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A model for nonspherical collapse and formation of black holes by emission of neutrinos, strings and gravitational waves†

by

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Abstract

We present a general relativistic exact model for the spacetime external to an axisymmetric bounded distribution of collapsing matter whose rate of mass decrease is a constant which, when vanishing yields Schwarzschild solution. Einstein's field equations actually requires the existence of a timelike shell in the equatorial plane, whose stress-energy tensor is constituted by two independently conserved terms, corresponding to the emission of neutrinos and strings on the shell. An extension of this model can be achieved to include the emission of gravitational waves, that provides a mechanism by which the rate of mass decrease goes to zero with the formation of a Schwarzschild black hole.

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Gravitational collapse is undoubtedly one of the most exciting areas of research in general relativity. The final configuration of the realistic gravitational collapse of a sufficiently massive body is assumed to be represented by black holes, since the cosmic censorship hypothesis (CCH) is considered correct[1]. Such final configurations, according to the no-hair theorem[2], are characterized by only three parameters: mass, angular momentum and charge.

The majority of analytical and numerical studies deals with spherically symmetric collapse, and the main objective is to look for physical counterexamples to the CCH. Nearly spherical collapse was first analyzed, in great detail, by Price[4], who showed that, from the viewpoint of an external observer, all zero-rest-mass integer-spin perturbations (scalar, electromagnetic, gravitational, etc) are radiated away or swallowed by the black hole, leaving only the three aforementioned properties. This conclusion was partially extended by Hartle[3] to include neutrino fields. Indeed, these are the mechanisms to support the no-hair theorem. On the other hand, exact nonspherical models are not so easily found in the literature; the field equations become rather complex to allow almost only numerical integration, besides the own fact that gravitational radiation has to be properly taken into account.

In this work, we exhibit an exact analytical model of nonspherical collapse, to the effect that the spacetime admits only one (rotational) Killing vector field. A remarkable feature of the model is that the field equations demand a flow of matter in the equatorial plane, which flow may be identified with neutrinos and strings ejected outward from the star. Notwithstanding, there are no gravitational waves, and their inclusion is necessary for a more complete and realistic model. We argue, in this way, that the exact model represents an intermediate stage of such a realistic collapse.

Our starting point will be to consider a family of Robinson-Trautman spacetimes[5] whose metric is given by (the units are such that $c = 8\pi G = 1$)

$$ds^2 = \alpha^2(u,r,\theta)d\,u^2 + 2d\,u\,d\,r - K^2(u,\theta)r^2\left(d\,\theta^2 + \sin^2\theta\,d\,\phi^2\right). \tag{1}$$

The components $G_{22} = 0$ and $G_{33} = 0$ of Einstein's equations in vacuum together with $G_{02} = 0 = G_{12}$, allows us to write

$$\alpha^2(u,r,\theta) = \lambda(u,\theta) + \frac{B(u)}{r} + 2\frac{\dot{K}(u,\theta)}{K(u,\theta)}r, \tag{2}$$

where a dot denotes $\partial/\partial u$, and λ and K are arbitrary functions of u and θ, and B is an arbitrary function of u. The remaining vacuum equations $G_{00} = G_{01} = G_{11} = 0$ suggest that we choose the arbitrary function $\lambda(u,\theta)$ as

$$\lambda(u,\theta) = \frac{1}{K^2} - \frac{K_{\theta\theta}}{K^3} + \frac{K_\theta^2}{K^4} - \frac{K_\theta}{K^3}\cot\theta, \tag{3}$$

where a subscript θ denotes now $\partial/\partial\theta$, yielding

$$3B(u)\frac{\dot K}{K} + \dot B(u) + \frac{(\lambda_\theta \sin\theta)_\theta}{2K^2\sin\theta} = 0. \tag{4}$$

Equations (3) and (4) are the basic ones. To integrate these equations we employ the following separation Ansätze: $\lambda(u,\theta) \equiv h(u)g(\theta)$ and $K(u,\theta) \equiv f(u)k(\theta)$, whereby we obtain

$$h(u)f^2(u) = c_1 = \frac{1}{k^2(\theta)g(\theta)}\left[1 - \frac{1}{\sin\theta}\left(\frac{\sin\theta k_\theta(\theta)}{k(\theta)}\right)_\theta\right], \tag{5}$$

$$\left(\frac{3B(u)\dot f(u)}{f(u)} + \dot B(u)\right)\frac{f^2(u)}{h(u)} = c_2 = -\frac{(g_\theta(\theta)\sin\theta)_\theta}{2k^2(\theta)\sin\theta}, \tag{6}$$

with c_1 and c_2 arbitrary separation constants. It will prove extremely useful furthermore to perform a change to new timelike σ and radial null l coordinates given by

$$l^2 := r^2 f^2(u), \quad \text{and} \quad d\sigma := \frac{du}{f(u)}. \tag{7}$$

The line element is thus cast into the form

$$ds^2 = \left[c_1 g(\theta) + \frac{X(\sigma)}{l}\right] d\sigma^2 + 2d\sigma dl - l^2 k^2(\theta)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{8}$$

where we have defined $X(\sigma) := B(u(\sigma))f^3(u(\sigma))$. An exact expression for such a function is easily obtained in terms of the new variable σ by integrating the *lhs* of eq. (6):

$$X(\sigma) = c\sigma - 2m. \tag{9}$$

Here, $c := c_1 c_2$ and m is an integration constant. The function $g(\theta)$ can be rescaled such that we can set $c_1 = 1$ without any loss of generality. In the remainder of the paper we assume that, when $c = 0$, $g(\theta)$ and $k(\theta)$ are equal to 1, so as to eliminate the possibility that the final static configuration be a black hole with a string. We note that, for $c \neq 0$, the spacetime is asymptotically flat (in the physical frame of Ref.[13], the curvature tensor goes to zero as $l \to \infty$) but not asymptotically Minkowskian.

Therefore, a complete characterization of the spacetime is accomplished after solving Eqs. (5) and (6) for the angular part, now written as

$$\begin{cases} 1 - \frac{1}{\sin\theta}\left(\frac{\sin\theta k_\theta(\theta)}{k(\theta)}\right)_\theta = k^2(\theta) g(\theta) \\ \\ \left(g_\theta(\theta)\sin\theta\right)_\theta = -2ck^2(\theta)\sin\theta. \end{cases} \tag{10}$$

Such a task turns out to be very difficult, but crucial physical information can be drawn without an exact integration of the above system. Based on the theory of differential equations[6], with the assumption that the solutions of (10) are analytic in c, one can show that these solutions are either singular at $\theta = 0$ and/or $\theta = \pi$, for $c \neq 0$. Just to provide a concrete example, we exhibit an approximate solution of the system (10) by imposing c is a very small parameter ($|c| \ll 1$) and the following relations

$$\begin{cases} g(\theta) = 1 + c\Delta(\theta) \\ \\ k^2(\theta) = 1 + cL(\theta) \end{cases} . \tag{11}$$

A convenient solution is

$$\begin{cases} \Delta_\pm(\theta) = -1 + 2\ln(1 \pm \cos\theta) \\ \\ L_\pm(\theta) = -2\ln(1 \pm \cos\theta) \end{cases} , \tag{12}$$

where the + solution is regular at $\theta = 0$, and the – solution is regular at $\theta = \pi$. Notice that, if $c = 0$, both solutions reduce to the Schwarzschild one.

Let us consider now the exact solution $\{g(\theta), k(\theta)\}$ singular at $\theta = 0$, say. Due to the symmetry of (10) under the change $\theta \to \pi - \theta$, the solution $\{g(\pi - \theta), k(\pi - \theta)\}$ is singular at $\theta = \pi$. In order to get rid of the undesirable singularities at $\theta = 0$ or $\theta = \pi$, we use the set $\{g(\theta), k(\theta)\}$ for $\pi/2 \leq \theta \leq \pi$, and $\{g(\pi - \theta), k(\pi - \theta)\}$ for $0 \leq \theta \leq \pi/2$ to cover the whole spacetime. This is carried out by matching both sets at the equatorial plane $\theta = \pi/2$, since they are continuous there. Noticing that the first derivatives of the metric are not continuous at $\theta = \pi/2$, a timelike shell must therefore be present at the equatorial plane. Applying the formalism developed by Israel,[7], the surface stress-energy tensor of the shell is given by

$$\hat{S}^{ab} = \frac{E(c)}{l\tilde{\alpha}^2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{G(c)}{l}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{13}$$

where $\hat{S}^{ab}$ is expressed in a convenient local triad basis,[8] $\tilde{\alpha}^2 := g(\pi/2) - 2m/l + c\sigma/l$, $E(c) := \frac{-[g_\theta]}{2k(\pi/2)}$ and $G(c) := \frac{[k_\theta]}{2k^2(\pi/2)}$, with $[A] := A^+(\pi/2) - A^-(\pi/2)$, the jump across the shell of a given quantity A. In the case of the above approximate solution, $E(c)$ and $G(c)$ are given by $E(c) \sim G(c) \sim 2c$. From Eq. (8) we see that the mass function[11] of the distribution $M(\sigma) = \frac{2m - c\sigma}{g(\theta)}$ decreases if $c > 0$. We have adequately splitted the surface stress-energy tensor in two parts so that the first one represents a null fluid[9]. In addition, both parts are separately conserved. Thus, we have a good indication that the matter content of the shell may be physically meaningful.

Before going through with the details concerning the nature of the shell, an important geometrical point is worth calling attention to. The apparent horizon is the outermost trapped surface of a given matter distribution, i. e., the three-surface in which outgoing null rays are momentarily stationary[10]. Such a surface is described by

$$\alpha^2 = 0. \tag{14}$$

For $c > 0$ the area of the apparent horizon decreases, whereas, for $c < 0$, it increases. Then, the former situation means that there is an outflux of matter taking place in the equatorial plane. In this way, our model is to be understood as a nonspherically symmetric collapse where matter of some kind is being ejected on the plane $\theta = \pi/2$. If, at some fixed time σ_c, the parameter c is taken equal to zero, the spacetime described by (8) becomes a black hole of mass $m - c\sigma_c/2$. The formation of this black hole is independent of the rate of mass ejection determined by c. For $c = 0$, the apparent horizon degenerates into the event horizon of the black hole. However, if we consider an interior solution matched to this spacetime on a surface beyond the apparent horizon, the formation of the black hole will now depend on the relative rate of mass decrease to the decrease of the radius of the matching surface.

We shall now prove that the matter distribution described by the first term of the *rhs* of (13) may be modelled by a radial flux of neutrinos ejected on the shell. To this end, let us examine the dynamics of massless neutrinos in the (1+3)-spacetime and also in the (1+2)-shell, which are basically distinct. Neutrinos in interaction with the gravitational field are described by spinor fields in the curved spacetime via the classic prescription of Brill and Wheeler[12]. In a convenient local tetrad basis, Dirac's equation for neutrinos is expressed as[13][14]

$$- i\gamma^A \left(e^{\alpha}_{(A)} \partial_\alpha - \Gamma_A \right) = 0, \tag{15}$$

where the Γ_A are the Fock-Ivanenko coefficients associated to the tetrad field. We restrict our considerations to radial neutrinos only, defined by

$$\gamma^0 \psi = \gamma^1 \psi, \tag{16}$$

such that the null four-current $J^A := \bar{\psi}\gamma^A\psi$ has components

$$J^A = \psi^\dagger \psi \, (1, 1, 0, 0). \tag{17}$$

It is straightforward to check that Dirac's equation (15) for radial neutrinos (16) has no solution, even as test particles. However, in the (1+2)-spacetime of the shell, radial neutrinos are admissible and generate the first part of the stress-energy of the shell. In the triad basis of Ref.[8], the general solution for these neutrinos is given by the 2-spinors[15]

$$\psi = \frac{1}{\sqrt{l\tilde{\alpha}}} \begin{pmatrix} -ib(\sigma) \\ b(\sigma) \end{pmatrix}, \tag{18}$$

where $b(\sigma)$ is an arbitrary complex function. The corresponding surface stress-energy tensor,

$${}_N\hat{T}_{ab} = i \left[\bar{\psi}\gamma_{(a}\nabla_{b)}\psi - \nabla_{(a}\bar{\psi}\gamma_{b)}\psi \right], \tag{19}$$

associated to (18), reads

$${}_N\hat{T}^{ab} = \frac{2i}{\tilde{\alpha}} \left(\psi^\dagger \dot{\psi} - \dot{\psi}^\dagger \psi \right) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

The above tensor models the first part of $\hat{S}^{ab}$ provided we identify

$$E(c) \equiv -4\, i \left(\dot{b}^* b - b^* \dot{b}\right). \tag{21}$$

As expected, this surface stress-energy tensor is independently conserved, so that these neutrinos have no interaction with the remaining part of $\hat{S}^{ab}$.

The second term appearing in the *rhs* of (13) is very similar to some kind of perfect fluid in which the pressure is negative (tension). Let us consider the following surface stress-energy tensor[16]

$$_S\hat{T}^{ab} = \rho_S \frac{\hat{\Sigma}^{ac}\hat{\Sigma}_c^b}{\sqrt{-\gamma}}, \tag{22}$$

that represents a gas of strings with energy density ρ_S. The skewsymmetric tensor $\hat{\Sigma}^{ab}$ represents the kinematics of the gas of strings and must satisfy the normalization condition $\hat{\Sigma}^{ab}\,\hat{\Sigma}_{ab} = 2\gamma$. As a consequence, the surface stress-energy tensor (22) assumes the form

$$_S\hat{T}^{ab} = \rho_S \sqrt{-\gamma} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

where, due to the symmetry of the shell, we have assumed that $\hat{\Sigma}^{01}$ is the only nonvanishing component of the tensor $\hat{\Sigma}^{ab}$. Thus, from the normalization condition and taking into account the conservation law,

$$\nabla_\mu \left(\rho_S\, \Sigma^{\mu\nu}\right) = 0, \tag{24}$$

we get

$$\sqrt{-\gamma}\, \rho_S = \frac{const}{l}. \tag{25}$$

Thus, to obtain the expression we want, it is necessary to identify the above constant to $G(c)$, whereupon the second term of the *rhs* of (13) stands for radial strings. The local triad defines a frame which has radial acceleration $\hat{a}^c = \left(0, \frac{(\tilde{\alpha}^2)^{\cdot}}{2\tilde{\alpha}^3} + \frac{(\tilde{\alpha}^2)'}{2\tilde{\alpha}}, 0\right)$, so that, if $c > 0$, the strings always have a positive radial acceleration.

The limitation of our model rests on the fact that, as long as the parameter c remains constant, the collapsing configuration goes on ejecting its mass-energy on the equatorial shell and, for sufficiently large retarded time variable σ, the spacetime would become a curved spacetime with a string-like topological defect. Therefore the applicability for the formation of black holes would be restricted to a finite interval of σ. This limitation could be circumvented by considering the emission of gravitational waves, which would amount to altering the linear behavior in σ of the mass function $X(\sigma)/g(\theta)$. This will be the subject of a forthcoming paper, but we may advance the following qualitative arguments for small c. We start from (8) and consider the more general configuration

$$\begin{aligned} ds^2 &= \left[\lambda(\theta) + \epsilon\, W(\sigma,\theta) + \frac{X(\sigma)+\epsilon Z(\sigma)}{l} + \frac{2\,l\,\epsilon}{k(\theta)}\,\frac{\partial Y(\sigma,\theta)}{\partial\sigma}\right] d\sigma^2 + 2d\sigma dl \\ &\quad -l^2\left[k(\theta)+\epsilon Y(\sigma,\theta)\right]^2 d\Omega^2, \end{aligned} \tag{26}$$

where ϵ is a small parameter and $Z(\sigma)$, $W(\sigma,\theta)$ and $Y(\sigma,\theta)$ satisfy two coupled differential equations similar to (3) and (4). If we examine the Weyl curvature of (26) in the physical basis of Ref. [13], we find out that it has an l^{-1}-term if and only if $(\partial^2/\partial\sigma\partial\theta)\,[k(\theta)Y(\sigma,\theta)] \neq 0$. As $l \to \infty$, this term has the expression $\frac{\epsilon}{l}N(\sigma,\theta)$, where $N(\sigma,\theta) := \frac{1}{2k(\theta)\sqrt{\lambda(\theta)}}\frac{\partial^2[k(\theta)Y(\sigma,\theta)]}{\partial\theta\partial\sigma}$, defining a gravitational wave zone. The function $N(\sigma,\theta)$ corresponds to the news function[17] of the gravitational wave field and is responsible for the mass variation due to the emission of gravitational waves. The σ-dependence of N determines the function $Z(\sigma)$, and N can in principle be fixed such that $M(\sigma) := X(\sigma) + \epsilon Z(\sigma)$ have the following behavior: $Z(\sigma) = 0$ for $\sigma_0 \leq \sigma \leq \sigma_1$ corresponding to the phase of pure shell emission (neutrinos and strings); and a subsequent phase of gravitational wave emission for $\sigma_1 \leq \sigma \leq \sigma_2$, when $N(\sigma,\theta) \neq 0$, leading to a third phase $\sigma \geq \sigma_2$ when $X(\sigma) + \epsilon\, Z(\sigma)$ becomes a constant function. This final configuration will correspond to a black hole of mass $m - \frac{c}{2}\,\sigma_2 + \frac{\epsilon}{2}\,Z(\sigma_2)$.

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[8] The following triad field is chosen

$$\begin{aligned} e^{\alpha}_{(0)} &= \left(\tilde{\alpha}^{-1}(\sigma, l), 0, 0, 0\right) \\ e^{\alpha}_{(1)} &= \left(-\tilde{\alpha}^{-1}(\sigma, l), \tilde{\alpha}(\sigma, l), 0, 0\right) \\ e^{\alpha}_{(3)} &= \left(0, 0, 0, \frac{1}{lk(\pi/2)}\right). \end{aligned}$$

with $e^{\alpha}_{(a)}\, e_{\alpha\,(b)} = diag(1, -1, -1)$, a,b=0,1,3.

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[13] The tetrad field $e^{\mu}_{(A)}$ is

$$
\begin{aligned}
e^{\mu}_{(0)} &= \left(\alpha^{-1}(\sigma,l,\theta),0,0,0\right)\\
e^{\mu}_{(1)} &= \left(-\alpha^{-1}(\sigma,l,\theta),\alpha(\sigma,l,\theta),0,0\right)\\
e^{\mu}_{(2)} &= \left(0,0,\frac{1}{lk(\theta)},0\right)\\
e^{\mu}_{(3)} &= \left(0,0,0,\frac{1}{lk(\theta)\sin\theta}\right).
\end{aligned}
$$

This set of vectors is unique in the sense that, by parallel transport along the null geodesics $\partial/\partial l$, it tends to, as $l \to \infty$ and $c = 0$:

$$
\begin{aligned}
e^{\mu}_{(0)} &= (1,0,0,0)\\
e^{\mu}_{(1)} &= (-1,1,0,0)\\
e^{\mu}_{(2)} &= \left(0,0,\frac{1}{l},0\right)\\
e^{\mu}_{(3)} &= \left(0,0,0,\frac{1}{l\sin\theta}\right).
\end{aligned}
$$

[14] γ^A are the constant 4×4 Dirac matrices in the Bjorken-Drell representation.

[15] For the (1+2)-spacetime of the shell, we use the following 2×2 representation for the algebra of Dirac matrices:

$$
\begin{aligned}
\gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\\
\gamma^1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\\
\gamma^3 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\end{aligned}
$$

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