}\ ,$$

$$\frac{\delta S}{\delta\ {}^0\Pi^{-1}_{D-4}} = d *\ {}^{-1}B^1_{D-3}\ ,\quad \frac{\delta S}{\delta\ {}^{-1}\Pi^1_{D-4}} = d *\ {}^0\overline{C}^{-1}_{D-3}\ ,\quad \cdots\ , \tag{4.17}$$

which, being linear, are not subject to renormalization.

For the sake of completeness, let us write the expansions of the various superfields present in this action:

$$
A = a + \theta\psi \ , \quad E = \chi + \theta\phi \ , \quad {}^{-1}B_{D-2-g}^{g} = {}^{-1}\bar{b}_{D-2-g}^{g} + \theta \ {}^{0}b_{D-2-g}^{g} \ ,
$$

$$
{}^{-2}\overline{\Phi}_{0}^{0} = {}^{-2}\bar{\phi}_{0}^{0} + \theta \ {}^{-1}\eta_{0}^{0} \ , \quad \Pi = \pi' + \theta\pi \ , \quad {}^{s}\Pi_{p}^{g} = {}^{s}(\pi')_{p}^{g} + \theta \ {}^{s+1}\pi_{p}^{g} \ , \tag{4.18}
$$

$$
{}^{s}\overline{C}_{p}^{g} = {}^{s}(\bar{c}')_{p}^{g} + \theta \ {}^{s+1}\bar{c}_{p}^{g} \ .
$$

Example 5 – Case $D = 3$

The BRST operator $\mathcal{S}$ is strictly nilpotent, and the complete action reads

$$
S = \mathrm{Tr} \int d^{1}\theta \Big({}^{-1}B_{1}^{0} \ F(A) + {}^{-2}\overline{\Phi}_{0}^{0} \ D(A) * \Psi
$$

$$
+ \Pi \ d * A - \overline{C} \ d * \mathcal{S}A + {}^{0}\Pi_{0}^{0} \ d * {}^{-1}B_{1}^{0} - {}^{0}\overline{C}_{0}^{-1} \ d * \mathcal{S} \ {}^{-1}B_{1}^{0} \Big) \ , \tag{4.19}
$$

which, in component fields, yields (see (3.16) for the θ-expansion of Ψ)

$$
S = \mathrm{Tr} \int \Big({}^{0}b_{1}^{0} \ F(a) + {}^{-1}\eta_{0}^{0} \ D(a) * (\psi + D(a)\chi) - {}^{-1}\bar{b}_{1}^{0} \ D(a)\psi
$$

$$
+ {}^{-2}\bar{\phi}_{0}^{0} \ (\ D(a) * (D(a)\phi - [\psi, \chi]) + [\psi, *(\psi + D(a)\chi)])
$$

$$
+ \pi \ d * a + \pi' \ d * \psi + {}^{1}\pi_{0}^{0} \ d * {}^{-1}\bar{b}_{1}^{0} - {}^{0}(\pi')_{0}^{0} \ d * {}^{0}b_{1}^{0} \tag{4.20}
$$

$$
- \bar{c} \ d * \mathcal{S}a - \bar{c}' \ d * \mathcal{S}\psi - {}^{1}\bar{c}_{0}^{-1} \ d * \mathcal{S} \ {}^{-1}\bar{b}_{1}^{0} + {}^{0}(\bar{c}')_{0}^{-1} \ d * \mathcal{S} \ {}^{0}b_{1}^{0} \Big) \ ,
$$

In the WZ gauge $\chi = 0$, this gives

$$
S = \mathrm{Tr} \int \Big({}^{0}b_{1}^{0} \ F(a) + {}^{-1}\eta_{0}^{0} \ D(a) * \psi - {}^{-1}\bar{b}_{1}^{0} \ D(a)\psi + {}^{-2}\bar{\phi}_{0}^{0} \ (\ D(a) * D(a)\phi + [\psi, *\psi])
$$

$$
+ \pi \ d * a + \pi' \ d * \psi + {}^{1}\pi_{0}^{0} \ d * {}^{-1}\bar{b}_{1}^{0} - {}^{0}(\pi')_{0}^{0} \ d * {}^{0}b_{1}^{0}
$$

$$
- \bar{c} \ d * \mathcal{S}a - \bar{c}' \ d * \mathcal{S}\psi - {}^{1}\bar{c}_{0}^{-1} \ d * \mathcal{S} \ {}^{-1}\bar{b}_{1}^{0} + {}^{0}(\bar{c}')_{0}^{-1} \ d * \mathcal{S} \ {}^{0}b_{1}^{0} \Big) \ ,
$$

$$\tag{4.21}$$

On may observe that the latter action contains the term $\pi' d * \psi$ which, compared with the term ${}^{-1}\eta_{0}^{0} \ D(a) * \psi$, shows that the fields π' and ${}^{-1}\eta_{0}^{0}$ are redundant and the quadratic part of the action, singular. This redundancy is an artifact of having written the action in the WZ gauge, where $\chi = 0$. In the supersymmnetric gauge yielding the action (4.20), the field ${}^{-1}\eta_{0}^{0}$ also couples to χ, and there is therefore no redundancy. When restricting to the WZ gauge, in order to get rid of this redundancy, one has to put $\pi' = 0$, too.

Example 6 – Case $D = 4$

Let us consider the case of a selfdual curvature, defined by the anti-selfduality condition (4.12) on the B-field and the BRST transformations (4.13). The action is

$$S = \text{Tr} \int d^1\theta \Big({}^{-1}B_2^0 \ F(A) + {}^{-2}\overline{\Phi}_0^0 \ D(A) * \Psi$$
$$+ \Pi \ d * A + {}^0\Pi_1^0 \ d * {}^{-1}B_2^0 - {}^{-1}\overline{C}_0^0 \ d * {}^0\Pi_1^0 \Big) + S_{\text{ghost}} \ , \tag{4.22}$$

where S_{ghost} is the part of the action depending on the fields of ghost number $\neq 0$, which we shall not write explicitly. In component fields, in the WZ gauge $\chi = 0$ and[7] $\pi' = 0$, this reads

$$S = \text{Tr} \int \Big({}^0b_2^0 \ F(a) + {}^{-1}\eta_0^0 \ D(a) * \psi + {}^{-1}\bar{b}_2^0 \ D(a)\psi + {}^{-2}\bar{\phi}_0^0 \ (\ D(a) * D(a)\phi + [\psi * \psi])$$

$$+ \pi \ d * a + {}^1\pi_1^0 \ d * {}^{-1}\bar{b}_2^0 \ + {}^0(\pi')_1^0 \ d * {}^0b_2^0$$

$$- {}^0\bar{c}_0^0 \ d * {}^0(\pi')_1^0 \ - {}^{-1}(\bar{c}')_0^0 \ d * {}^1\pi_1^0 \Big) + S_{\text{ghost}} \ . \tag{4.23}$$

We can see from the actions (4.19) and (4.22) given in the two examples above, that the non-ghost part of the action constructed using the "super-BF gauge fixing" procedure coincides, in the WZ gauge, with the action (4.2, 4.6) given by the Blau-Thompson procedure. In $D = 4$ dimensions, for instance, the Blau-Thompson action is given by (2.5) and the super-BF like action by (4.23). They are almost identical, up to changes in the notation:

$$^1\lambda_1 \rightarrow \ ^1\pi_1^0 \ , \quad {}^0\bar{\psi}_1 \rightarrow \ ^0(\pi')_1^0 \ , \quad {}^0\lambda_0 \rightarrow \ ^0\bar{c}_0^0 \ , \quad {}^{-1}\bar{\psi}_0 \rightarrow \ ^{-1}(\bar{c}')_0^0 \ ,$$

and up to the presence of simple derivatives in the latter action instead of covariant derivatives in the former one.

In the latter action the supermultiplets[8] $\{ \ ^0(\pi')_1^0, \ ^1\pi_1^0\}$ and $\{ \ ^{-1}(\bar{c}')_0^0, \ ^0\bar{c}_0^0\}$ appear naturally as Lagrange multipliers and antighosts within the Batalin-Vilkovisky scheme, with couplings fixed uniquely by the gauge conditions (4.17). Hence, due to this and to the gauge invariance of the BF type defined by (4.4), the action (4.22) is uniquely defined – up to an irrelevant renormalization of the superfields ${}^{-1}B_2^0$ and ${}^{-2}\overline{\Phi}_0^0$, thus guarantying an unambiguous quantum extension of the theory. In contrast, $\{ \ ^0(\pi')_1^0, \ ^1\pi_1^0\}$ and $\{ \ ^{-1}(\bar{c}')_0^0, \ ^0\bar{c}_0^0\}$ appear in the Blau-Thompson approach as independent supermultiplets introduced together with their couplings in an ad hoc way, with the consequence that the action (4.7) of [4] is not the most general supersymmetric and gauge invariant one. Indeed, forgetting the BF-type invariance and the character of Lagrange multiplier and antighost of ${}^0\Pi_1^0$ and ${}^{-1}\overline{C}_0^0$, one would have to consider possible (counter)terms involving these fields, such as those given by

---

[7]See the remark at the end of the preceding example.
[8]Denoted in equations (4.6, 4.7) of [4] by $\{V, \ \bar{\psi}\}$ and $\{\bar{\eta}, \ u\}$, respectively.

(4.7) – in the notation of subsection 4.1.2 – which are gauge invariant, supersymmetric and of the same power counting dimension 4 as the action.

Of course, these considerations apply as well to the general case of an arbitrary dimension and also to the models with an arbitrary number of supersymmetry generators considered in next subsection 4.2.

Let us also repeat that the action as originally given by Witten [1] in the 4-dimensional case, would correspond to adding to the action (4.23) the term

$$\frac{1}{2}\text{Tr}\int d^1\theta \left({}^{-1}B_2^0 \, \partial_\theta \, {}^{-1}B_2^0\right) = \frac{1}{2}\text{Tr}\int \left({}^0b_2^0\right)^2 \ , \tag{4.24}$$

and substituting the now auxiliary field ${}^0b_2^0$ by its equation of motion ${}^0b_2^0 = -P_-F(a)$, where P_- is the anti-selfduality projector defined in (A.6), thus leading to the term

$$\frac{1}{2}\text{Tr}\int \left(P_-F(a)\right)^2 \ .$$

This would amount to go from a "gauge fixing" of the Landau type for the local shift symmetry, to one of the Feynman type. However, such a term (4.24) is not allowed in our scheme since it is not invariant under the BF type gauge transformation, as we have discussed above.

4.2   Action for Any N_T

The generalization for any number N_T of supersymmetry generators is straightforward. The θ-expansions of the superfield components A and E_I of the superconnection $\hat{A}$ (3.6) and of the superghost C are given in (3.7, 3.8). Their BRST transformations are given in (3.9, 3.10). The Lagrange multiplier superfields, associated to the zero curvature (or selfduality) condition and to the fixing of the zero mode of ψ_I, read ${}^{-N_T}B_{D-2}^0$ and ${}^{-N_T-1}(\overline{\Phi}^I)_0^0$, respectively. The supersymmetric and supergauge invariant action is given by

$$S_{\text{inv}} = \text{Tr}\int d^{N_T}\theta \ \left({}^{-N_T}B_{D-2}^0 \, F(A) + {}^{-N_T-1}(\overline{\Phi}^I)_0^0 \, D(A) * \Psi_I\right) \ , \tag{4.25}$$

whith the supercurvature components $F(A)$ and Ψ_I defined by (3.12). We shall not spell out this expression, nor the following ones, in components. The ghosts and ghosts for ghost of ${}^{-N_T}B_{D-2}^0$ are shown together with their antighosts in the BV triangle

$$\begin{array}{ccccccc}
 & & & {}^{-N_T}B_{D-2}^0 & & & \\[2ex]
 & & {}^0\overline{C}_{D-3}^{-1} & & {}^{-N_T}B_{D-3}^1 & & \\[2ex]
 & {}^{-N_T}\overline{C}_{D-4}^0 & & {}^0\overline{C}_{D-4}^{-2} & & {}^{-N_T}B_{D-4}^2 & \\[2ex]
 & \cdots & \cdots & & \cdots & & \cdots
\end{array}$$

and the corresponding Lagrange multipliers in the triangle

$$^0\Pi_{D-3}^0$$

$$^{-N_T}\Pi_{D-4}^1 \qquad\qquad ^0\Pi_{D-4}^{-1}$$

$$\cdots \qquad\qquad \cdots \qquad\qquad \cdots$$

The BRST transformations (4.14, 4.15) hold, and the total action reads, as much as its linear part in the ghost fields is concerned:

$$S_{\text{(linear part in the ghosts)}} = \text{Tr} \int d^{N_T}\theta \left({}^{-N_T}B_{D-2}^0 \, F(A) + {}^{-N_T-1}(\overline{\Phi}^I)_0^0 \, D(A) * \Psi_I + \Pi \, d * A \right.$$

$$- \overline{C} \, d * \mathcal{S}A + {}^0\Pi_{D-3}^0 \, d * {}^{-N_T}B_{D-2}^0 + {}^0\Pi_{D-4}^{-1} \, d * {}^{-N_T}B_{D-3}^1 + {}^{-N_T}\Pi_{D-4}^1 \, d * {}^0\overline{C}_{D-3}^{-1}$$

$$\left. - {}^0\overline{C}_{D-3}^{-1} \, d * \mathcal{S} \, {}^{-N_T}B_{D-2}^0 - {}^0\overline{C}_{D-4}^{-2} \, d * \mathcal{S} \, {}^{-N_T}B_{D-3}^1 - {}^{-N_T}\overline{C}_{D-4}^0 \, d * \mathcal{S} \, {}^0\overline{C}_{D-3}^{-1} + \cdots \right) , \tag{4.26}$$

with $F(A)$ and Ψ_I given by (3.20). The couplings of the Lagrange multipliers are still defined by the gauge conditions (4.17), with the obvious SUSY-number substitution $-1 \to -N_T$ in due place.

Example 7 – Case $N_T = 2$, $D = 3$

The complete action is

$$S = \text{Tr} \int d^2\theta \left({}^{-2}B_1^0 \, F(A) + {}^{-3}(\overline{\Phi}^I)_0^0 \, D(A) * \Psi_I \right.$$
$$\left. + \Pi \, d * A - \overline{C} \, d * \mathcal{S}A + {}^0\Pi_0^0 \, d * {}^{-2}B_1^0 - {}^0\overline{C}_0^{-1} \, d * \mathcal{S} \, {}^{-2}B_1^0 \right) . \tag{4.27}$$

We can write this action in components, in the WZ gauge $\chi_I = 0$, $\phi_{IJ} - \phi_{JI} = 0$, using the θ-expansions defined in (3.17) and by

$$^{-1}B_{D-2-g}^g = \bar{b}(x) + \theta^I b_I(x) + \frac{1}{2}\theta^2 b(x) , \qquad {}^{-2}(\overline{\Phi}^I)_0^0 = \bar{\phi}^I + \theta^J \bar{\phi}_J^I + \frac{1}{2}\theta^2 \bar{\phi}_F^I ,$$

$$\Pi = \pi + \theta^I \pi_I + \frac{1}{2}\theta^2 \pi_F , \qquad\qquad {}^0\Pi_0^0 = \pi' + \theta^I \pi_I' + \frac{1}{2}\theta^2 \pi_F' , \tag{4.28}$$

The result is, restricted to the quadratic terms:

$$S_{\text{quadr}} = -\text{Tr} \int \left(bf(a) - \epsilon^{IJ} b_I d\psi_J + \bar{b}d\alpha + \bar{\phi}_F^I d * \psi_I + \epsilon^{JK} \bar{\phi}_J^I d * (\epsilon_{IJ}\alpha + d\phi_{IJ}) + \bar{\phi}^I d * d\eta_I \right.$$

$$\left. + \pi_F d * a + \epsilon^{IJ} \pi_I d * \psi_J + \pi d * \alpha + \pi_F' d * \bar{b} + \epsilon^{IJ} \pi_I' d * b_J + \pi' d * b \right) . \tag{4.29}$$

As in the $N_T = 1$ case, one has redundancy in some of the fields, which must be eliminated by putting $\pi = \pi_I = 0$.

One can see that this action – like in the examples 5 and 6 – also corresponds to an action written by Blau and Thompson (eq. (4.5) of [4]).

5 Examples of Observables

It has been shown in [7] that all the observables for $N_T = 1$, defined as BRST cohomology classes of supersymmetric field polynomials, are given from the Chern classes associated to the superconnection $\hat{A}$ (3.6), and that the result is equivalent to the result of Witten given in subsection 2.2. We shall give here the generalization for any value of N_T, however without proving that this still gives the complete set of observables [15]. The observables are completely determined from the general solution of the superdescent equations

$$\mathcal{S}\hat{\Omega}_D + \hat{d}\hat{\Omega}_{D-1}^1 = 0 \ , \quad \mathcal{S}\hat{\Omega}_{D-1}^1 + \hat{d}\hat{\Omega}_{D-2}^2 = 0 \ , \quad \cdots \ , \quad \mathcal{S}\hat{\Omega}_0^D = 0 \ . \tag{5.1}$$

where $\hat{\Omega}_D(x, \theta)$ are superforms of ghost number 0 and superform degree D which are nontrivial elements of the cohomology $H(\mathcal{S}|\hat{d})$ of $\mathcal{S}$ modulo $\hat{d}$ in the space of the superforms,

$$\mathcal{S}\hat{\Omega} = 0 \quad (\text{modulo } \hat{d}) \ , \quad \text{but} \quad \hat{\Omega} \neq \mathcal{S}\hat{\Phi} \quad (\text{modulo } \hat{d}) \ .$$

Expanding $Q^{N_T}\hat{\Omega}_D = (\partial_\theta)^{N_T}\hat{\Omega}_D$ according to the space-time form degree p:

$$Q^{N_T}\hat{\Omega}_D = \sum_{p=0}^{D} w_{p; I_1 \ldots I_{D-p}} d\theta^{I_1} \cdots d\theta^{I_{D-p}} \ , \tag{5.2}$$

one identifies the space-time forms w_p as the desired solutions. Indeed,

$$\mathcal{S}w_p(x) = 0 \ (\text{modulo } d) \quad (n \geq 1) \ , \quad Qw_0(x) = 0 \ ,$$

which follows from applying the operator Q^{N_T} to the first of the superdescent equations (5.1), and using the identities $Q^{N_T}\hat{d} = Q^{N_T}d = (-1)^{N_T}d\,Q^{N_T}$, which are direct consequences of the definitions.

The general result for (5.1) is [7]

$$\hat{\Omega}_D = \theta_{r_1}^{\mathrm{CS}}(\hat{A})f_{r_2}(\hat{F}) \cdots f_{r_L}(\hat{F}) \ , \quad \text{with} \quad D = 2\sum_{i=1}^{L} m_{r_i} - 1 \ , \quad L \geq 1 \ , \tag{5.3}$$

where $f_r(\hat{F})$ is the supercurvature invariant of degree m_r in $\hat{F}$ corresponding to the gauge group Casimir operator of degree m_r, and $\theta_r^{\mathrm{CS}}(\hat{A})$ is the associated super-Chern-Simons form:

$$\hat{d}\theta_r^{\mathrm{CS}}(\hat{A}) = f_r(\hat{F}) \ . \tag{5.4}$$

We note that the superform degree of the solution (5.3) is odd.

Example 8 – Maximum degree $D = 3$

The superdescent equations read as

$$\mathcal{S}\hat{\Omega}_3 + \hat{d}\hat{\Omega}_2^1 = 0 \ , \quad \mathcal{S}\hat{\Omega}_2^1 + \hat{d}\hat{\Omega}_1^2 = 0 \ , \quad \mathcal{S}\hat{\Omega}_1^2 + \hat{d}\hat{\Omega}_0^3 = 0 \ , \quad \mathcal{S}\hat{\Omega}_0^3 = 0 \ .$$

The unique nontrivial solution is

$$\hat{\Omega}_3 = \text{Tr}\left(\hat{A}\hat{d}\hat{A} + \tfrac{2}{3}\hat{A}^3\right) , \quad \hat{\Omega}_2^1 = \text{Tr}\left(\hat{A}\hat{d}C\right) , \quad \hat{\Omega}_1^2 = \text{Tr}\left(C\hat{d}C\right) , \quad \hat{\Omega}_0^3 = -\tfrac{1}{3}\,\text{Tr}\,C^3 .$$

Note that $\hat{\Omega}_3$ is the Chern-Simons superform associated to the quadratic Casimir operator of the gauge group. Following (5.1) we get, for $N_T = 1$

$$
\begin{aligned}
w_0 &= \text{Tr}\left(\phi^2 + 2\phi\chi^2\right) , & w_1 &= 2\text{Tr}\left(\psi\phi + \psi\chi^2 + \phi D(a)\chi\right) , \\
w_2 &= \text{Tr}\left(\psi^2 + 2\phi F(a) + 2\psi D(a)a\chi\right) , & w_3 &= 2\text{Tr}\left(\psi F(a)\right) .
\end{aligned}
\tag{5.5}
$$

The observables are the integrals of these forms (and of $\text{Tr}\,F(a)^2$) on closed submanifolds of appropriate dimension.

In the Wess-Zumino gauge $\chi = 0$:

$$w_0 = \text{Tr}\left(\phi^2\right) , \quad w_1 = 2\text{Tr}\left(\psi\phi\right) , \quad w_2 = \text{Tr}\left(2\phi F(a) + \psi^2\right) , \quad w_3 = 2\text{Tr}\left(\psi F(a)\right)$$

which corresponds to Witten's result up to total derivatives.

For $N_T = 2$ we obtain (in the WZ gauge $\chi_I = 0$, $\phi_{[IJ]} = 0$):

$$
\begin{aligned}
w_0 &= 2\text{Tr}\,\eta_{(I}\phi_{JK)} , \\
w_1 &= \text{Tr}\left(2\alpha\phi_{(IJ)} + 2\psi_{(I}\eta_{J)} + \epsilon^{KL}\phi_{(I|K}D(a)\phi_{L|J)}\right) , \\
w_2 &= 2\text{Tr}\left(\alpha\psi_I + F(a)\eta_I + \epsilon^{JK}\phi_{IJ}D(a)\psi_K\right) , \\
w_3 &= \text{Tr}\left(2\alpha F(a) + \epsilon^{IJ}\psi_I D(a)\psi_J\right).
\end{aligned}
$$

6   Absence of Radiative Corrections

The Feynman rules in the general case are deduced from the action (4.26). It is useful to work directly in superspace. The nonzero superpropagadores are

$$
\begin{aligned}
&\langle A(1),\ ^{-N_T}B_{D-2}^0(2)\rangle , \quad \langle A(1), \Pi(2)\rangle , \quad \langle\overline{C}(1), C(2)\rangle , \\
&\langle E_I(1),\ ^{-N_T-1}(\overline{\Phi}^I)_0^0(2)\rangle , \quad \langle E_I(1), \Pi(2)\rangle , \\
&\langle\ ^{-N_T}B_{D-g-2}^g(1),\ ^0\Pi_{D-g-3}^{-g}(2)\rangle , \quad \langle\ ^{-N_T}B_{D-g-2}^g(1),\ ^0\overline{C}_{D-g-2}^{-g}(2)\rangle \quad (g \geq 0) , \\
&\langle\ ^0\overline{C}_{D-g-2}^{-g}(1),\ ^{-N_T}\Pi_{D-g-3}^g(2)\rangle \quad (g \geq 1) , \\
&\langle\ ^{-N_T}\overline{C}_{D-g-4}^g(1),\ ^0\Pi_{D-g-5}^{-g}(2)\rangle , \quad \langle\ ^{-N_T}\overline{C}_{D-g-4}^g(1),\ ^0\Pi_{D-g-3}^{-g}(2)\rangle \quad (g \geq 0) , \\
&\langle\ ^0\overline{C}_{D-g-4}^{-g}(1),\ ^{-N_T}\Pi_{D-g-5}^g(2)\rangle , \quad \langle\ ^0\overline{C}_{D-g-4}^{-g}(1),\ ^{-N_T}\Pi_{D-g-3}^g(2)\rangle \quad (g \geq 0) , \\
&\text{etc. } ,
\end{aligned}
\tag{6.1}
$$

where we are using the notation $\varphi(n)$ for $\varphi(x_n, \theta_n)$. With one irrelevant exception shown hereafter, all these propagators have as factor a θ-space δ-function $(\theta_1 - \theta_2)^{N_T}$. For instance, the first one reads

$$\left\langle A_\mu(1), \ ^{-N_T} B^0_{\nu_1 \cdots \nu_{D-2}}(2) \right\rangle \sim \Delta^{-1} \, \epsilon_{\mu \nu_1 \cdots \nu_{D-2} \rho} \partial^\rho \delta(1,2) \ , \tag{6.2}$$

up to some numerical factor, where Δ^{-1} is the inverse of the Laplace operator $\Delta = *d * d + d * d*$, and

$$\delta(1,2) = \delta^D(x_1 - x_2) \frac{(-1)^{N_T+1}}{N_T!} (\theta_1 - \theta_2)^{N_T}$$

is the (D, N_T)-superspace Dirac distribution. The exception is the propagator

$$\langle E_I(1), \ \Pi(2) \rangle \sim \Delta^{-1} \, \frac{\partial}{\partial \theta_1^I} \delta(1,2) \ , \tag{6.3}$$

which is of degree $N_T - 1$ in $\theta_1 - \theta_2$. However the latter does not contribute to any 1-particle irreducible (1PI) graph since the Lagrange multiplier superfield Π has no interaction in virtue of the gauge conditions (4.17).

Now, repeating a well known argument of superspace diagrammatic [10, 16], we observe that, since all contributing propagators have a factor $(\theta_m - \theta_n)^{N_T}$, the integrant of a nontrivial 1PI graph with N vertices will be homogeneous of degree $N \times N_T$ in the differences $\theta_m^I - \theta_n^I$. On the other hand, having $N \times N_T$ independent Grassmann coordinates, we can only form $(N-1) \times N_T$ independent differences. Hence, due to the anticommutativity of the θ's, the integrant will vanish. We thus conclude to the complete absence of radiative corrections.

7 Conclusion

We have developed a general scheme, based on superspace formalism, which allows for a systematic construction of topological Yang-Mills theories for arbitrary numbers of shift supersymmetry generators and space-time dimensions. The main advantage of this scheme, beyond its systematic character, is that it leads to an unambiguous determination of the respective actions, thanks to the introduction of a BF theory type supergauge invariance, which has been fixed accordingly to the Batalin-Vilkovisky prescriptions. Moreover, the ultraviolet finiteness – in fact the absence of radiative corrections – follows, in the supersymmetric gauge fixing we have chosen, directly from the superspace Feynman rules.

Appendices. Notations and Conventions

A Differencial calculus

Here, "space-time" is an arbitray D-dimensional smooth manifold, equipped with a Riemannian background metric $(g_{\mu\nu})$, of determinant $g > 0$. Space-time objects are differential forms such as $a = a_\mu dx^\mu$, etc. We shall call an object even or bosonic (respectively, odd or fermionic) if it obeys to commutation (respectively, anticommutation) relations.

The bracket $[\cdot , \cdot]$ in general denotes the graded bracket

$$[X, Y] = XY + YX \quad \text{if both } X \text{ and } Y \text{ are odd,}$$
$$[X, Y] = XY - YX \quad \text{otherwise.} \tag{A.1}$$

The fields (forms, superfields, etc.) appearing in this paper are all taken in the Lie algebra of the gauge group G, which we assume to be compact. A field φ is then a matrix $\varphi^a \tau_a$, where the generators τ_a obey the Lie algebra commutation relations and trace property

$$[\tau_a , \tau_b] = f_{ab}{}^c \tau_c , \quad \text{Tr}\, \tau_a \tau_b = 2\delta_{ab} . \tag{A.2}$$

The *Hodge dual* of a p-form ω is the $(D - p)$-form $*\omega$ defined by [17]

$$*\omega = \frac{1}{(D-p)!} \tilde{\omega}_{\mu_1 \dots \mu_{D-p}} dx^{\mu_1} \dots dx^{\mu_{d-p}}$$
$$\text{where} \quad \tilde{\omega}_{\mu_1 \dots \mu_{D-p}} = \frac{1}{p!} \frac{1}{\sqrt{g}} \epsilon_{\mu_1 \dots \mu_D} \omega^{\mu_{D-p+1} \dots \mu_D} . \tag{A.3}$$

Here and elsewhere in the text, the wedge product symbol has been omitted. Moreover, the background metric $(g_{\mu\nu})$, as well as the totally antisymmetric tensor of Levi-Civita:

$$\epsilon_{\mu_1 \dots \mu_D} = g_{\mu_1 \nu_1} \cdots g_{\mu_D \nu_D} \epsilon^{\nu_1 \dots \nu_D} , \quad \epsilon^{1 \dots D} = 1 , \quad \epsilon_{1 \dots D} = g . \tag{A.4}$$

The following formulas are quite useful [17]:

$$* * \omega_p = (-1)^{p(D-p)} \omega_p \quad , \quad \omega_p * \phi_p = \phi_p * \omega_p . \tag{A.5}$$

Since the Hodge star operator maps a form of degree p to a form of total degree $D - p$, it represents an even operator if the space-time dimension D is even and an odd operator otherwise. For $D = 4$, a *selfdual* or *anti-selfdual* 2-form ω_2 is defined by the condition $*\omega_2 = \pm\omega_2$. Projectors on selfdual or anti-selfdual 2-forms are given by

$$P_\pm = \frac{1}{2}(1 \pm *) . \tag{A.6}$$

B N_T- supersymmetry and superspace

(D, N_T)-superspace bosonic coordinates are denoted by x^μ, $\mu = 0, \ldots, D-1$, the fermionic (Grassmann, or anticommuting) coordinates being denoted by θ^I, $I-1, \ldots, N_T$. The N_T supersymmetry generators Q_I are represented on superfields $F(x, \theta)$ by

$$Q_I F = \partial_I F \equiv \frac{\partial}{\partial \theta^I} F \ ,$$

where, by definition, $\partial_K \theta^J = \delta_K^J$. Further conventions and properties about the θ-coordinates are the following:

$$\theta^{N_T} = \epsilon_{I_1 \cdots I_{N_T}} \theta^{I_1} \cdots \theta^{I_{N_T}} = N_T! \, \theta^1 \cdots \theta^{N_T} \ ,$$

$$(\partial_\theta)^{N_T} = \epsilon^{I_1 \cdots I_{N_T}} \partial_{I_1} \cdots \partial_{I_{N_T}} = N_T! \, \partial_1 \cdots \partial_{N_T} \ ,$$

$$(\partial_\theta)^{N_T} \theta^{N_T} = -(N_T!)^2 \ ,$$

where $\epsilon^{I_1 \cdots I_{N_T}}$ is the completely antisymmetric tensor of rank N_T, with the conventions

$$\epsilon^{1 \cdots N_T} = 1 \ , \quad \epsilon_{I_1 \cdots I_{N_T}} = (-1)^{N_T + 1} \, \epsilon^{I_1 \cdots I_{N_T}} \ .$$

One may define the conserved supersymmetry number – *SUSY number* – attributing the value 1 to the generators Q_I, hence -1 to the θ-cordinates. The SUSY number of each field component is then deduced from the SUSY number given to each superfield.

Superspace integration of a superfield form $\Omega_p(x, \theta)$ is defined by integrals

$$\int d^{N_T} \theta \; \Omega_p(x, \theta) = \int_{M_p} \int d^{N_T} \theta \; \Omega_p(x, \theta) \ ,$$

where the x-space integral is made on some p-dimensional (sub)manifold M_p, and the θ-space integral is the Berezin integral defined by

$$\int d^{N_T} \theta \; \cdots = -\frac{1}{(N_T!)^2} (\partial_\theta)^{N_T} \cdots \ , \quad \text{such that} \quad \int d^{N_T} \theta \; \theta^{N_T} = 1 \ .$$

In the specal case of $N_T = 2$, the antisymmetric tensors ϵ^{IJ} and ϵ^{IJ} may be used for raising and lowering the indices:

$$\theta_I = \epsilon_{IJ} \theta^J \ , \quad \theta^I = \epsilon^{IJ} \theta_J \ , \quad \epsilon_{IJ} = -\epsilon^{IJ} \ , \quad \epsilon^{12} = 1 \ , \quad \epsilon^{IJ} \epsilon_{JK} = \delta_K^I \ ,$$

and one has the useful formulas

$$\theta^2 = \theta^I \theta_I = -\theta_I \theta^I, \quad \theta^I \theta^J = -\frac{1}{2} \epsilon^{IJ} \theta^2 \ , \quad \theta_I \theta_J = \frac{1}{2} \epsilon_{IJ} \theta^2 \ .$$

$N_T = 1$ and $N_T = 2$ superfields have the conventional expansions

$$\Phi(x, \theta) = \phi(x) + \theta \phi'(x) \quad (N_T = 1) \ ,$$

$$\Phi(x, \theta) = \phi(x) + \theta^I \phi_I(x) + \frac{1}{2} \theta^2 \phi_F \quad (N_T = 2) \ .$$

References

[1] E. Witten, Commun. Math. Phys. 117 (1988) 353; Int. J. Mod. Phys. A 6 (1991) 2775;

[2] J.H. Horne, Nucl. Phys. B 318 (1989) 22 34;

[3] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rep. 209 (1991) 129;

[4] M. Blau and G. Thompson, *Nucl. Phys.* B492 (1997) 545 [hep-th/9612143];

[5] B. Geyer and D. Mulsch, *Nucl. Phys.* B616 (2001) 476 [hep-th/0108042]; *Nucl. Phys.* B662 (2003) 531 [hep-th/0211061].

[6] C. Lucchesi, O. Piguet and K. Sibold, *Nucl. Phys.* B395 (1993) 325 [hep-th/9208047].

[7] J.L. Boldo, C.P. Constantinidis, F. Gieres, M. Lefrançois and O. Piguet, to be published in *Int. J. Mod. Phys.* A () [hep-th/0303053]; *Int. J. Mod. Phys.* A18 (2003) 2119 [hep-th/0303084].

[8] S. Marculescu, *Nucl. Phys. B (Proc.Suppl.)* 56B (1997) 148; H.D. Dahmen, S. Marculescu and T. Portmann, *Nucl. Phys.* B462 (1996) 493 [hep-th/9506162].

[9] S. Donaldson, J. Diff. Geom. 30 (1983) 289, Topology 29 (1990) 257;

[10] S.J. Gate, M.T. Grisaru, M. Roček and W. Siegel, *Superspace*, Frontiers of Physics, v.58 (1983);

[11] D. Birmingham, M. Blau and G. Thompson, *Int. J. Mod. Phys.* A5 (1990) 4721;

[12] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **B 69** (1977) 309; I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **B 102** (1981) 27; I.A. Batalin and G.A. Vilkovisky, Phys. Rev. **D 28** (1983) 2567;

[13] D. Bellisai, F. Fucito, G. Travaglini and A. Tanzini, *J. High Energy Phys.* 0007 (2000) 017 [hep-th/0003272].

[14] O. Piguet and S.P. Sorella, Algebraic Renormalization, Springer-Verlag (1995)

[15] J.L. Boldo, C.P. Constantinidis, J.A. Nogueira and O. Piguet, in preparation;

[16] O. Piguet and K. Sibold, *Renormalized Supersymmetry*, series *"Progress in Physics"*, vol. 12 (Birkhäuser Boston Inc., 1986);

[17] R. A. Bertlmann, *Anomalies in quantum field theory* (Clarendon Press, Oxford 1996);