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Inhomogeneous Multidimensional Cosmologies

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Abstract

Einstein's equations for a $4+n$-dimensional inhomogeneous space-time are presented, and a special family of solutions is exhibited for an arbitrary n. The solutions depend on two arbitrary functions of time. The time development of a particular member of this family is studied. This solution exhibits a singularity at $t = 0$ and dynamical compactification of the n dimensions. It is shown that the behaviour of the system in the 4-dimensional (*i.e.* post-compactification) phase is constrained by the way in which the compactified dimensions are stabilized. The fluid that generates this solution is analyzed by means of the energy conditions.

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I. INTRODUCTION

Over the last two decades, increasing attention has been paid to theories that unify the fundamental interactions in more than three spatial dimensions. The story of this kind of theories started in the 20's, when Kaluza [1] and Klein [2] augmented the dimensionality of space to describe both gravity and electromagnetism as manifestations of geometry, using the degrees of freedom available from the 5-dimensional metric tensor [3]. The idea was renewed in the 60's by deWit [4] who tried to incorporate non-abelian interactions into the scheme. The original idea of Kaluza and Klein turned out to be incomplete for several reasons, but it still pervades in one way or another many of the unifying schemes currently thought to be viable (most notably in the case of string theory. See for instance [5]). However, if we are willing to accept any of these theories in which space has more than three dimensions, we are faced with several questions, particularly on the cosmological side. Perhaps the most obvious one is related to the fact that we live in a 4-dimensional space-time, so every theory formulated in more than 4 dimensions must say something about the fate of the extra dimensions. A convenient working hypothesis would be to assume that they have been compactified up to some small size. From a theoretical point of view, the most satisfactory way to achieve the compactification of the extra dimensions would be the dynamical one. This means that the theory has solutions in which the size of the extra dimensions diminishes as the universe evolves. Solutions of this type have been found for the more symmetric cases both in 4+1 and 4+n dimensions [3], but only a few with some degree of inhomogeneity can be found in the literature, and always for the 4+1 case (see [6] and references therein). Here instead a $4+n$ dimensional model with arbitrary n will be studied. This case may have a paramount importance, as shown by the recent work of Arkani-Hamed *et al* [7], in which the existence of n sub-millimeter dimensions (with $n \geq 2$) yields a new framework for solving the hierarchy problem, which does not rely on supersymmetry or technicolor. The central idea of this scheme is that the existence of these extra dimensions brings quantum gravity to the Tev scale through the relationship between the Planck scales of the $4+n$ dimensional theory and the long-distance 4-dimensional theory. It must be remarked that the extra dimensions are supposed to have a characteristic length of less than a millimeter, in accordance with the lower bound at which gravity has been tested up to date [8]. In this framework, the fields of the Standard Model are localised on a 3-brane in the higher dimensional space. Some of the important consequences of these ideas in phenomenology, astrophysics, and cosmology can be found in [9].. Many papers related to these matters have appeared lately; we mention here only a few. Argyres *et al* [10] have studied the properties of black holes with Schwarszchild radius smaller than the size of the extra dimensions, and concluded that the spectrum of primordial black holes in a 4+n dimensional spacetime differs from the usual one. Moreover, these primordial black holes would provide dark matter candidates and seeds for early galaxy and QSO formation. Mirabelli *et al* [11] have recently analyzed the missing-energy signatures that should be present in high-energy particle collisions due to the radiation of gravitons if gravity is important at TeV scale. They argue that collision experiments provide the strongest present constraint on the size of the extra dimensions. Nath and Yamaguchi [12] have explored the effect of the excitations associated with extra dimensions on the Fermi constant. They give stringent constraints on the compactification radius from current precision determinations of the Fermi constant, of the fine structure constant, an of the mass of the W and the Z bosons.

A salient feature of the model we propose here is its inhomogeneity. This type of models might describe an early phase of the universe, or may be of use on a super-horizon scale, as suggested by chaotic inflation [13]. Besides, the work of Mustapha *et al* [14] indicates that there is no unquestionable observational evidence for spatial homogeneity. This makes worthwhile analysing models that are isotropic but exhibit some degree of inhomogeneity [15].

The aim of this paper is then to show the existence of analytical solutions in inhomogeneous cosmological models in $4+n$ dimensions. Although some exact solutions for the 4+1 dimensional inhomogeneous case have been worked out [6] [16], the case dealt with here has not been studied previously. Due to the complexity of the equations of motion (which have not been displayed before in the literature in the case of an arbitrary n), the inhomogeneity has been restricted to the n internal dimensions. It will be shown here that there exist solutions for which the 4-dimensional spacetime is expanding while the extra dimensions conpactify due to the evolution of the system. Also, some remarks will be made on the dependence of the evolution of the system after the compactification on the stabilization of the extra dimensions. Finally, the matter content of the system in the multidimensional phase will be characterized by the study of the strong and weak energy conditions (SEC and WEC respectively).

II. FIELD EQUATIONS AND SOLUTIONS

The starting point is the $4+n$ dimensional metric, given by

$$ds^2 = -dt^2 + e^{2\lambda(t,r)}(dr^2 + r^2\, d\Omega^2) + e^{2\mu(t,r)}dy^2, \tag{1}$$

where $d\Omega^2$ is the surface element on the 2-sphere, and $dy^2 \equiv \sum_{i=4}^{n} dy_i^2$. For simplicity we will work with a plane 3-space, and we assume a single scale factor for the internal dimensions.

We adopt the following stress-energy tensor for the matter content of the model:

$$T_{\mu\nu} = \mathrm{diag}(\rho, -p_3, -p_3, -p_3, -p_n, ..., -p_n), \tag{2}$$

with p_n the internal pressure. The nonvanishing field equations in this $4+n$ dimensional space-time are then

$$-2\dot{\lambda}' - n\dot{\mu}' + n\dot{\lambda}\mu' - n\dot{\mu}\mu' = 0, \tag{3a}$$

$$2\ddot{\lambda} + 3\dot{\lambda}^2 + 2n\dot{\lambda}\dot{\mu} + n\ddot{\mu} + \frac{n(n+1)}{2}\dot{\mu}^2 - e^{-2\lambda}\left[\frac{2\lambda'}{r} + \frac{2n}{r}\mu' + 2n\lambda'\mu' + \lambda'^2 - \frac{n(1-n)}{2}\mu'^2\right] = -8\pi p_3, \tag{3b}$$

$$2\ddot{\lambda} + 3\dot{\lambda}^2 + 2n\dot{\lambda}\dot{\mu} + n\ddot{\mu} + \frac{n(n+1)}{2}\dot{\mu}^2 - e^{-2\lambda}\left[\lambda'' + \frac{\lambda'}{r} + n\mu'' + \frac{n}{r}\mu' + \frac{n(n+1)}{2}\mu'^2\right] = -8\pi p_3, \tag{3c}$$

$$\begin{aligned}(n-1)\ddot{\mu} + \frac{n(n-1)}{2}\dot{\mu}^2 + 3(n-1)\dot{\lambda}\dot{\mu} + 3\ddot{\lambda} + 6\dot{\lambda}^2 - e^{-2\lambda}\big[(n-1)\lambda'\mu' + \\ (n-1)\mu'' + \frac{n(n-1)}{2}\mu'^2 + \frac{2(n-1)}{r}\mu' + 2\lambda'' + \frac{4}{r}\lambda' + \lambda'^2\big] = -8\pi p_n,\end{aligned} \tag{3d}$$

$$3\dot{\lambda}^2 + \frac{n(n-1)}{2}\dot{\mu}^2 + 3n\dot{\lambda}\dot{\mu} - e^{-2\lambda}\left[2\lambda'' + n\mu'' + \frac{4}{r}\lambda' + n\lambda'\mu' + \frac{n(n+1)}{2}\mu'^2 + \frac{2n}{r}\mu' + \lambda'^2\right] = 8\pi\rho \tag{3e}$$

(as usual, a dot denotes derivative with respect to time, and a prime, with respect to the radial coordinate).

The restriction to the case $\lambda = \lambda(t)$ gives the following equations of motion:

$$-\dot{\mu}' + \dot{\lambda}\mu' - \dot{\mu}\mu' = 0, \tag{4a}$$

$$2\ddot{\lambda} + 3\dot{\lambda}^2 + 2n\dot{\lambda}\dot{\mu} + n\ddot{\mu} + \frac{n(n+1)}{2}\dot{\mu}^2 - e^{-2\lambda}\left[\frac{2n}{r}\mu' - \frac{n(1-n)}{2}\mu'^2\right] = -8\pi p_3, \tag{4b}$$

$$2\ddot{\lambda} + 3\dot{\lambda}^2 + 2n\dot{\lambda}\dot{\mu} + n\ddot{\mu} + \frac{n(n+1)}{2}\dot{\mu}^2 - e^{-2\lambda}\left[n\mu'' + \frac{n}{r}\mu' + \frac{n(n+1)}{2}\mu'^2\right] = -8\pi p_3, \tag{4c}$$

$$(n-1)\ddot{\mu} + \frac{n(n-1)}{2}\dot{\mu}^2 + 3(n-1)\dot{\lambda}\dot{\mu} + 3\ddot{\lambda} + 6\dot{\lambda}^2 - e^{-2\lambda}\left[(n-1)\mu'' + \frac{n(n-1)}{2}\mu'^2 + \frac{2(n-1)}{r}\mu'\right] = -8\pi p_n, \tag{4d}$$

$$3\dot{\lambda}^2 + \frac{n(n-1)}{2}\dot{\mu}^2 + 3n\dot{\lambda}\dot{\mu} - e^{-2\lambda}\left[n\mu'' + \frac{n(n+1)}{2}\mu'^2 + \frac{2n}{r}\mu'\right] = 8\pi\rho. \tag{4e}$$

In the case $n = 1$, these equations reduce to the ones given in Chaterjee *et al* [6].

It is easy to show that Eq.(4a) can be rewritten as

$$\mu'' + \mu'^2 + \frac{1}{2}\mu'\Phi(r) = 0, \tag{5}$$

where $\Phi(r)$ is an arbitrary function. Besides, by substracting Eq.(4b) to Eq.(4c) we get

$$\mu'' + \mu'^2 - \frac{1}{r}\mu' = 0. \tag{6}$$

So for the last two equations to be compatible we must choose $\Phi(r) = -\frac{2}{r}$. Eq.(6) is integrable and the result is

$$e^{\mu(t,r)} = \beta(t)r^2 + \gamma(t), \tag{7}$$

where $\beta(t)$ and $\gamma(t)$ are arbitrary functions of time. Now from Eqs.(4a) and (7) we get $\dot{\lambda} = \frac{\dot{\beta}}{\beta}$, which yields

$$e^{\lambda(t)} = \frac{1}{b}\,\beta(t), \tag{8}$$

where b is an arbitrary constant. This solution is a generalization of the one obtained by Chatterjee *et al* for the case $n = 1$ [6]. However, it should be emphasized that due to the presence of an additional term (proportional to $1 - n$) in Eq.(4b) for the case of an arbitrary n, this generalization is by no means trivial.

Let us remark at this point that very little is known about the behaviour of matter at extreme conditions of density and pressure in a multidimensional spacetime. So instead of adopting any particular and arbitrary equation of state, Eqs.(4b), (4d), and (4e) shall be taken as definitions of $p_3(t,r)$, $p_n(t,r)$, and $\rho(t,r)$, respectively. The type of matter requires to achieve dynamical compactification in the case under consideration shall be discussed below.

The model may display several different features according to the explicit form of the functions β and γ. In the following a particular expression for these functions will be chosen, but first certain quantities that will be of interest in the subsequent analysis are listed: the scalar curvature of the $3+n$ space, the Kretschmann scalar, the expansion scalar, and the shear scalar.

$$R^{(3+n)} = -9\ddot{\lambda} - 21\dot{\lambda}^2 - 3n\ddot{\mu} - n(2n+1)\dot{\mu}^2 - 12n\dot{\lambda}\dot{\mu} + ne^{-2\lambda}\left[2\mu'' + (1+n)\mu'^2 + \frac{4}{r}\mu'\right], \tag{9}$$

$$\begin{aligned} K = {} & 24\dot{\lambda}^4 + 24\ddot{\lambda}\dot{\lambda}^2 + 12n\dot{\lambda}^2\dot{\mu}^2 + 2n(n+1)\dot{\mu}^4 + 12\ddot{\lambda}^2 + 4n\ddot{\mu}^2 + 8n\ddot{\mu}\dot{\mu}^2 + e^{-2\lambda}\Big[-8n\dot{\lambda}^2\mu'^2 + \\ & 16n\dot{\mu}'\dot{\lambda}\mu' - 16n\dot{\mu}'\dot{\mu}\mu' - \frac{16n}{r}\dot{\lambda}\dot{\mu}\mu' - 8n\dot{\mu}'^2 - 4n(n+1)\dot{\mu}^2\mu'^2 + 8n\mu'^2\dot{\lambda}\mu' - 8n\mu''\dot{\lambda}\dot{\mu}\Big] + \\ & e^{-4\lambda}\left[\frac{8n}{r^2}\mu'^2 + 2n(n+1)\mu'^4 + 4n\mu^{2''} + 8n\mu''\mu'^2\right], \end{aligned} \tag{10}$$

$$\theta = 3\dot{\lambda} + n\dot{\mu}, \tag{11}$$

$$\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu} = (n+4)\dot{\lambda}^2 + n\frac{n(n+1)+9}{9}\dot{\mu}^2 + 2\frac{n(n+1)}{3}\dot{\mu}\dot{\lambda}. \tag{12}$$

The scalar curvature of the $3+n$ space was calculated by means of the expression [17]

$$R^{\mu}{}_{\mu} = R^{(3+n)} + \dot{\theta} + \theta^2 - 2\omega^2 - \dot{v}^{\mu}{}_{;\mu} \tag{13}$$

(It can be seen that in this model the last two terms of this expression are null).

We would like the model to describe a 4-dimensional "macroscopic" expanding space-time plus n "microscopic" dimensions. We must impose in consequence the conditions $\dot{\lambda} > 0$, and $\dot{\mu} < 0$ on the metric functions (the latter must be valid at least for some part of the evolution of the universe). This can be easily achieved by taking advantage of the freedom in the arbitrary functions β and γ. Let us take as an example

$$\beta(t) = -b\ln(1+t^{\alpha}), \qquad\qquad \gamma(t) = a\ln(1+t^{\alpha}) - kt^{\delta}. \tag{14}$$

With this choice, the functions appearing in the metric are

$$e^{\lambda} = \ln(1+t^{\alpha}), \qquad\qquad e^{\mu} = (a-br^2)\ln(1+t^{\alpha}) - kt^{\delta} \tag{15}$$

Note that the constant b is a measure of the inhomogeneity of the model. The positively defined and arbitrary constants a, b, k, α, and δ are to be chosen in a convenient way.

Next, some comments on the behaviour of the scale factors. Due to the election made for $\beta(t)$ and $\gamma(t)$, the big bang is synchronous for both scales. Besides, in order that $\dot{\mu}$ be negative from some point of the evolution onwards, we have to demand that $a - br^2 > 0$. In this case, the scale factor of the three space is monotonically increasing, while the scale factor of the internal dimensions grows until a time $t_{\max}$, given for each r by

$$r^2 = \frac{a}{b} - \frac{k\delta}{b\alpha}\, t_{\max}^{\delta-\alpha}\,(1+t_{\max}^{\alpha}), \tag{16}$$

and compactifies dynamically to zero size at different times t_0 for each r, given by

$$r^2 = \frac{a\ln(1+t_0^{\alpha}) - kt_0^{\delta}}{b\ln(1+t_0^{\alpha})}. \tag{17}$$

We move now to the analysis of the asymptotic behaviour of the model. From the explicit expression of K and R it is seen that both of them diverge as $t \to 0$, the first one as t^{-4}, and the second one as t^{-2}. The existence of the initial singularity is confirmed by the divergence at $t = 0$ of ρ, p_3 and p_n. At $t = t_0$, all the matter functions, the curvature scalars and the shear scalar diverge. It has been argued however [18] that there might exist some sort of stabilization mechanism (probably due to quantum gravity effects) which could prevent the formation of the final singularity [19]. This would allow the evolution of the ordinary 3-space independently of the internal (and microscopic) space, as can be seen from the equation of the conservation of the energy. However, one must be careful at this point. It must be emphasized that no matter which the stabilization mechanism is, the resulting function μ, along with λ, must still be a physically sensible solution of the $4+n$ dimensional equations of motion after the compactification (see [3] and [18]). For instance, if we assume that $\mu = \mu_0 =$ constant after the compactification [18], then the pressure in the internal space must be constant in the post-compactification phase. However, this contradicts Eq.(4d) which implies than p_n is a function of t if $\mu = \mu_0$. Obviously, the ultimate stabilization mechanism (if any) will be determined by the still elusive quantum theory of gravitation. In the meantime, any claim about the post-compactification phase of the system must be in agreement with the physical consequences of the stabilization mechanism chosen. This fact has been frequently overlooked in the literature on this subject [20].

Finally we analyze the behaviour of the fluid in the light of the strong and weak energy conditions. In the case of SEC, the quantity of interest is

$$R_{\mu\nu}v^\mu v^\nu = \frac{8\pi G}{2+n}[(1+n)\rho + 3p_3 + np_n], \tag{18}$$

where $R_{\mu\nu}$ is the $4+n$-dimensional Ricci tensor and v^μ is the velocity of the fluid †. To simplify the calculations, we adopt the particular case in which $\alpha = \delta = 1$. In this case,

$$R_{\mu\nu}v^\mu v^\nu = \frac{(n+3)(a-br^2)\ln(1+t) - 3kt}{(1+t)^2\ln(1+t)}\; e^{-\mu}. \tag{19}$$

The fact that this expression is positive for all the values of t in the interval $(0, t_0)$ implies that the matter satsfies SEC for all t and r in the $4+n$-dimensional phase.

In the case of WEC, the important quantity is the matter density, given by

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†The expression (18) plays an important role in the problem of singularities in higher-dimensional spacetimes. See [21].

$$\rho(r,t) = \frac{[R_1(t)\, r^4 + R_2(t)\, r^2 + R_3(t)]\ln(1+t) + 6k^2t^2}{4(1+t)^2\ln(1+t)}\, e^{-2\mu}, \tag{20}$$

where

$$R_1(t) = 2b^2(n^2+5n+6), \tag{21}$$

$$R_2(t) = -8nb^2(n+2)\, t^2 + 2b[2n^2(k-4b) + n(7k-16b) + 6k]\, t + 4b[n^2(k-a-2b) + 2n(k-5a-2b) - 6a], \tag{22}$$

$$R_3(t) = -12nkbt^3 + 2n[12b(a-k) + k^2(n+2)]\, t^2 + 2[2n^2k(k-a) + n(24ab + k^2 - 7ka - 6bk) - 6ka]\, t + 2[n^2(a-k)^2 + n(12ab + 5a^2 - 4ka - k^2) + 6a^2]. \tag{23}$$

To completely avoid WEC violation, the numerator of Eq.(20) must be positive for all values of the variables r and t. The fulfillment of this conditions depends crucially on the values of the constants a, b and k. However, we expect that for a given t, the $3+n$-dimensional space could be resolved in two types of regions, according to whether the matter in each region satisfies WEC or not. It follows from Eq.(20) that the distribution of these regions will be inhomogeneous.

III. CONCLUSIONS

It was shown that there exists a family of solutions, parameterized by the functions β and γ, for the very complex system of equations corresponding to the case of a $4+n$-dimensional inhomogeneous model. The matter content of the model satisfies, in the pre-compactification phase, the SEC for every value of t and r, and the WEC in some regions of spacetime. A general feature of these solutions is that the time at which dynamical compactification of the extra dimensions begins is different for each value of the r coordinate. The particular example that was analyzed here evolves from a $4+n$-dimensional into a 4-dimensional spacetime, the features of which depend on the stabilization method. It is worth pointing out again that any claim about the evolution after the compactification must be consistent with the adopted compactification scheme and with the higher-dimensional equations of motion.

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