Constructive proposals for QFT based on the crossing property and on lightfront holography

Dedicated to Jacques Bros on the occasion of his 70th birthday

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Abstract

The recent concept of modular localization of wedge algebras suggests two methods of classifying and constructing QFTs, one based on particle-like generators of wedge algebras using on-shell concepts (S-matrix, formfactors. crossing property) and the other using the off-shell simplification of lightfront holography (chiral theories).

The lack of an operator interpretation of the crossing property is a serious obstacle in on-shell constructions. In special cases one can define a "masterfield" whose connected formfactors constitute an auxiliary thermal QFT for which the KMS cyclicity equation is identical to the crossing property of the formfactors of the master field.

Further progress is expected to result from a conceptual understanding of the role of on-shell concepts as particle states and the S-matrix within the holographic lightfront projection.

1 History of the crossing property

The so-called crossing property of the S-matrix and formfactors[1] is a deep and important, but at the same time incompletely understood structure in particle physics. As a result of its inexorable link with analyticity properties in the quantum field theoretic setting of scattering theory, crossing is not a symmetry in the standard sense (of Wigner), even though it is often referred to as "crossing symmetry".

In contrast to the underlying causality principles which are "off-shell", i.e. are formulated in terms of local observables or fields with unrestricted Fourier transforms, the crossing property is "on-shell", that is to say it refers to particle states which are described by wave functions on the forward mass hyperboloid $p^2 = m^2, p^0 \geq 0$. Particle properties are intrinsic to a theory, whereas fields are (point-like [1] or string-like[2][3][4]) "coordinatizations" of local algebras; only local equivalence classes of fields or the local algebras generated by fields are truely "intrinsic"[2]. The use of the notion of "intrinsicness" in local quantum physics (LQP) is reminiscent of the use of "invariant" (as opposed to coordinate-dependent) in geometry; in this analogy the coordinates in geometry correspond to the coordinatization of spacetime-indexed algebras by pointlike field generators. More specifically, the use of pointlike fields is analogous to the use of singular coordinates (coordinate systems which become singular somewhere) since quantum fields are "operator-valued distributions" which require smearing with test functions.

In the Lagrangian quantization approach to QFT, as well as in the more intrinsic algebraic approach to LQP, crossing plays no significant role. Only in formulations of particle physics which start with on-shell quantities and aim at the construction of spacetime-indexed local algebras or local equivalence classes of fields, the crossing becomes an important structural tool.

Examples par excellence of pure on-shell approaches are the various attempts at S-matrix theories which aim at direct constructions of scattering data without the use of local fields and local observables. The motivation behind such attempts was for the first time spelled out by Heisenberg [5] and amounts to the idea that by limiting oneself to particles and their mass-shells, one avoids (integration over) fluctuation on a scale of arbitrarily small spacelike distances which are the cause of ultraviolet divergencies.

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[1] In the setting of formfactors i.e. matrix elements of operators between multiparticle ket in-states and bra out-states the S-matrix is a special case of a (generalized) formfactor associated with the identity operator.

[2] The individuality of classical fields is lost in QFT where e.g. a meson field is any local (relative local with respect to the local obervables of the theory) covariant object with a nonvanishing matrix element between the vacuum and a one-meson state ("interpolating field").

This idea of giving constructive prominence to "on-shell" aspects is quite different and certainly more conservative than attempts at improving short-distance properties by introducing non-local interactions in a field theoretic framework (for a historical review of non-local attempts see [6]) which generally causes grave problems with the causality properties underlying particle physics. The main purpose of approaches using scattering concepts ("on-shell") is to avoid such inherently singular objects as pointlike fields in calculational steps, which is a reasonable aim independent of whether one believes that a formulation of interactions in terms of singular pointlike fields exists in the mathematical physics sense or not.

Heisenberg's S-matrix proposal can be seen as the first attempt in this direction. It incorporated unitarity, Poincaré invariance and certain analytic properties, but already run into problems with the implementation of cluster factorization properties for the multiparticle scattering.

There exists a more recent scheme of "direct particle interaction" which solved this cluster factorization problem for the multi-particle representations of the Poincaré group in the presence of interactions by an iterative construction [7]. To understand the problem with clustering, it is helpful to recall that in multiparticles Schrödinger quantum mechanics the step from n to n+1 particles by simply adding the two-particle interactions of the new particle with the n previous ones manifestly complies (for sufficiently short range interactions) with the cluster factorizability of the unitary representors of the 10-parametric Galilei-group; the system and its symmetries factorizes into previously constructed subsystems. But this infinite "Russian matrushka" picture of particle physics (iteratively adding particles together with their interactions and in turn recovering the previous smaller systems by translating one of the particle to infinity) runs into serious problems in the relativistic context. In mathematical terms there exists a mismatch between the adding-on of particles and their L-covariant interactions on the one hand, and the cluster factorizability property i.e. the tensor factorization of the representation into the representation of the previously encountered multi-particle subsystems on the other hand. For the two-particle systems there is no problem with clustering if one defines the interaction in terms of an additive modification of the invariant two-particle mass operator as first proposed by Bakamijan and Thomas [8]. However the iteration of this B-T procedure to 3 particles leads to a Poincaré covariant representation which fails to cluster (the Hamiltonian and the L-boosts are not asymptotically additive); although the 3-particle S-matrix[3] does cluster [9]. Adding a fourth particle in the B-T way would also lead to the breakdown of the 4-particle S-matrix clustering. The solution to this obstruction was later found in [7]; it consisted in modifying the 3-particle system by adding a connected 3-particle interaction in such a way that the 3-particle S-matrix does not change. This is done by a so-called "scattering equivalence"[10] i.e. a unitary transformation which changes the (Bakamijan-Thomas) 3-particle representation without affecting the 3-particle S-matrix[4].

It turns out that this process of adding on interactions to the mass operator and then enforcing clustering by invoking scattering equivalence works iteratively [7] and yields an n-particle interacting representation of the Poincaré group; in particular one obtains Møller operators and an S-matrix which fulfill the cluster factorization property. There is a prize to pay, namely the use of scattering equivalences prevent the use of a second quantization formalism known from Schrödinger QM, thus separating relativistic direct particle interactions from QFT even on a formal level. Nevertheless it does secure the macro-locality expressed by the (rapid in case of short range interactions) fall-off properties of the connected parts of the representation of the Poincaré group and the S-matrix. Different from the mass superselection rule in Galilei invariant quantum mechanics, there is no selection rule involving particle masses which requires the absence of particle creation processes coming from Poincaré symmetry in this relativistic direct particle interaction formalism [7]. This poses the interesting question whether by coupling channels which lead to an increasing number of created particles one can approximate field theoretic models by mathematically controllable direct particle interactions. After this interlude about the feasibility of macro-causal relativistic particle theory (for a more detailed presentation see [6]) we now return to the setting of QFT.

Since the early 1950s, in the aftermath of renormalization theory, the relation between particles and

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[3]The possibility of two-particle bound states entering as incoming particles requires the use of the framework of rearrangement collisions in which the space of (noninteracting) fragments is distinguished from the (Heisenberg) space on which the interacting Poincaré group is represented [7].

[4]Whereas in QFT the permitted field changes which maintain the local net of algebras are described by the local equivalence classes (Borchers classes), the scattering equivalences in the C-P scheme form a much bigger nonlocal class of changes.

fields received significant elucidation through the derivation of time-dependent scattering theory. It also became clear that Heisenberg's S-matrix proposal had to be amended by the addition of the crossing property i.e. a prescription of how to analytically continue particle momenta on the complex mass shell in order to relate matrix elements of local operators between incoming ket and outgoing bra states with a fixed total sum of in + out particles in terms of one "masterfunction". In physical terms it allows to relate matrix elements with particles in both the incoming ket- and outgoing bra-states to the vacuum polarization matrix elements where the ket-state (or the bra state) is the vacuum vector.

Whereas Heisenberg's requirements on a relativistic S-matrix can be implemented in a direct particle interaction scheme, the implementation of crossing is conceptually related to the presence of vacuum polarization for which QFT with its micro-causality is the natural arena. At this point it should be clear to the reader why we highlighted the little known direct particle interaction theory; if one wants to shed some light on the mysterious crossing symmetry, it may be helpful to contrast it with theories of relativistic particle scattering in which this property is absent.

The LSZ time-dependent scattering theory and the associated reduction formalism relates such a matrix element (referred to as a generalized formfactor) in a natural way to one in which an incoming particle becomes "crossed" into an antiparticle on the backward real mass shell; it is at this point where analytic continuation from a physical process enters. The important remark here is that the use of particle states requires the restriction of the analytic continuation to the complex mass shell ("on-shell"). If one were to allow off-shell analytic continuations, the derivation of the crossing would be much easier since it would then follow from off-shell spectral representations of the Jost-Lehmann-Dyson kind or perturbatively from Feynman diagrams and time-ordered functions. In this paper the notion of crossing will only be used in the restrictive on-shell analytic continuation as it is needed for on-shell relation between formfactors.

A rigorous on-shell derivation for two-particle scattering amplitude has been given by Bros[5], Epstein and Glaser [11]. The S-matrix is the formfactor of the identity operator. In the special case of the elastic scattering amplitude, the crossing of only one particle from the incoming state has to be accompanied by a reverse crossing of one of the outgoing particles in order to arrive at a physical process allowed by energy-momentum conservation. This crossing of a pair of particles from the in/out elastic configuration is actually the origin of the terminology "crossing" and was the main object of rigorous analytic investigations [11]. A derivation of crossing in the setting of QFT for general multi-particle scattering configurations and for formfactors, as one needs it for the derivation of a bootstrap-formfactor program (see later) from the general principles of local quantum physics, does not yet exist. It is not clear to me whether the present state of art in QFT would permit to go significantly beyond the old and still impressive results quoted before [11].

The crossing property became the cornerstone of the so-called bootstrap S-matrix program and several ad hoc representations of analytic scattering amplitudes were proposed (Mandelstam, Regge...) in order to incorporate crossing in a more manageable form.

An interesting early historical chance to approach QFT from a different direction by using on-shell global objects without short distance singularities was wasted when the S-matrix bootstrap approach ended in a verbal cleansing rage against QFT[6] instead of serving in its construction as attempted in this paper.

Some of the S-matrix boot strap ideas were later used by Veneziano [12] in the construction of the "dual model". But there is an essential difference in the way crossing was implemented. Whereas the field theoretic crossing involves a finite number of particles with the scattering continuum participating in an essential way, the dual model implements crossing without the continuum by using instead as a start discrete infinite "particle tower" with ever increasing masses (the origin of what was later called "stringyness"). This tower structure was afterwards interpreted in terms of the particle excitations of a relativistic string. It is important to note that Veneziano's successful mathematical experiment to implement crossing with properties of Gamma functions was more than a mathematical invention. In the late 60s there some of the dominant phenomenological ideas about Regge poles called for a one-particle

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[5]Since the issue of crossing constitutes the main subject of the BEG paper, I find it particularly appropriate to dedicate this work to Jacques Bros on the occasion of his 70th birthday.

[6]One glance at the old conference proceedings and review articles of the Chew S-matrix school reveals that I am not exaggerating. Nowadays the ideological fervor against QFT is hard to understand, in particular in view of the fact that almost all the concepts originated from QFT.

"saturation" of the crossing property in the setting of Mandelstam's representation of the 2-particle scattering amplitude. The popularity which the dual model enjoyed before QCD appeared on the scene was more related to these phenomenological aspects rather then to its role in carrying some of the legacy of the S-matrix bootstrap approach.

There is some irony in the fact that Chew and his followers, who tried to find a philosophical basis for their S-matrix bootstrap ideas to attain the status of a theory of everything (TOE), did not succeed in these attempts[7], whereas Veneziano, who had no such aims, laid the seeds of string theory. Contrary to the original phenomenological intentions of the dual model, its string theoretical re-interpretation elevated it in the eyes of some physicist to the status of a TOE (this time including gravity). Whether one subscribes to such view or not, there can be little doubt that string theory became quite speculative and acquired a somewhat ideological stance. Contrary to the bootstrap of the Chew school however, it led to significant mathematical enrichments even though its role for particle physics became increasingly mysterious.

The main reason why the old bootstrap approach ended in the dustbin of history was its clinging to its dismissive view of QFT even at a time when the success of gauge theories was already obvious. On a deeper level and in and in relation to the content of the present paper it is obvious that it did not succeed in its own terms since it was unable convert the analyticity based bootstrap ideas by a mathematically well-defined operator formalism which incorporates the crossing property in a natural way.

In recent years the similarity of the cyclic crossing property of formfactors with the better understood cyclic KMS condition for wedge-localized algebras (the Rindler Unruh thermal aspect) led to the conjecture that the former is an on-shell consequence of the latter. Whereas this turns out to be true for d=1+1 factorizing models, the nature of the connection between these two cyclic properties in the general setting remains obscure and needs further clarifications.

The content of the paper is organized as follows. In the next section we set the stage for the concept of modular localization which will be our main new constructive tool. Whereas without interactions there is a complete parallelism between particle- and field- modular localization, the presence of interactions has a de-localizing effect on the side of particles as a result of interaction-caused vacuum polarization. A useful concept which captures this de-localization aspect is that of vacuum-**p**olarization-**f**ree **g**enerators (PFG) which highlights the wedge localization as representing the best compromise between particle- and field- localization. In the third section we recall that the requirement of translation invariant domains for PFGs ("tempered" PFGs) essentially leads to the Zamolodchikov-Faddeev algebra structure which characterize d=1+1 factorizing theories. This is a modest realization of the old "bootstrap dream", but now as a valuable constructive tool of QFT without the unfounded claim of a TOE.

In the fourth section the idea of a "masterfield" will be set forth whose connected formfactors define a nonlocal QFT in momentum space for which the KMS condition is identical with crossing. Whereas for factorizing models this idea reduces to Lukyanov's "free field representations", in a more general setting the hypothesis remains a matter of interesting speculation and a subject for future research.

Finally modular localization is used to formulate "algebraic lightfront holography" which relates massive quantum field theories to generalized chiral models on the lightfront. As a result of its firm anchoring in AQFT and is conceptual tightness, one would expect this new idea to play an important role in future construction methods. Its confrontation with the setting for d=1+1 factorizing models reveals that the massive particle aspects including crossing and scattering data and the chiral conformal field based holographic properties coexist as two descriptions of the same theory in one and the same Hilbert space.

2   Modular Localization for Particles and Fields

The concept of modular localization, which will be reviewed in this section, has significantly enriched ideas about the relation between particles and fields. In particular it has led to a profound understanding of those properties in the particle-field relation which persist in the presence of interactions and which in turn are important in an intrinsic understanding of interaction; this is the understanding which, borrowing

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[7] The S-matrix bootstrap returned many years later as a valuable tool (but not a TOE) of the "formfactor program" in the limited context of d=1+1 factorizing models of QFT[13].

an aphorism of Pascual Jordan [14], does not rely on "classical crutches", as does the standard Lagrangian quantization.

Historically the first step into a direction of intrinsic formulation of relativistic quantum physics was undertaken by Wigner when in 1939 he identified relativistic particle states with irreducible positive energy representations of the Poincaré group. These representations come with two localization concepts: the Newton-Wigner localization [15] and the more recent modular localization [16][17][18].

The N-W localization is the result of the adaptation of Born's quantum mechanical localization probability density to Wigner's relativistic setting. This localization is important in relativistic scattering theory since it leads to the probability interpretation of cross sections, which was actually the setting in which Born introduced probabilities into QM (the x-space probability interpretation of the Schroedinger wave function appears later in Pauli's Handbuch article). It is not Lorentz-covariant nor local[8] for finite distances, but the fact that it acquires these two properties in the asymptotic region is sufficient for obtaining a relativistic asymptotic particle description and in particular a Poincaré invariant S-matrix [23]. It should not come as a surprise that its use for propagation over finite distances leads to nonsensical results on the feasibility of superluminal propagation [24].

On the other hand the modular localization is the localization which is implicit in the formalism of local quantum field theory. It is well known that if one applies smeared fields with localized $\mathcal{O}$-support of the smearing function $suppf \subset \mathcal{O}$ to the vacuum, the resulting vectors will belong to a dense subspace $H(\mathcal{O})$[9] which will change its position in the ambient space with the change of the localization region

$$A(f)\Omega \in H(\mathcal{O}) \subseteq H \tag{1}$$

Modular localization theory is a relatively new conceptual framework which places this kind of relation between spacetime regions of vacuum excitations and positions of dense subspaces on a more intrinsic and rigorous footing, so that it becomes independent of the use of field coordinatizations. This is done by trading the subspace generated by smeared fields with the domain of the $\mathcal{O}$-dependent Tomita S-operator $H(\mathcal{O}) \equiv domS_{\mathcal{O}}$ (see next section) which is directly associated with the localized algebra and does not refer to its coordinatization in terms of fields.

This encoding of Minkowski spacetime localization into relative position of subspaces (or equivalently in terms of real subspaces (3) of which $H(\mathcal{O})$ turns out to be the complex combination) is a characteristic phenomenon of local quantum physics; it essentially depends on the presence of a finite maximal causal propagation speed and hence has no counterpart in the Schrödinger QM. The denseness of the localization spaces prevents a description in terms of projectors onto complex subspaces and hence evades the assumptions of the mentioned no-go theorem [22].

This unusual situation, which goes somewhat against quantum mechanical intuition, is inexorably linked with a structural change of the local algebras as compared to the algebraic structure of quantum mechanics. Whereas the algebra of QM has minimal projectors (corresponding to best observations), the structure of projection operators within local relativistic algebras is very different from that of projectors in the global algebra associated with the entire Minkowski spacetime. All these changes can be traced back to the omnipresence of vacuum polarizations which in turn are inexorably related to relativistic causality in the setting of quantum theories.

The difference between quantum mechanical and modular localization shows up in a dramatic fashion in a famous Gedankenexperiment which Fermi proposed [25] in order to show that the velocity of light remains the limiting propagation velocity in the quantum setting of relativistic field theory. An updated argument confirming Fermi's conclusion which takes into account the conceptual progress on the issue of causal localization and mathematical rigor can be found, as mentioned before, in [24]. Although all quantum mechanical situations associated with Bell's inequalities can be transferred to QFT with the help of the split property, there are problems with achieving the vacuum polarization free two-particle state postulated by Fermi [10]. This does however not affect the conclusion that localized exitations of the vacuum cannot propagate with a superluminal speed.

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[8] Far from being a a peculiar shortcoming of the Newton-Wigner localization, there exist a general No-Go theorem which rules out the existence of any Poincaré-covariant localization in terms of projectors and probabilities in theories with positive energy [22].

[9] The denseness of this subspace is the main content of the Reeh-Schlieder theorem [23].

[10] I am indebted to Larry Landau for reminding me of the problems of using the split property in connection with the realization of the Fermi Gedankenexperiment in the relativistic setting.

The modular localization theory associated with localized algebras in QFT has a simpler spatial counterpart which can be directly applied to the Wigner representation theory of the Poincaré group. In the next section we will study this spatial modular localization. In addition of being interesting in its own right, this will facilitate the subsequent presentation of algebraic modular localization theory which is indispensable in order to incorporate interactions in a field-coordinatization independent way.

2.1 Modular localization in the absence of interactions

Modular localization as an intrinsic concept of local quantum physics (i.e. without reference to any pointlike field coordinatization), has its origin in the Bisognano-Wichmann theorem for wedge-localized algebras in QFT [26][27]. In the context of Wigner's description of elementary relativistic systems in terms of irreducible positive energy representations of the Poincaré group, the construction of this localization proceeds as follows [16] [17][18]

1. Fix a reference wedge region, e.g. $W_R = \left\{ x \in \mathbb{R}^4; x^1 > \left| x^0 \right| \right\}$ and use the Wigner representation of the W_R-affiliated boost group $\Lambda_{W_R}(\chi)$ and the $x^0 - x^1$−reflection[11] along the edge of the wedge j_{W_R} in order to define the following antilinear unbounded closable operator (with $closS = clos\Delta^{\frac{1}{2}}$). Retaining the same notation for the closed operators, one defines

$$S_{W_R} := J_{W_R}\Delta^{\frac{1}{2}} \tag{2}$$
$$J_{W_R} := U(j_{W_R}), \ \Delta^{it} := U(\Lambda_{W_R}(2\pi t))$$

The commutativity of J_{W_R} with Δ^{it} together with the antiunitarity of J_{W_R} yield the property which characterize a Tomita operator[12] $S_{W_R}^2 \subset 1$ whose domain is identical to its range. Such operators are well-known to be equivalent to their real standard subspaces of the Wigner representation space H which arise as their closed real +1 eigenspaces $K(W)$

$$K(W_R) := \{\psi \in H, S_{W_R}\psi = \psi\} \tag{3}$$
$$\overline{K(W_R) + iK(W_R)} = H, \ K(W_R) \cap iK(W_R) = 0$$
$$J_R K(W_R) = K(W_R)^{\perp}$$

The real subspace $K(W_R)$ is closed in H, whereas the complex subspace spanned together with the -1 eigenspace $iK(W_R)$ is the dense domain of the Tomita operator S_{W_R} and forms a Hilbert space in the graph norm of S_{W_R}. The denseness in H of this span $K(W_R) + iK(W_R)$ and the absence of nontrivial vectors in the intersection $K(W_R) \cap iK(W_R)$ is called "standardness". The right hand side in the third line refers to the symplectic complement i.e. a kind of "orthogonality" in the sense of the symplectic form $Im(\cdot, \cdot)$.

Additional comments. The denseness of the complex spans of modular localization spaces is a one-particle analog of the Reeh-Schlieder theorem [23]. Each Tomita operator S_W encodes physical information about localization into the position of its dense domain (which equals its dense range) within H. Equivalently real standard subspaces or their complex dense span determine uniquely an abstract Tomita operator (which in general is not related to geometry or group representation theory). The application of Poincaré transformations to the reference situation generates a consistent family of wedge spaces $K(W) = U(\Lambda, a)K(W_R)$ if $W = (\Lambda, a)W_R$.

One of the surprises of this modular localization setting is the fact that it already preempts the spin-statistics connection on the level of one-particle representation theory by producing a mismatch between the symplectic and the geometric complement which is related to the spin-statistics factor [17][18]

$$K(W)^{\perp} = ZK(W') \tag{4}$$
$$Z^2 = e^{2\pi is}$$

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[11] In certain cases the irreducible representation has to be doubled in order to accomodate the antiunitary (time is inverted) reflection. This is always the case with zero mass finite helicity representations and more generally if particles are not selfconjugate.

[12] Operators with this property are the corner stones of the Tomita-Takesaki modular theory [29] of operator algebras. Here they arise in the spatial Rieffel van Daele spatial setting of modular theory from a realization of the geometric Bisognano-Wichmann situation within the Wigner representation theory.

Another surprising fact is that the modular setting prepares the ground for the crossing property, since the equation characterizing the real modular localization subspaces in more details reads

$$\left(J\Delta^{\frac{1}{2}}\psi \right)(p) = \Sigma\overline{\psi_c(-p)} = \psi(p) \tag{5}$$

i.e. the complex conjugate of the analytically continued wave function (but now referring to the charge-conjugate situation) is up to a matrix Σ which acts on the spin indices equal to the original wave function.

2 The sharpening of localization is obtained by intersecting wedges in order to obtain real subspaces as causally closed subwedge regions:

$$K(\mathcal{O}) := \cap_{W\supset\mathcal{O}}K(W) \tag{6}$$

The crucial question is whether they are "standard". According to an important theorem of Brunetti, Guido and Longo [16] standardness holds for spacelike cones $\mathcal{O} = \mathcal{C}$ in all positive energy representations. In case of finite spin/helicity representations the standardness also holds for (arbitrary small) double cones D. The double cone regions D are conveniently envisaged as intersections of a forward cone with a backward cone whose apex is inside the forward cone; the simplest description of a spacelike cone C with apex a is in terms of a scaled up double cone $C = a + \cup_{\lambda \geq 0}\lambda D$ where D is spacelike separated from the origin. Both regions are characteristic for simply connected Poincaré-invariant causally closed families of compact or noncompact extension resulting from intersecting wedges in Minkowski spacetime. In those cases where the double cone localized spaces with pointlike "cores" are trivial (massless infinite spin, massive d=1+2 anyons), the smallest localization regions are spacelike cones with semiinfinite strings as cores.

Additional comments. Although the connection between standard real subspaces and Tomita operators S holds in both directions (and hence standard intersections always have an associated Tomita operator S), the components of their polar decomposition Δ^{it} and J have generally no relations to diffeomorphisms of the underlying spacetime. While leaving the localization regions invariant (or transforming them into their causal disjoint) and hence still encoding the full information of localization, their actions within $\mathcal{O}$ as well on its causal complement $\mathcal{O}'$ are "fuzzy", which at best may be expressed (in the Wightman setting of QFT) in terms of actions on test function spaces with fixed localization supports (see below 12).

3 In the absence of interactions the transition from free particles to algebras of fields is most appropriately done in a functorial way by applying the Weyl (CCR) (or in case of halfinteger spin the CAR functor) to the localization K-spaces[13]:

$$\mathcal{A}(\mathcal{O}) := alg\left\{ Weyl(\psi)|\ \psi \in K(\mathcal{O}) \right\} \tag{7}$$

$$Weyl(f) := expi\left\{ \int a^*(p,s)\psi(p,s)\frac{d^3p}{2\omega(p)} + h.a. \right\}$$

The functorial relation between real subspaces and von Neumann algebras preserves the causal localization structure [19] and commutes with the improvement of localization through intersections (6) (denoted by $\cap$) as expressed in the following commuting square

$$\begin{array}{ccc} K_W & \longrightarrow & \mathcal{A}(W) \\ \downarrow\cap & & \downarrow\cap \\ K_{\mathcal{O}} & \longrightarrow & \mathcal{A}(\mathcal{O}) \end{array} \tag{8}$$

i.e. without interactions there is a perfect match between particle- and field- localization[14]. For later purposes we introduce the following definition [28].

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[13]To maintain simplicity we limit our presentation to the bosonic situation and refer to [17][18] for the general treatment.

[14]We retain the traditional word "field" in the sense of carriers of causal localization even though the present construction avoids the explicit use of pointlike operator-valued distributions.

Definition 1 *A vacuum-polarization-free generator (PFG) for a region $\mathcal{O}$ is an operator affiliated with the algebra $\mathcal{A}(\mathcal{O})$ which created a vacuum-polarization-free one-particle vector*

$$G \,\eta\, \mathcal{A}(\mathcal{O}) \tag{9}$$
$$G\Omega = 1 - particle$$

It is easy to see that (in case of Bosons) PFGs are necessarily unbounded operators. In the absence of interactions they turn out to consist precisely of those $\mathcal{O}$-localized operators which are linear in the Wigner creation/annihilation operators. In that case a denumerable covariant pointlike basis of PFGs is conveniently described in terms of the well-known set of interwining functions $u(p,s)$ (and their charge conjugates $v(p,s)$) which relate the given canonical (m,s) Wigner representation with the various tensorial (spinorial) covariant free fields

$$A(x) = \int \left\{ e^{-ipx} \sum u(p,s_3) a(p,s_3) + e^{ipx} \sum v(p,s_3) b^*(p,s_3) \right\} \frac{d^3p}{2p^0} \tag{10}$$
$$p^0 = \sqrt{\vec{p}^2 + m^2} \tag{11}$$

Whereas the (m,s) Wigner creation/annihilation operators $a^\#(p,s)$ and the above localized algebras are unique, there exists an denumerable set (labeled by pairs of undotted/dotted spinorial indices) of covariant intertwinwers for fixed (m,s) [20]. Their main role with respect to the issue of modular localization consists in relating the quantum concept of modular localization to the more classical notion of localization via support properties of test functions

$$K(\mathcal{O}) = clos \left\{ E_m \tilde{f}(p) u_k(p,s) \,|\, supp f \subset \mathcal{O}, k = 1...N \right\} \tag{12}$$

where $E_m f(p)$ stands for the mass-shell projection[15] of the Fourier transform of the real test function f and the closure is taken in the linear span with i runs over all Lorentz (spinorial) components N and f running over all $\mathcal{O}$-supported test functions; as before the closure within the Wigner representation space is restricted to real linear combinations. This way of relating modular localization to classical test function supports is (whenever it is possible) the easiest way to show the standardness property. When the appearance of massless infinite spin representations only allows standardness of spacelike cone-localized spaces, the analogs of the above intertwiners lead to semiinfinite spacelike string-localized fields $A(x,e)$ (with e being a spacelike unit vector [2]) which have no interpretation in terms of Lagrangian quantization (and should not be confused with objects of string theory).

As expected, the crossing relation for connected matrix elements (connected formfactors) of a wedge-localized operator $B \in \mathcal{A}(W)$ ($\bar{p}$ denotes the charge conjugate particle with momentum p)

$$\langle p_1, ..., p_k \,|B|\, p_{k+1}, ..., p_n \rangle_{conn} \tag{13}$$
$$= \langle -\bar{p}_n, p_k, ..., p_1 \,|B|\, p_{k+1}, ..., p_{n-1} \rangle_{conn}$$

results from the KMS property of the wedge-restricted vacuum state ($supp f_i \subset W$)

$$\langle A(f_1)^* ... A(f_k)^* B A(f_{k+1})...A(f_n) \rangle \tag{14}$$
$$= \langle Ad\Delta(A(f_n)) A(f_1)^* ... A(f_k)^* B A(f_{k+1})...A(f_{n-1}) \rangle$$

by taking the connected part and using the density of the W-supported product of test functions in the multiparticle tensor-product Wigner spaces.

Since it is very convenient to consider the later lightfront holography (section 6) as part of modular wedge localization, we will briefly explain in the following in a pedestrian way how this is done for a massive Hermitian free field $A(x)^* = A(x)$. Using the previous notation (12) one has for real test

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[15]As a result of the mass shell restriction a Wigner wave function (and the smeared fields) is represented in terms of an equivalence class of test functions.

functions with $supp f \subset W$

$$A(f) = \int \left(a^*(p) E_m \tilde{f}(p) + h.c. \right) \frac{d^3 p}{2p_0} = \int \left(a^*(p) E_m \tilde{f}(p) + h.c. \right) \frac{d\theta}{2} d^2 p_\perp \tag{15}$$

$$[a(p), a^*(p')] = 2p^0 \delta(\vec{p} - \vec{p}') = 2\delta(\theta - \theta') \delta(p_\perp - p'_\perp)$$

$$with \ p = (m_{eff} \cosh \theta, m_{eff} \sinh \theta, p_\perp), \ m_{eff} = \sqrt{m^2 + p_\perp^2}$$

where the $x^0 - x^1$ localization in the 0-1 reference wedge implies that $E_m \tilde{f}(p)$ of the real test function f is a vector in the dense subspace $K_r(W) + iK_r(W)$ of boundary value of analytic functions in the θ−strip with respect to the measure $\frac{d\theta}{2} d^2 p_\perp$. Since product functions $E_m f(p) = \tilde{f}_+(\theta) \tilde{f}_\perp(p_\perp)$ with $\tilde{f}_+(\theta)$ strip-analytic are dense in $K_r(W) + iK_r(W)$ it is convenient to use them in the following way $(p_- \equiv e^\theta)$

$$\int \left(a^*(p) \tilde{f}_+(\theta) \tilde{f}_\perp(p_\perp) + h.c. \right) \frac{d\theta}{2} d^2 p_\perp = A(f_+ f_\perp) \tag{16}$$

$$= \int A_{LF}(x) f_+(x_+) f_\perp(x_\perp) dx_+ dx_\perp$$

$$f_+(x_+) \equiv \frac{1}{2\pi} \int_0^\infty \tilde{f}_+(\ln p_-) e^{ip_- x_+} \frac{dp_-}{2p_-}$$

$$A_{LF}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left(a^*(p) e^{ip_- x_+ + ip_\perp x_\perp} + h.c. \right) dp_- d^2 p_\perp$$

$$[a(p), a^*(p')] = 2p_- \delta(p_- - p'_-) \delta(p_\perp - p'_\perp)$$

$$\curvearrowright \langle A_{LF}(x) A_{LF}(x') \rangle = \int e^{-ip_-(x_+ - x'_+)} \frac{dp_-}{2p_-} \cdot \delta(x_\perp - x'_\perp)$$

where in the last line the two-point function has been rewritten in the new lightfront variables. As a consequence of the strip analyticity in θ the function $f_+(x_+)$ is supported on the positive x_+ axis. Note that the vanishing of the Fourier transform at $p_- = 0$ is not imposed but results from the square integrability of $\tilde{f}_+(\theta)$ which forces the $\tilde{f}_+(\ln p_-)$ to vanish at the lower boundary $p_- = 0$ (this also holds without the specialization to product functions).

Without this vanishing property the infrared divergence in the Fourier representation for $A_{LF}(x)$ would not be compensated and the expression would not be equal to the original one. The relevant testfunction spaces for lightcone quantization were first introduced (without referring to modular localization) in [21]. Note also that the Fourier transformed lightfront test functions $f_+(x_+) f_\perp(x_\perp)$ (unlike their original counterpart $f(x)$) are not subject to any mass shell restriction i.e. the lightfront localization relates the smeared fields with individual functions on the lightfront rather than mass shell equivalence classes[16] of ambient test functions..

The terminology "lightfront restriction" for this rewriting becomes more comprehensible in terms of the following formal steps $(r = \sqrt{x_1^2 - x_0^2})$

$$A(x)|_W = A(r \sinh \chi, r \cosh \chi, x_\perp)$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left(a^*(p) e^{im_{eff} r \sinh(\chi - \theta) + ip_\perp x_\perp} + h.c. \right) \frac{d\theta}{2} d^2 p_\perp \tag{17}$$

$$\overset{\chi = \ln r, r \to 0}{\longrightarrow} \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left(a^*(p) e^{im_{eff} e^\theta x_+ + ip_\perp x_\perp} + h.c. \right) \frac{d\theta}{2} d^2 p_\perp \tag{18}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^2 p_\perp \int_0^\infty \left(a^*(p) e^{ip_- x_+ + ip_\perp x_\perp} + h.c. \right) \frac{dp_-}{2p_-} = A_{LF}(x) =: A(x)|_{LF}$$

where in the last line we have absorbed m_{eff} into the definition of the integration variable p_-. Although we obtain the same formula as before, the formal way requires to add the restriction on test function

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[16] We are referring to the fact that the relation between testfunctions f and their wave functions $E_m \tilde{f}$ in the Wigner one-particle space is an equivalence class relation.

spaces whose Fourier transforms vanish at $p_- = 0$ "by hand". For d=1+1 the transverse $x_\perp$ and $p_\perp$ are absent.

Lightfront restriction does not mean pointwise restriction of the correlation functions i.e. $\langle A(x)A(x')\rangle_{x_-=0=x'_-} \neq \langle A(x)A(x')\rangle|_{LF} \equiv \langle A|_{LF}(x)A|_{LF}(x')\rangle$. This point was the source of occasional confusion in the literature on lightcone quantization. In fact already the terminology "lightcone quantization" creates the impression that one is aiming at a different quantization leading to a possibly different theory, whereas in reality the physical problem is to describe the ambient local theory in terms of a different locality structure associated with the lightfront. This LF locality structure, although being local in its own right, is relatively nonlocal with respect to the ambient locality structure. The pivotal problem of how these two structures are related was not addressed in the old approach.

In the absense of interactions the lightfront restriction $A|_{LF}$ shares with the ambient free field A the vanishing of higher than two-point correlations. As a consequence there is only one ambient theory associated with the above lightfront field. As will be argued in section 6, in the presence of interactions one expects the relation of the ambient theories to their holographic projection to be many to one i.e. the concept of "holographic universality classes" becomes important in inverse holography (reconstruction of ambient theories from a given LF description).

The important observation in the context of localization is that the algebras generated by smearing $A(x)|_W$ and $A(x)|_{LF}$ with the corresponding test function spaces are identical

$$alg\left\{A(f) \mid suppf \subset W\right\} = alg\left\{A_{LF}(f_+f_\perp) \mid suppf_+ \subset \mathbb{R}_+\right\} \tag{19}$$

Although the equality of the wedge- with the lightfront- localized algebra turns out to be a general feature of QFT[17], it is only in the free field case that one can describe the localization aspects of the lightfront algebra by the above process of a restriction of the ambient free field. For interacting fields the local net structure on the lightfront has to be recovered in an algebraic manner referred to as "algebraic lightfront holography", which will be presented in section 6. This new approach demystifies and corrects to a considerable degree the old ideas on lightcone quantization.

2.2 Modular localization in the presence of interactions

There is a drastic weakening in the relation between particle- field localization when interactions are present. The parallelism expressed in the above commuting square is lost. In particular interactions destroy the possibility of having subwedge-localized PFGs[18]. Quantum fields also loose that kind of "individuality" (associated with the measurement of field strength) which fields enjoy in classical physics; the role of quantum fields (besides being the non-intrinsic implementers of the relativistic locality principle) is restricted to interpolate particles and to "coordinatize" (in the sense of singular generators) local nets of algebras. Hence it is somewhat surprising that there are two remarkable and potentially useful properties which survive the presence of interactions. As in the framework of LSZ scattering theory, in the following we are assuming the existence of a mass gap.

1. Wedge algebras $\mathcal{A}(W)$ have the smallest localization region which still permits affiliated PFGs [47], i.e. to every wedge-localized one-particle wave function $\psi \in K(W) + iK(W)$ there exists a $G_\psi \eta \mathcal{A}(W)$ with

$$G_\psi \Omega = \psi \tag{20}$$
$$G_\psi^* \Omega = S\psi$$

   This is the best compromise between particles and fields in the presence of interactions; any improvement on the level of particles (e.g. construction of n-particle states for n>1) would only be possible in the completely de-localized global algebra (which contains e.g. the creation/annihilation operators). Vice versa any improvement in the localization by passing to subwedge algebras would lead to the admixture of interaction-induced vacuum polarization (states with ill-defined particle

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[17] The only exception is the case of massless theories in d=1+1.

[18] The J-S theorem can easily be generalized to subwedge-localized PFGs [3].

number) to the one-particle component. Hence the presence of this kind of vacuum polarization clouds for subwedge regions is an intrinsic signal of the presence of interactions. This raises the interesting question whether there is some common feature to interaction-induced vacuum polarization clouds which permits a finer classification of interactions; this is a problem which certainly must be solved if one wants to use this intrinsic characterization of interactions as a constructive alternative to the more extrinsic field-coordinatization dependent standard Lagrangian quantization approach.

2. In asymptotically complete QFT, the S-matrix S_{scat} is a relative modular invariant between the interacting and the free incoming wedge algebras

$$S = J\Delta^{\frac{1}{2}} \tag{21}$$
$$\Delta^{it} = \Delta^{it}_{in}, \ J = J_{in}S_{scat}$$

This follows from the TCP-invariance of the S-matrix and the fact that the modular J differs from TCP by a spatial π-rotation [23] which (as all connected Poincaré transformations) commutes with the scattering matrix. This structural property relates the position of the dense wedge-localized subspace $H_F(W)$ within the Fock space H_F (defined by e.g. the out-operators) to the S-matrix.

3. The split property [23] permits to formulate the notion of "statistical independence" (well-known from quantum mechanics) which concerns the construction of interacting states with independently prescribed local components. This is needed in order to control the strong vacuum fluctuations which result from sharp spacetime localization and leads to a partial return of quantum mechanical structures. Although the split property has up to now not played a direct role in model constructions, it is believed to be important in securing the standardness of intersections of wedge algebras and hence the nontriviality of models [30].

Additional comments. The interpretation of the scattering operator as a relative modular invariant associated with the wedge region leads to rather strong consequences if one assumes that the connected part of the formfactors fulfill the following crossing relations

$$^{out}\langle p_1, ..., p_k | B | p_{k+1}, ..., p_n \rangle^{in}_{conn} = \tag{22}$$
$$^{out}\langle -\bar{p}_n, p_k, ..., p_1 | B | p_{k+1}, ..., p_{n-1} \rangle^{in}_{conn}$$

where B is an operator affiliated with $\mathcal{A}(W)$. It has the same form as in the free case (13) except that the particles in the bra/ket vectors are referring to the different out/in particle states. Evidently this property permits to relate the vacuum polarization components

$$\langle p_n, ..., p_1 | B\Omega \rangle \tag{23}$$

with the general formfactor by a succession of crossings. The position of the dense subspace generated by all operators $B\eta\mathcal{A}(W)$ affiliated with $\mathcal{A}(W)$ from the vacuum is determined by the domain of the Tomita operator S which is in turn determined by the scattering operator S_{scat}. Assume that a given crossing symmetric scattering operator S_{scat} would admit two different wedge algebras $\mathcal{A}_i(W)$, $i = 1, 2$. Since these algebras must have the same Tomita operator for each $B_1\eta\mathcal{A}_1(W)$ there must exist an operator $B_2\eta\mathcal{A}_1(W)$ such that $B_1\Omega = B_2\Omega$ which means that the vacuum polarization components (23) are identical. But then the crossing property (22) lift this identity to the general formfactors which requires $B_1 = B_2$ and hence the desired equality $\mathcal{A}_1(W) = \mathcal{A}_2(W)$. Since the net of localized algebras is uniquely fixed in terms of intersections of wedge algebras, this would imply the uniqueness of the inverse scattering problem [52]. Note however that the crossing property of formfactors in the general interacting case is presently an additional assumption[19]; only for d=1+1 factorizing models crossing it can be shown to follow from the KMS property for the restriction of the vacuum to wedge algebras in a similar fashion as

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[19] The assumption of crossing for formfactors as one needs it for the uniqueness of inverse scattering seems to go beyond what has derived by the analyticity techniques in [11], but a definite conclusion on this matter can probably not obtained without updating these old but still impressive methods.

for free fields (see section 4). Without assuming the crossing property for formfactors it does not appear to be possible to derive the uniqueness of the inverse scattering problem from the standard postulates of QFT [31].

The prerequisites for formfactor crossing are obtained from the LSZ scattering theory and in particular from the resulting reduction formulas in terms of time-ordered products. For the connected formfactors one obtains

$$^{out}\langle q_1, q_2, ...q_m |B| p_n, ...p_2, p_1 \rangle^{in}_{conn} = \tag{24}$$

$$-i \int {}^{out}\langle q_2, ...q_m |K_y TBA^*(y)| p_1, p_2...p_n \rangle^{in}_{conn} d^4y e^{-iq_1 y}$$

$$= -i \int {}^{out}\langle q_1, q_2, ...q_m |K_y TBA(y)| p_2...p_n \rangle^{in}_{conn} d^4y e^{ip_1 y}$$

Here the time-ordering T involving the original operator $B \in \mathcal{A}(\mathcal{O})$ and the pointlike interpolating Heisenberg field[20] $A(x)$. The latter appears in the reduction of a particle from the bra- or ket state. For the definition of the time ordering between a fixed finitely localized operator B and a field with variable localization y we may use $TBA(y) = \theta(-y)BA(y) + \theta(y)A(y)B$, however as we place the momenta on-shell, the definition of time ordering for y near $locB$ fortunately turns out to be irrelevant[21]. These on-shell reduction formulas remain valid if one used as interpolating operators instead of pointlike fields the translates of bounded compactly localized operators [32]. Each such reduction is accompanied by another disconnected contribution in which the creation operator of an outgoing particle $a^*_{out}(q_1)$ changes to an incoming annihilation $a_{in}(q_1)$ acting on the incoming configuration; there is a corresponding contraction term if we would reduce a particle from the incoming state vector. These disconnected terms (which contain formfactors with two particle less in the bra- and ket- vectors) have been omitted since they do not contribute to generic nonoverlapping momentum contributions and to the analytic continuations (and hence do not enter the connected part).

Under the assumption that there is an analytic path from $p \to -p$ (or $\theta \to \theta - i\pi, p_\perp \to -p_\perp$ in the rapidity parametrization of the standard wedge), the comparison between the two expressions gives the desired crossing property that is to say a particle of momentum p in the incoming ket state within the formfactor is crossed into an outgoing bra antiparticle at the analytically continued momentum -p (here denoted as -p̄) and the connected and the connected formfactor remains invariant.

Reduction formulas and the crossing property are characteristic for pointlike localized fields (cor-responding to double cone localization in the algebraic setting), their derivation breaks down [33] if interacting fields only permit string-like localization (corresponding to the singular limit of spacelike cone localization). The reason for this is that it is not enough to control the localization of endpoints but one also must take care of the spacelike string direction; but the kinematical requirement for having convergence to outgoing asymptotic multi-particle states is different from that for incoming states so that there exist no single interpolating field which converges in both asymptotic directions. The particle-field relation and the constructions derived from it exclude string-localized fields. However this does not nec-essarily exclude string theory since there is no indication that string theory is string-localized (see also the concluding remarks).

In order to obtain an analytic path on the complex mass-shell for e.g. the $2 \to 2$ scattering amplitude it is convenient to pass from time ordering T to retardation R

$$TBA(y) = RBA(y) + \{B, A(y)\} \tag{25}$$

The unordered (anticommutator) term does not have the pole structure on which the Klein-Gordon operator K_y can have a nontrivial on-shell action and therefore drops out. The application of the JLD spectral representation puts the p-dependence into the denominator of the integrand of an integral representation from where the construction of an analytic path interpolating the formfactors with its

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[20]The notion of interpolating fields and associated reduction formulas cease to exist if the in/out particles require the application of semiinfinite string-like Heisenberg operators to the vacuum.

[21]For far separated y we may consider $locB$ to be near zero; then $\theta(y) \approx \theta(y - locB)$ agrees approximately with the relative timelike distance θ-function used for pointlike localization.

crossed counterpart proceeds in an analog fashion to the derivation of crossing for the S-matrix [34][32]. Whereas it is fairly easy to find an off-shell analytic path, the construction of an on-shell path i.e. one which remains in the complex mass shell is a significantly more difficult matter [11]. The LSZ reduction formalism is suggestive of crossing but for themselves too weak for securing the mathematical existence of paths on the complex mass shell which link real forward and backward mass shells.

The simplifications of the LSZ formalism resulting from factorizability of models can be found in an appendix of [35]

The result of the comparison between the reduction (24) applied to outgoing and incoming configurations may be written in the following suggestive way (for spinless particles)

$$
\begin{aligned}
&^{out}\left\langle p_1, p_2, ...p_l \left| B \right| q_{k,k-1} ...q_2, q_1 \right\rangle^{in} = \\
&\underset{q_c \to -q_1}{a.c.} \ ^{out}\left\langle q_c, p_1, p_2, ...p_l \left| B \right| q_k, q_{k-1}...q_2 \right\rangle^{in} + c.t \\
&=:\ ^{out}\left\langle -q_1, p_1, p_2, ...p_l \left| B \right| q_k, q_{k-1}...q_2 \right\rangle^{in} + c.t.
\end{aligned}
\tag{26}
$$

where the contraction terms c.t. involve momentum space δ-functions (which are part of the LSZ reduction theory) and the last line denotes a shorthand notation for the analytic continuation to the real negative mass shell. Instead of crossing from incoming ket to outgoing bras one may of course also cross in the reverse direction from bras to kets. The important physical role of the crossing property is to relate the vacuum polarization components of an operator to the connected part of the transition it causes between in and out scattering states via iterated crossing

$$
^{out}\left\langle p_1, p_2, ...p_n \left| B \right| \Omega \right\rangle \overset{iteration}{\longrightarrow} \ ^{out}\left\langle p_k, p_{k+1}, ...p_n \left| B \right| -\bar{p}_{k-1}... -\bar{p}_2, -\bar{p}_1 \right\rangle^{in}_{conn}
\tag{27}
$$

Note that the vacuum polarization components are always connected. It is very important to realize that the simplicity of the crossing property occurs only for the connected part of the matrix elements; in order to write down the relation for the full matrix elements one must keep track of all the momentum space contraction terms in the iterative application of the LSZ formalism. It is the connected part which is described by one analytic "master function" whose different boundary values correspond to the connected part of the different matrix elements. This already indicates that one should expect problems if one wants to understand crossing as an operational property in the original theory of operators since taking connected parts of correlation functions is not expressible as an operator algebraic property. Indeed attempts to relate crossing to the cyclicity property of thermal expectation values in KMS states on operator algebras within the general setting of QFT failed[22].

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3   The bootstrap-formfactor program in d=1+1 factorizing QFT

As mentioned in the introduction, a modest but in its own right very successful version of the S-matrix bootstrap with strong field theoretic roots emerged in the second half of the 70s from some prior quasiclassical integrability discoveries [38]. These seemingly exact quasiclassical observations on the special two-dimensional as the "Sine-Gordon" model of QFT required an explanation beyond quasiclassical approximations [39]. This line of research led finally to a general program of a bootstrap-formfactor construction of so-called d=1+1 factorizable models [13][41][42]. From this new nonperturbative scheme for constructing a particular class of field theories came a steady flux of new models and it continues to be an important innovative area of research. Our interest in the present setting lies in the potential messages it contains with respect to a mass-shell based constructive approach without the "classical crutches" which underlie the Lagrangian quantization approach. In particular we are interested in a better understanding of formfactor crossing.

This formfactor program uses the very ambitious original S-matrix bootstrap idea in the limited context of a d=1+1 S-matrix Ansatz in which S factorizes into 2-particle elastic components $S^{(2)}$. A

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[22]The structural similarity between the cyclicity of the crossing- with the KMS-property has lured many authors (including the present author [36]) into conjectures that crossing has a KMS interpretation in the setting of wedge-localized algebras of the original theory. These conjectures (including "proofs" [37]) are incorrect.

consequence of this simplification is that the classification and calculation of factorizing S-matrices [43] can be separated from the problem from the construction of the associated off-shell QFT. Hence the S-matrix bootstrap becomes the first step in a bootstrap-formfactor program, followed by a second step which consists in calculating generalized formfactors of fields and operators beyond the identity operator (which represents the S-matrix). One does not expect such a two-step approach to be possible beyond factorizable models, rather the construction of the S-matrix (which may be considered as the special formfactor of the identity operator between in-out multi-particle states) is expected to have to be carried out as part of the formfactor construction.

It is interesting to note that the calculated formfactors of those factorizing models which possess continuously varying coupling parameters turn out to be analytic functions (below the threshold of formation of bound states) with a finite radius of analyticity around zero coupling strength [35][44]. For the correlation function on the other hand one does not expect expandability into a power series since their perturbative structure is not visibly different from that of other strictly renormalizable models and there exist general arguments against the convergence of perturbative series. This raises the interesting question of whether such a dichotomy between perturbatively converging on-shell objects versus nonconverging (at best asymptotic) series for off-shell correlation functions may continue to hold in general. It would be quite startling if on-shell quantities as formfactors in renormalizable field theories have improved perturbative convergence properties which are not shared by correlation functions.

The main motivating idea in favor of an on-shell approach, namely the total avoidance of ultraviolet divergences, is convincingly vindicated in the setting of factorizing models. The pointlike fields, which in the present state of development of factorizing models are only known via their multi-particle formfactors [44], have an interesting interaction-induced vacuum structure in that they possess no PFGs localized in subwedge regions. In other words despite their lack of real particle creation through scattering processes, they nevertheless have the full vacuum polarization structure which one expects in an interacting QFT and which in turn is the prerequisite for the appearance of interaction-caused anomalous short distance dimensions. In this respect of short distance behavior factorizing models are more realistic than the (non-factorizing) polynomial interactions in d=1+1 whose complete mathematical control was achieved with the methods of "constructive QFT" [45] (see also[46] for recent applications in a more algebraic QFT setting). It seems that the "hard analysis" methods of the constructivists are restricted to superrenormalizable models whose short distance behavior is not worse than that of free fields, whereas presently the modular methods, which avoid using singular field coordinatizations altogether, work best for factorizing models.

Contrary to the cluster property and macro-causality which, as we have seen, are also implemented in the relativistic particle-based theory of "direct particle interactions" [7]), crossing is the characteristic imprint which relativistic micro-causality leaves in on-shell restrictions of QFT. Although the on-shell aspects of d=1+1 factorizing models appear at first sight associated with a kind of one-dimensional relativistic particle-conserving quantum mechanics (due to the absence of real particle creation via scattering), a closer look reveals a significant difference which already makes itself felt on the level of the particle-conserving S-matrix. Its crossing property leads to a bound state picture which has become known under the name "nuclear democracy" as opposed to the quantum mechanical hierarchy with respect to the issue of bound versus elementary issue. Nuclear democracy is the statement that in interacting QFT all stable one-particle one-particle states are on the same footing apart from their superselected charges which are subject to hierarchical fusion laws. The hierarchy is that between fundamental and composite (fused) superselected charges, whereas the particles are the asymptotically stable carries of unconfined charges.

If, as e.g. in the case of the Sine-Gordon model, one still misses operators which carry fundamental charges which cannot be obtained by fusion (but rather permit to represent the charges of the known particles as fused fundamental charges), then the representation theoretical approach of the superselection theory in the setting of AQFT reconstructs the missing charges and particles. The reason why the presence of the latter is easily overlooked in the standard formalism is that these more fundamental particles do not appear directly, but only manifest themselves through particle-antiparticle vacuum polarization "clouds" in intermediate states of correlation functions. The theory of superselection sectors extends the original theory in such a way that these new charges and their possible particle carriers are naturally incorporated so that their scattering can be described in terms of interpolating fields. It is the principle of locality which permits the construction of full-fledged field algebras from observable algebras and arrive in this

way at a fundamental understanding of the concept of internal symmetries as a consequence of the local representation theory of observable nets of operator algebras [23].

The fact that the bootstrap-formfactor approach to factorizable models does not need special prescriptions, but that its "axioms" [42] follow from general principles of QFT becomes particularly transparent if the construction is placed into the setting of Tomita-Takesaki modular theory of operator algebras as adapted to the local quantum physics setting (also referred to as the method of modular localization) [28][47][48]. This will be illustrated in some detail in the next section.

This setting also highlights the "existence problem of QFT" in a new and promising fashion [30][49]. Here we remind the reader that even after almost eight decades after its discovery, and despite impressive perturbative and asymptotic successes, the description of interacting particles by covariant fields in 4-dimensional Minkowski spacetime remained part of mathematically as well as conceptually uncharted territory. This applies in particular to the "standard model" which is a source of a very specific permanent discomfort unknown in other areas of theoretical physics. The predictive success of this model, if anything, highlights the seriousness of this problem which without that success would be of a more academic nature.

The algebraic basis of the bootstrap-formfactor program for the special family of d=1+1 factorizable theories is the validity of a momentum space Zamolodchikov-Faddeev algebra [40]. The operators of this algebra are close to free fields in the sense that their Fourier transforms are on-shell (see 28 in next section) objects, but they are non-local in the pointlike sense. A closer look reveals that they are localizable in the weaker sense of generating wedge algebras[23] [28][48]. In fact the existence of "tempered" (existence of a well-defined Fourier transform) wedge localized PFGs which implies the absence of real particle creation through scattering processes [47] turns out to be the prerequisite for the success of the bootstrap-formfactor program for factorizable models in which one uses only formfactors and avoids (short-distance singular) correlation functions.

According to an old structural theorem which is based on certain analytic properties of a field theoretic S-matrix [50][47], virtual particle creation without real particle creation is only possible in d=1+1 theories. This in principle leaves the possibility for direct 3- or higher- particle elastic processes beyond two particle scattering. An argument by Karowski (private communication) based on formfactor crossing shows that this is inconsistent with the absence of real particle creation. In this sense the Z-F algebra structure, which is at the heart of factorizing models, turns out to be a consequence of special properties of PFG for modular wedge-localization, a fact which places the position of the factorizing models within QFT into sharper focus. The crossing property is encoded into the two-particle scattering amplitude from where it is subsequently passed on to the formfactors. In line with the previous unicity argument of inverse scattering based on crossing, the bootstrap formfactor approach associates precisely one local equivalence class of fields (one net of localized operator algebras) to a factorizing S-matrix. It also goes a long way in securing the existence of operators whose matrix elements in multi-particle states give rise to these explicitly computed formfactors.

In agreement with the philosophy underlying AQFT, which views pointlike fields as coordinatizations of generators of localized algebras, the bootstrap-formfactor construction for d=1+1 factorizing models primarily aims to determine coordinatization-independent double-cone algebras by computing intersections of wedge algebras. The nontriviality of a theory is then tantamount to the nontriviality ($\neq \mathbb{C}\mathbf{1}$) of such intersections[24]. The computation of a basis of pointlike field generators of these algebras is analogous but more involved than the construction of the basis of composites of free fields which are the Wick polynomials. Even for noninteracting theories the functorial description of the algebras (7) based on modular localization is conceptually simpler than the use of free fields (10) and their local equivalence class of Wick-ordered composites.

The crossing property is the crucial property which links scattering data with off-shell operators spaces. As explained in the previous section, it relates the multiparticle component of vectors obtained by one-time application of a local (at least wedge-localized) operator to the vacuum with the connected formfactors of this operator. It is importanr to note that in factorizing models crossing is not an assump-

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[23] An operator which is localizable in a certain causally closed spacetime region is automatically localized in any larger region but not necessarily in a smaller region. The unspecific terminology "non-local" in the literature is used for any non pointlike localized field.

[24] See a recent review [51] in which the minimal formfactor contributions, which are a joint property of the local equivalence class, have been separated from the polynomial contributions (the "p-functions") which distinguish between the vacuum polarization contributions of individual fields.

tion but rather follows from the properties of tempered PFGs for wedge algebras.

It is not easy to think of a formfactor approach beyond factorizing models. We will present an operational idea of crossing which in principle does not suffer from the above limitations of temperate PFGs, although one is presently only able to test it in the d=1+1 factorizing setting. It is based on the working hypothesis that each quantum field theory possesses a distinguished field called a "masterfield" whose connected parts of its formfactors defines a global (i.e. no local substructure) quantum field theory in the on-shell momentum space variables. This auxiliary theory is in a thermal state at the KMS Hawking temperature in such a way that the cyclic KMS property (the thermal aspect of modular theory) is identical with the cyclic crossing property. By construction this theory obeys momentum space cluster decomposition properties in the rapidity variables. The simplicity of d=1+1 factorizing models finds its expression in the fact that the auxiliary operator, whose KMS correlation functions are identified with the connected formfactors of the masterfield, is an exponential of a bilinear expression in free creation/annihilation operators. There is a good chance that this structure is characteristic for factorizing models.

The subsequent content of the paper is organized as follows. The next section recalls some details about the role of the Zamolodchikov-Faddeev algebra in the generation of the modular wedge-localized operator algebra. After that we will present two ideas which could be important in modular localization-based constructions without assuming factorizability. One of these ideas consists in postulating the already mentioned "masterfield" which generalizes observations on cluster properties in momentum-rapidity space [51] as well as observations on "free field representations" of formfactors in factorizing models [53]. Another less speculative idea is to classify and construct theories from their holographic lightfront projections, which will be the subject of the last section before we present some conclusions.

4 The Zamolodchikov-Faddeev algebra and its relation to modular localization

In this section we recall some details about how the modular localization formalism supports the bootstrap-formfactor construction.

It has been my Leitmotiv for a number of years [36] that the spirit behind Wigner's representation theoretical approach enriched with the concept of modular localization (as presented in the second section) could lead to a truly intrinsic constructive approach in QFT which avoids those classical quantization crutches which already the protagonist of field quantization Pascual Jordan wanted to overcome. It was natural to test this idea first in models which are similar to free field models in that their wedge-localized algebras can be generated by fields which posses on-shell Fourier transforms.

In the previous section we learned that this class is related with the Zamolodchikov-Faddeev algebra structure. In the simplest case of a scalar chargeless particle without bound states[25] the wedge generators are of the form [28]

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int (e^{ip(\theta)x(\chi)} Z(\theta) + h.c.)d\theta \tag{28}$$

$$Z(\theta)Z^*(\theta') = S^{(2)}(\theta - \theta')Z^*(\theta')Z(\theta) + \delta(\theta - \theta')$$

$$Z(\theta)Z(\theta') = S^{(2)}(\theta' - \theta)Z(\theta')Z(\theta)$$

Here $p(\theta) = m(ch\theta, sh\theta)$ is the rapidity parametrizations of the d=1+1 mass-shell and $x = r(sh\chi, ch\chi)$ parametrizes the right hand wedge in Minkowski spacetime; $S^{(2)}(\theta)$ is a structure function of the Z-F algebra which is a nonlocal *-algebra generalization of canonical creation/annihilation operators. The notation preempts the fact that $S^{(2)}(\theta)$ is the analytic continuation of the physical two-particle S-matrix $S^{(2)}(|\theta|)$ which via the factorization formula determines the general scattering operator S_{scat} (31). The unitarity and crossing of S_{scat} follows from the corresponding two-particle properties which in terms of

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[25]A situation which in case of factorizing models with variable coupling (as e.g. the Sine-Gordon theory) can always be obtained by choosing a sufficiently small coupling. Bound state poles in the physical θ-strip require nontrivial changes (e.g. the ϕ-generator is only wedge localized on the subspace of Z-particles) which will be dealt with eleswhere.

the analytic continuation are $S^2(z)^* = S^{(2)}(-z)$ (unitarity) and $S^{(2)}(z) = S^{(2)}(i\pi - z)$ (crossing) [43]. The $Z^*(\theta)$ operators applied to the vacuum in the natural order $\theta_1 > \theta_2 > ... > \theta_n$ are by definition equal to the outgoing canonical Fock space creation operators whereas the re-ordering from any other ordering has to be calculated according to the Z-F commutation relations e.g.

$$Z^*(\theta)a^*(\theta_1)a^*(\theta_2)...a^*(\theta_n)\Omega = \prod_{i=1}^{k} S^{(2)}(\theta - \theta_i)a^*(\theta_1)a^*(\theta_2)..a^*(\theta).a^*(\theta_n)\Omega \tag{29}$$

where $\theta < \theta_i\ i = 1..k$, $\theta > \theta_i\ i = k+1,..n$. The general Zamolodchikov-Faddeev algebra is a matrix generalization of this structure.

It is important not to identify the Fourier transform of the momentum with a localization variable. Although the x in $\phi(x)$ behaves covariantly under Poincaré transformations, it is not marking a causal localization point; in fact it is non-local variable in the sense of the standard use of this terminology[26]. It is however wedge-localized in the sense that the generating family of operator for the right-hand wedge W Wightman-like (polynomial) algebra $alg\{\phi(f), suppf \subset W\}$ commutes with the TCP transformed algebra $alg\{J\phi(g)J, suppg \subset W\}$ which is the left wedge algebra [48]

$$[\phi(f), J\phi(g)J] = 0 \tag{30}$$
$$J = J_0 S_{scat}$$

Here J_0 is the TCP symmetry of the free field theory associated with $a^\#(\theta)$ and S_{scat} is the factorizing S-matrix which on (outgoing) n-particle states has the form

$$S_{scat}a^*(\theta_1)a^*(\theta_2)...a^*(\theta_n)\Omega = \prod_{i<j} S^{(2)}(\theta_i - \theta_j)a^*(\theta_2)...a^*(\theta_n)\Omega \tag{31}$$

if we identify the $a^\#(\theta)$ with the incoming creation/annihilation operators. It is then possible to give a rigorous proof [48] that the Weyl-like algebra generated by exponential unitaries is really wedge-localized and fulfills the Bisognano-Wichmann property

$$\mathcal{A}(W) = alg\left\{e^{i\phi(f)} \mid suppf \subset W\right\} \tag{32}$$
$$\mathcal{A}(W)' = J\mathcal{A}(W)J = \mathcal{A}(W')$$

where the dash on operator algebras is the standard notation for their von Neumann commutant and the dash on spacetime regions stands for the causal complement. Within the modular setting the relative position of the causally disjoint $A(W')$ depends via S_{scat} on the dynamics. The operator TCP operator J is the (antiunitary) angular part of the polar decomposition of Tomita's algebraically defined unbounded antilinear S-operator with the following characterization

$$SA\Omega = A\Omega,\ A \in \mathcal{A}(W) \tag{33}$$
$$S = J\Delta^{\frac{1}{2}},\ \Delta^{it} = U(\Lambda(-2\pi t))$$

with $\Lambda(\chi)$ being the Lorentz boost at the rapidity χ.

At this point the setup looks like relativistic quantum mechanics since the $\phi(f)$ (similar to genuine free fields if applied to the vacuum) do not generate vacuum polarization clouds. The advantage of the algebraic modular localization setting is that vacuum polarization is generated by algebraic intersections which is in agreement with the intrinsic definition of the notion of interaction presented in terms of PFGs in the previous section

$$\mathcal{A}(D) \equiv \mathcal{A}(W) \cap \mathcal{A}(W_a') = \mathcal{A}(W) \cap \mathcal{A}(W_a)' \tag{34}$$
$$D = W \cap W_a'$$

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[26]The world local is reserved for "commuting for spacelike distances". In this work we are dealing with non-local fields which are nevertheless localized in causally complete subregions (wedges, double cones) of Minkowski spacetime.

This is the operator algebra associated with a double cone D (which is chosen symmetric around the origin by intersecting suitably translated wedges and their causal complements). Note the difference from the quantization approach, where pointlike localized fields are used from the outset and the sharpening of localization of smeared products of fields is simply achieved by the classical step of restricting the spacetime support of the test functions. The problem of computing intersected von Neumann algebras is in general not only difficult (since there are no known general computational techniques) but also very unusual as compared to functional integral representation methods related to Lagrangian quantization.

The problem becomes more amenable if one considers instead of operators their formfactors i.e. their matrix elements between incoming ket and outgoing bra state vectors. In the spirit of the old LSZ formalism one can then make an Ansatz in form of a power series in $Z(\theta)$ and $Z^*(\theta) \equiv Z(\theta - i\pi)$ (corresponding to the power series in the incoming free field in LSZ theory). In a shorthand notation which combines both frequency parts we may write

$$A = \sum \frac{1}{n!} \int_C ... \int_C a_n(\theta_1, ...\theta_n) : Z(\theta_1)...Z(\theta_n) : d\theta_1...d\theta_n \qquad (35)$$

where each integration path C extends over the upper and lower part of the rim of the $(0, -i\pi)$ strip in the complex θ-plane. The strip-analyticity of the coefficient functions a_n expresses the wedge-localization of A[27]. It is easy to see that these coefficients on the upper part of C (the annihilation part) are identical to the vacuum polarization form factors of A

$$\langle \Omega \, |A| \, p_n, ..p_1 \rangle^{in} = a_n(\theta_1, ...\theta_n), \quad \theta_n > \theta_{n-1} > ... > \theta_1 \qquad (36)$$

whereas the crossing of some of the particles into the left hand bra state (see the previous section) leads to the connected part of the formfactors

$$^{out}\langle p_1, ..p_l \, |A| \, p_n, ..p_{l+1}\rangle^{in}_{conn} = a_n(\theta_1 + i\pi, ...\theta_l + i\pi, \theta_{l+1}, ..\theta_n) \qquad (37)$$

Hence the crossing property of formfactors is encoded into the notation of the operator formalism (35) in that there is only one analytic function a_n which describes the different possibilities of placing θ on the upper or lower rim of C. This is analogous to the Glaser-Lehmann-Zimmermann expansion formulas [54] of the interacting Heisenberg fields in terms of free fields in which the nth term is the on-shell value of the Fourier transform of a retarded function which combines the different formfactors for fixed n.

In terms of the formfactors the relative commutant (34) results from restricting the series (35) by requiring that the $A's$ commute with the generators of the shifted algebra $\mathcal{A}(W_a)$

$$[A, U(a)\phi(f)U(a)^*] = 0 \qquad (38)$$

Thanks to the simplicity of the wedge generators $\phi(f)$, the Z series of the commutator can be computed in terms of the a_n. The linearity of $\phi(f)$ in the $Z's$ results in the nth term being a linear combination of a_{n-1} and a_{n+1}. The denseness of W localized functions and the analyticity in the open strip finally lead to the equivalence of the vanishing of this commutator with the famous "kinematical pole condition", namely the a_{n-1} function can be expressed as a residuum of a pole in a_{n+1}

$$Res_{\theta_{12}=i\pi} a_n(\theta_1, \theta_2, ...\theta_n) = 2ia_{n-2}(\theta_3, ...\theta_n)(1 - S_{2n}...S_{23}), \; \theta_{12} = \theta_1 - \theta_2 \qquad (39)$$

This relation was first postulated as one of the construction recipes by Smirnov [42]; it is the only relation between different components in the absence of bound states. This together with the Payley-Wiener Schwartz analytic characterization of the localization region and the crossing property (which links the crossed formfactor to the analytic continuation between the two rims of the θ-strip $\mathbb{R} + i(0, \pi)$) characterizes the *space of formfactors* associated with the algebra $\mathcal{A}(D)$. Attempts to improve the localization by restricting the support of f in the $\mathcal{A}(W)$ generators $\phi(f)$ to a smaller region $suppf \subset \mathcal{D} \subset W$ would fail; the generator continues to be wedge-localized and by sharpening test function supports one can only enlarge but not reduce the localization region.

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[27]Compact localization leads to coefficient functions which are meromorphic outside the open strip [35].

The multiplicative structure[28] is outside of mathematical control as long as one is unable to take care of the convergence of the infinite sums; in this respect the situation is at first sight not better than that of the old GLZ expansion formulas [54] for interpolating Heisenberg fields in terms of out/in free fields in which the coefficient functions are on-shell restrictions of retarded correlation functions. The linear space of formfactors can be parametrized in terms of a covariant basis which corresponds to the formfactors of a basis of "would be" composite fields. It turns out that the dependence on the individual composite field in the Borchers class of relatively local fields can be encodes into a polynomial factor [35] (after splitting off a common factor which is the same for all fields in the same class). This tells us that if we knew that those operator subalgebras characterized by the vanishing of the relative commutant (38) are nontrivial, then the associated quantum field theory exists as a algebraically nontrivial theory and we have a nonperturbative formalism to compute formfactors of pointlike fields or of more general operators in $\mathcal{A}(D)$.

Since the formalism only involves formfactors but avoids correlation functions of pointlike fields, it is free of ultraviolet problems (and a fortiori does not require renormalization of infinities). Hence the world of factorizing models is a candidate for the first explicit illustration of Pascual Jordan's envisaged paradise of local quantum physics where one is able to walk without classical crutches.

There has been extensive work on the calculation of formfactors of composite fields. Similar to Wick polynomials there exists a basis of (composite) fields in the same superselection sector. As mentioned, the formfactors of fields from the same local equivalence class contain one factor which is common to all of them (the so-called minimal formfactor [35]); this factor is associated with the "core" of the vacuum polarization cloud which is common to all states created by operators from the same spacetime region and with the same charge. It is this factor which carries the interaction; the remaining polynomial factor is in the exponential of the rapidities carries the information about the different fields in the local equivalence class; this is analogous to the different Wick polynomials of free fields.[29] The polynomial factors actually complicate the calculation of correlation functions as convergent series over formfactors. In fact apart from two-point functions in very special cases, the program of controlling correlation functions of pointlike fields was without much success, despite many attempts. The short distance aspects, which were banned thanks to the on-shell nature of the bootstrap formfactor program, enter through the back door in the form of convergence problems for the series (35).

In this context it is very interesting to note that recently Buchholz and Lechner [30] proposed an elegant criterion for the nontriviality of $\mathcal{A}(D)$ in terms of an operator algebraic property of the wedge algebra $\mathcal{A}(W)$ which allows to bypass the problem of controlling formfactor series altogether. They found that the "nuclear modularity" of $\mathcal{A}(W)$ insures the nontriviality of the $\mathcal{A}(D)$ intersection and its standardness (the Reeh-Schlieder property) with respect to the vacuum. Lechner tested this criterion in the case of the Ising field theory [49]. There seems to be a well-founded hope that the already impressive calculational results of the bootstrap-formfactor program for factorizing models will be backed up by a structural argument of the existence of their local algebras without having to control the convergence of infinite sums over formfactors. Although the knowledge of wedge algebras already determines the algebras associated with intersections uniquely, the Buchholz-Lechner idea applies only to d=1+1 theories.

In the following two sections I will present ideas by which one hopes to generalize the formfactor bootstrap approach.

5   The hypothesis of a Masterfield

For factorizable models, the crossing relation of the analytic coefficient functions in the series representation (35) is a consequence of the algebraic properties of the $Z's$. Since there are no Z-F operators for models with non factorizing S-matrices, one must look for a more general operational formulation of crossing. In order to obtain an idea in what direction to look for, let us first recall the precise conceptual position of factoring models within the general setting of massive models with a mass gap (to which scattering theory applies).

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[28] If the formfactors are matrixelements of operators, they must also have a multiplicative structure which corresponds to sums over infinitely many multi-particle states.

[29] This factor is different for bounded operators $A \in \mathcal{A}(D)$ where one obtains a decrease for large momenta which may help in the control of the convergence in (35).

As was mentioned in the second section, PFGs with generating properties for wedge-localized algebras only exist for d=1+1 theories with S-matrices which factorize into 2-particle contributions $S^{(2)}$. This is a very peculiar situation in which cluster separability does not distinguish between the two contributions in $S^{(2)} = 1 + T^{(2)}$ since they carry the same energy-momentum delta functions.

So the crucial question is how can one get an operational formulation of crossing in formfactors[30] beyond such special situations? We already dismissed the idea of interpreting crossing as KMS property in the same theory as incorrect. The only alternative idea which maintains a KMS interpretation of crossing, would consist in declaring simply the formfactors of an operator A to be correlation functions in a KMS state at the Hawking-Unruh temperature 2π of (nonlocal) operators $R^{(A)}$ in rapidity momentum space (the auxiliary $R's$ will be referred to as "Rindler operators")

$$\langle \theta_1, ...\theta_n \,|A|\, \Omega \rangle \overset{?}{=} \left\langle R^{(A)}(\theta_1)...R^{(A)}(\theta_n) \right\rangle \tag{40}$$

But this idea only works if we find special operators A in the original theory whose formfactors define a system of positive R-correlation functions, since then the GNS reconstruction would lead to a global auxiliary operator field theory. A necessary condition for such an interpretation is the validity of the cluster separation property. It is known that this property holds also in global operator algebras (i.e. algebras without a local net substructure) as long as the operator algebra is a von Neumann factor in which case it is related to the property of asymptotic abelieness [23][55]. In many factorizable models one was able to identify such fields with rapidity space clustering [56]. We will formulate a requirement, which we call the hypothesis of a "masterfield"

Definition 2 *A masterfield $M(x)$ associated to a QFT is a distinguished scalar Boson field within the Borchers class of locally equivalent fields whose connected formfactors defines a thermal auxiliary "Rindler QFT" at the Hawking temperature $\beta = 2\pi$ in terms of a nonlocal field $R(\theta, p_\perp)$ in the sense of the above formula (40) with A being the masterfield $M(x)$ at x=0.*

The KMS relation in θ reads

$$\left\langle R^{(M)}(\theta_1, p_{1\perp})...R^{(M)}(\theta_{n-1}, p_{n-1\perp})R^{(M)}(\theta_n, p_{n\perp}) \right\rangle_{\beta=2\pi} \tag{41}$$
$$= \left\langle R^{(M)}(\theta_n - 2\pi i, p_{n\perp})R^{(M)}(\theta_1, p_{1\perp})...R^{(M)}(\theta_{n-1}, p_{n-1\perp}) \right\rangle_{\beta=2\pi}$$
$$= (R^{(M)}(\theta_n - 2\pi i, p_{n\perp})^* \Omega_{\beta=2\pi}, R^{(M)}(\theta_1, p_{1\perp})...R^{(M)}(\theta_{n-1}, p_{n-1\perp})\Omega_{\beta=2\pi})$$
$$= (JR^{(M)}(\theta_n - \pi i, p_{n\perp})\Omega_{\beta=2\pi}, R^{(M)}(\theta_1, p_{1\perp})...R^{(M)}(\theta_{n-1}, p_{n-1\perp})\Omega_{\beta=2\pi})$$

where in the last two lines we used the more convenient state-vector notation for the thermal expectation values and modular theory in order to convert $\Delta^{\frac{1}{2}}$ into J. The identification of this expression with the crossing property of the formfactor of $M(0)$

$$\langle 0 \,|M(0)|\, p_1, ...p_n \rangle = \langle -\bar{p}_n \,|M(0)|\, p_1, ...p_{n-1} \rangle \tag{42}$$
$$= (JR^{(M)}(\theta_n - \pi i, p_{n\perp})\Omega_{\beta=2\pi}, R^{(M)}(\theta_1, p_{1\perp})...R^{(M)}(\theta_{n-1}, p_{n-1\perp})\Omega_{\beta=2\pi})$$

requires the action of the auxiliary J as $JR^{(M)}(\theta_n - \pi i, p_{n\perp})\Omega_{\beta=2\pi} = R^{(M)}(\theta_n - 2\pi i, -p_{n\perp})^*\Omega_{\beta=2\pi}$. In d=1+1 the interpretation of crossing in terms of KMS of an auxiliary theory simplifies, since there is no transverse momentum $p_\perp$.

It is important to notice that the auxiliary field theory associated with the formfactors of the master field is not subject to the restriction of wedge-localized PFGs which led to factorizable models. In fact being a global (i.e. without a local net structure) KMS theory, the concept of particles and in particular the concept of PFG becomes meaningless.

Let us first look at the rather trivial illustration of a free master field namely

$$M(x) \equiv : e^{\gamma A(x)} :, \; A(x) = free\,field \tag{43}$$
$$(\Omega \,|M(0)|\, p(\theta_1, p_{1,\perp})...p(\theta_n, p_{n,\perp})) = e^{c\gamma}$$

---

[30]We always mean the connected part of the formfactors which is what one gets by starting with the outgoing components of the one field (or operator from a local algebra) state and crossing from outgoing bras to incoming kets.

where the positive constant c is related to the vacuum-one particle normalization of A. Clearly among all composites of the free field which lead to θ-independent connected formfactors, the only case with the correct combinatorics complying with clustering is the above exponential field. The auxiliary algebra of $R^{(M)}$ is the trivial abelian algebra which permits states for every KMS temperature. The free field is also the only model in which the formfactors of the masterfield define an abelian auxiliary theory; a nontrivial S-matrix prevents abelienness.

The masterfield hypothesis remains nontrivial even in the setting of factorizable models. In the following we use two quite different models to illustrate its working. We first recall some formalism of KMS states on free fields.

For bosonic quasifree KMS states at the KMS temperature β one obtains

$$\langle c(q)c(q')\rangle_\beta = e^{\beta q}\left\{\langle c(q)c(q')\rangle_\beta - [c(q),c(q')]\right\} \tag{44}$$

$$\curvearrowright \langle c(q)c(q')\rangle_\beta = \frac{e^{\beta q}}{e^{\beta q}-1}\varepsilon(q)\delta(q+q')$$

For the first illustration we take the Sinh-Gordon theory. The field which leads to formfactors which have the cluster factorization property in the rapidity variable is again an exponential $M(x)=e^\varphi$ operator in terms of the basic Sinh-Gordon field φ [56]. They are known to have the following structure

$$(\Omega\,|M(0)|\,p(\theta_1)...p(\theta_n)) = K_n(\underline{\theta})\prod_{i<j\leq n}F(\theta_{ij}) \tag{45}$$

$$K_n(\underline{\theta}) = \sum_{l_1=0,1}...\sum_{l_n=0,1}(-1)^{\sum l_i}\prod_{i<j}(1+(l_i-l_j)\frac{i\sin\pi\nu}{\sinh\theta_{ij}})\prod_k Ce^{i\pi\frac{\gamma}{\beta}(-1)^{l_k}}$$

where the coupling strength β and ν are related by $\frac{1}{\nu}=\frac{8\pi}{\beta^2}$. The product factor involves the 2-particle formfactor F and has the combinatorics of an exponential which is bilinear in $c(q)$ free Boson operators. This suggests to start from the complex exponential

$$C(\theta) = e^{ia(\theta)} \tag{46}$$

$$a(\theta) = \int dq w(q)c(q)e^{iq(\theta-i\frac{\pi}{2})}$$

and look for a Rindler operator R[31] as a Hermitian combination of the form

$$R(\theta) = N\left\{e^{i\gamma}C(\theta-i\frac{\pi}{2})+h.a.\right\} \tag{47}$$

$$C(\theta)C(\theta') = S^{(2)}(\theta-\theta')C(\theta')C(\theta),\ S^{(2)}(\theta)=exp\int_0^\infty dqf(q)sinhq\frac{\theta}{i\pi}$$

$$\left\langle C(\theta-i\frac{\pi}{2})C(\theta'-i\frac{\pi}{2})\right\rangle_{2\pi} = exp\int_0^\infty dqf(q)\frac{1-chq(\pi+i(\theta-\theta'))}{2sh\pi q}$$

$$\equiv F_{min}(\theta-\theta')$$

The function $F_{min}(\theta)$ is the so-called minimal 2-particle formfactor of the model i.e. the unique function which obeys $F(\theta)=S^{(2)}(\theta)F(-\theta)$ and is holomorphic in the strip. For the present model without bound states it agrees with F. In the last line in (47) we used the fact that the KMS state at the inverse temperature 2π fixes the quasi-free state on the Rindler creation/annihilation operator algebra which in turn determines the thermal expectations of the C-operators.

The Sinh-Gordon S-matrix

$$S_{sh}(\theta) = \frac{th\frac{1}{2}(\theta-i\kappa)}{th\frac{1}{2}(\theta+i\kappa)},\ \kappa = \frac{\pi\beta^2}{8\pi+\beta^2} = \pi B \leq \pi \tag{48}$$

---

[31]We use the letter Z for the particle physics representation of the Zamolodchikov algebra (the Minkowski spacetime operators which are related to the PFG wedge generators) whereas R is used the thermal Rindler representation.

fixes the quasifree commutation relation of the Rindler operators $R(\theta)$ with

$$f(q) = \frac{2sh\frac{\kappa q}{2}sh\frac{\kappa q 8\pi}{2\beta^2}}{qch\frac{\pi q}{2}} = \frac{2sh\frac{q\pi}{2}Bsh\frac{q\pi}{2}(2-B)}{qch\frac{\pi q}{2}} \tag{49}$$

The n-point function

$$\left\langle C(\theta_1 - i\frac{\pi}{2})....C(\theta_n - i\frac{\pi}{2})\right\rangle_{2\pi} \sim \prod_{i<k} F_{min}(\theta_{ik}) \tag{50}$$

fulfills the commutation relation of the R-algebra (which is identical to that of the C-algebra as well as the KMS condition. Our interest lies in the Hermitian field operator R. For convenience we have adjusted our notation to the resulting combinatorics for the thermal Z-expectation which are sums of terms with different $C_l(\theta) := C(\theta - il\frac{\pi}{2})$, $l = \pm$

$$\langle C_{l_n}...C_{l_1}\rangle_{2\pi} = \prod_{i<k} F_{min}(\theta_{ik}) \left\{1 - (l_k - l_i)\frac{isin\kappa}{sh\theta_{ki}}\right\}, \quad l_i = \pm 1 \tag{51}$$

$$\langle R(\theta_n)...R(\theta_1)\rangle_{\beta=2\pi} \sim \sum \langle C_{l_n}...C_{l_1}\rangle_{2\pi} \, exp i\pi\alpha(l_1 + l_2 + ...l_n)$$

where α depends on the numerical pre-factors.

With the present construction of an auxiliary global Rindler QFT for the formfactors of the masterfield we have reproduced a curious observation by Lukyanov [53] which is known in the literature on factorizing models as "free field representations" (for a recent account see also [57]). The difference to Lukyanov is in the underlying concepts and not in the actual computation. The thermal state turned out to be a Rindler-Unruh KMS state at a fixed Hawking temperature rather than a tracial Gibbs states in a heat bath setting. Unique KMS states on operator algebras lead to von Neumann factors which in turn fulfill weak cluster property [55] and it was the cyclicity of crossing together with the somewhat mysterious cluster properties in the rapidity variables [56] which suggested this operator KMS interpretation of the crossing property for the formfactors of a masterfield.

Since fields whose formfactors cluster have been found in many similar factorizing models of Toda type [32], one would expect that the idea of an auxiliary Rindler theory in momentum space works in all of them. Moreover it would be tempting to conjecture that the simplifying feature of factorizing models consists in the auxiliary formfactor theory being bilinear exponential in $c^{\#}(q)$ creation/annihilation operators. This conjecture draws also support from a recent observation by Babujian and Karowski who observed that a suitably generalized form of clustering for also holds in statistics changing Z_n-models [58] of which the lowest one is the Ising field theory. In that case a combination of disorder/order field formfactors leads to clustering [51].

In the following we briefly show that the masterfield idea also works in the Ising model; in that case the relevant state is a "twisted" KMS state at the temperature $\beta = \pi$. The twisting consists in changing the KMS formula by a -sign.

$$\langle c(q)c(q')\rangle_{\beta} = -e^{\beta q}\left\{\langle c(q)c(q')\rangle_{\beta} - [c(q), c(q')]\right\} \tag{52}$$

$$\langle c(q)c(q')\rangle_{\beta} = \frac{e^{\beta q}}{1 + e^{\beta q}}\varepsilon(q)\delta(q + q')$$

$$\langle c(q)c(q')\rangle_{\pi} = \frac{e^{\frac{\pi}{2}q}}{2cosh\frac{\pi}{2}q}\varepsilon(q)\delta(q + q')$$

where the third line contains the KMS two-point function at $\beta = \pi$ which we are going to use together with the following definition of $a(\theta)$

$$a(\theta) = \int c(q)dq \tag{53}$$

$$\langle a(\theta)a(\theta')\rangle_{\beta=\pi} = \int_0^{\infty} \frac{\sinh q(i\theta - i\theta' + \frac{\pi}{2})}{\cosh\frac{\pi}{2}q}\frac{dq}{q} = \ln i\tanh\frac{\theta - \theta'}{2}$$

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[32] In [58] it was shown that distinguished fields with clustering formfactors exist for all A_{n-1} affine Toda field theories of which the Sinh-Gordon is the lowest member.

which finally leads to the well-known disorder/order Ising formfactors which is given by a combinatorial expression in the two-particle formfactor of the disorder operator (which correspond to an even number of particles)

$$\left\langle e^{a(\theta)} e^{a(\theta')} \right\rangle_\pi = \tanh \frac{\theta - \theta'}{2} \tag{54}$$

As the Sinh-Gordon model is the simplest representative of the class of A_{n-1} affine Toda models [58], the Ising field theory is the first in the family of Z_n models. These models are more difficult as a consequence of their preferred Z_n braid group statistics and a candidate for a masterfield is not immediately visible. The suggestion from the Ising case would be that a suitable combination of all disorder/order operators would be a candidate for a field which fulfills some generalized clustering (i.e. adjusted to the exotic statistics).

The important point underlying the idea of a masterfield is that there exists an analytic correlation function (41) whose different boundary values in momentum rapidity space (41) correspond to different operator ordering. For factorizing S_{scat} matrices the close relation between transpositions and actions of $S^{(2)}$ suggested how to relate the different orderings to on-shell operator data. For general S_{scat}-matrices we could the opposite θ-order with the action of S_{scat}

$$|n, n-1, ...2, 1\rangle = S_{scat} |1, 2, ...n-1, n\rangle \tag{55}$$

but it is not obvious what kind of operator relation one should use for other orderings. Perhaps the cluster property leads to further restrictions which together with the KMS property permit to determine the auxiliary R-theory. In any case it seems to me that an operator interpretation of the different rapidity orderings in formfactors and the KMS property is an indispensable part of a deeper operator understanding of crossing and a (perturbative) on-shell construction.

The on-shell bootstrap-formfactor idea is not the only possibility to avoid short-distance problems resulting from the use of field coordinatizations and their singular correlations. Another less speculative but by no means simpler idea will be presented in the next section.

6 Lightfront Holography as a constructive tool?

In the previous sections we have used modular theory together with on-shell concepts in order to analyze wedge algebras in the presence of interactions. In this section I will present a recent proposal which also uses modular localization ideas in order to simplify the problem of classifying and constructing QFTs. But instead of particle concepts, as e.g. PFGs for wedge algebras and formfactors, it is based on the good understanding of chiral theories which are related to the actual theory by a process of "algebraic lightfront holography" (ALH).

Of course no non-perturbative approach to higher dimensional interacting QFT can achieve miracles; simplification just means the partition of a complex dynamical problem into a sequence of less complicated single steps. Perhaps the following comparison with the canonical formalism sheds additional light on this point. This ETC formalism tries to classify and construct QFTs by assuming the validity of canonical equal time commutation relations (ETCR). The shortcomings of that approach are well-known. Apart from the fact that in higher dimensional relativistic QFT the ETC structure is inconsistent with the presence of interactions, ETCR are not useful as a starting point for a rough intrinsic distinction between different (universality) classes of interactions since ETCR are totally universal.

Lightfront holography tries to address this imbalance by replacing the ETCR by the much richer structure of chiral theories on the lightfront. Starting from a richer "kinematical" setting than ETCR, one may hope for a more accessible "dynamical" side. The holographic projection may map different ambient theories to the same chiral image, but similar to the better known scale invariant short distance universality classes, the holographic universality classes allow for more realizations than the unique ETCR structure. However in contradistinction from scaling limits, holographic projections live in the same Hilbert space as the ambient theory; in fact they just organize the spacetime aspects of a shared algebraic structure in a radically different way.

Let us briefly recall the salient points of ALH[33].

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[33] We add this prefix "algebraic" in order to distinguish the present notion of holography from the gravitational holography of t'Hooft [60]. More on similarities and differences between the two can be found in the concluding remarks,

ALH may be viewed as a kind of conceptually and mathematically updated "lightcone quantization" (or "$p \to \infty$ frame" description). Whereas the latter approaches never faced up to the question of how the new fields produced by the lightfront quantization prescriptions are related to the original local fields i.e. in which sense the new description addresses the original problems posed by the ambient theory, the ALH is conceptually precise and mathematically rigorous on this points. It turns out that the idea of restricting fields to the lightfront is limited to free fields and certain superrenormalizable interacting models with finite wave function renormalization (which only can be realized in d=1+1). Theories with interaction-caused vacuum polarization which leads to Kallen-Lehmann spectral functions with diverging wave function renormalization factors do not permit lightfront restrictions for the same reason as they do not have equal time restrictions; e.g. for scalar fields on has[34]

$$\langle A(x)A(y) \rangle = \int_0^\infty \rho(\kappa^2) i \Delta^{(+)}(x-y, \kappa^2) d\kappa^2 \tag{56}$$

$$\langle A(x)A(y) \rangle |_{LF} \sim \int_0^\infty \rho(\kappa^2) d\kappa^2 \delta(x_\perp - y_\perp) \int_0^\infty \frac{dk}{k} e^{-ik(x_+ - y_+)}$$

where in passing to the second line we used the rule (18) which replaces the ambient two-point function of mass κ by its zero mass lightfront restriction in the sense of the second section. As explained there, the infrared-divergence in the longitudinal factor is spurious if one views the lightfront localization in the setting of modular wedge localization. On the other hand the obstruction resulting from the large κ divergence of the K-L spectral function (short distance regime of interaction-caused vacuum polarization) is shared with that in ETCR i.e. in both cases the process of restriction is meaningless.

However whereas equal time restricted interacting fields in d=1+3 simply do not exist, there is no such limitation on the short distance properties of generalized chiral conformal fields which turn out to generate the ALH. What breaks down is only the idea that these lightfront generating fields can be gotten simply by restricting the fields of the ambient theory, as was the case in the example of free fields in the second section.

It turns out that in algebraic lightfront holography the connection between the ambient theory and its holographic projection requires the use of modular theory. Although the ambient theory may well be given in terms of pointlike fields and the ALH may also allow a pointlike description (see 62), there is no direct relation between these fields. This also sheds light on the old difficulties with lightcone quantization which posed an obstacle to generations of physicists; even in the interaction-free case when the restriction works, the ALH net of algebras is nonlocal relative to the ambient algebra and hence the recovery of the ambient from the LF operators involves nonlocal steps. Whereas lightcone quantization was not able to address those subtle problems, ALH solves them.

The intuitive physical basis of this algebraic approach is a limiting form of the causal closure property. Let $\mathcal{O}$ be a spacetime region and $\mathcal{O}''$ its causal closure (the causal disjoint taken subsequently taken twice) then the causal closure property is the following equality

$$\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}'') \tag{57}$$

In the case of free fields this abstract algebraic property is inherited via quantization from the Cauchy propagation in the classical setting of hyperbolic differential equations. The lightfront is a limiting case (characteristic surface) of a Cauchy surface. Each lightray which passes through $\mathcal{O}$ either must have passed or will pass through O''. For the case of a $x^0 - x^3$ wedge W and its $x^0 - x^3 = 0$ (upper) causal lightfront boundary $LFB(W)$ (which is half of a lightfront) the relation

$$\mathcal{A}(LFB(W)) = \mathcal{A}(W) \tag{58}$$

is a limiting situation of the causal shadow property; a lightlike signal which goes through this boundary must have passed through the wedge (or in the terminology of causality, the wedge is the backward causal completion of its lightfront boundary). Classical data on the lightfront define a characteristic initial value problem and the smallest region which casts an ambient causal shadow is half the lightfront as in (58); for

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[34]It is important to realize that the LF restriction is not a pointwise procedure. The best understanding is achieved within the setting of modular localization (see below).

any transversely not two-sided infinite extended subregion, as well as for any region which is bounded in the lightray direction, the causal completion is trivial i.e. $\mathcal{O} = \mathcal{O}''$. This unusual behavior of the lightfront is related the fact that as a manifold with its metric structure inherited from the ambient Minkowski spacetime it is not even locally hyperbolic.

Some of the symmetries which the lightfront inherits from the ambient Poincaré group are obvious. It is clear that the lightlike translation together with the two transverse translation and the transverse rotation are leaving the lightfront invariant and that the longitudinal Lorentz boost, which leaves the wedge invariant, acts as a dilatation on the lightray in the lightfront. There are two additional invariance transformations of the lightfront which are less obvious. Their significance in the ambient space is that of the two "translations" in the 3-parametric Wigner little group $E(2)$ (a Euclidean subgroup of the 6-parametric Lorentz group) which leave the lightray in the lightfront invariant. Projected into the lightfront these "translations" look like transverse Galilei transformations in the various $(x_\perp)_i - x_+$ planes.

The resulting 7-parametric symmetry group of the lightfront is used to construct the modular localization structure of the ALH. For the longitudinal localization in the lightray direction the construction is based on the inclusion [59][61]

$$\mathcal{A}(W) \supset \mathcal{A}(W_{e_+}) \equiv AdU(e_+)\mathcal{A}(W) \tag{59}$$

where $\mathcal{A}(W_{e_+})$ is the image of $\mathcal{A}(W)$ under a translation e_+ along the lightray. This inclusion is known to be "half-sided modular" (hsm) i.e. the modular group of the larger algebra Δ_W^{it} compresses the smaller one for $t < 0$ (+ half-sided modular)

$$Ad\Delta_W^{it}\mathcal{A}(W_{e_+}) \subset \mathcal{A}(W_{e_+}), \ \ t < 0 \tag{60}$$

It is well-known [63] that such inclusions lead to Moebius covariant chiral nets precisely if they are "standard" i.e. if

$$\mathcal{A}(W_{e_+})' \cap \mathcal{A}(W)\Omega \ is \ dense \ in \ H \tag{61}$$

The lightlike inclusion is the limit of spacelike inclusions which in compactly localizable theories are evidently standard (but not hsm). This property is known to hold in the absence of interactions where it can be traced back to the spatial standardness of the respective subspaces of the Wigner representation space [62]. For factorizing models in d=1+1 this algebraic requirement is the prerequisite for the existence of pointlike fields in the bootstrap formfactor approach. The fact that the short distance behavior of these fields admit arbitrary high inverse powers suggests that this standardness assumption (unlike the lightcone quantization and the above lightfront restriction method) is not affected by short distance properties. Since our aim is the classification and construction of models, the range of validity of our method is at the end decided by its future success.

The interpretation of the chiral net obtained from the hsm inclusion for d>1+1 is that of a system of algebras associated with transverse "slices" (stripes in case of d=1+2) i.e. regions of finite longitudinal and two-sided infinite transverse extension. Note that the conformal rotation (or the proper conformal transformation), which requires the one-point compactification of the longitudinal coordinate, does not arise from the holographic projection of the Poincaré transformations, but rather results from the symmetry-improving aspect of lightfront holography [70].

In order to obtain the complete local resolution on the lightfront we still have to find a mechanism which generates a transverse localization structure. This is done with the help of "modular intersections". For this purpose we now use the aforementioned two "translations" in Wigner's little group $E(2)$. These transformations tilt the wedge W in such a way that its upper boundary remains inside the lightfront. The thickness of the slice in the lightray direction is maintained whereas the transverse directions are tilted in the sense of Galilei group actions. It is easy to see that the intersection of the algebras localized in the original slice with those of the tilted slice defines an algebra localized in a finite region. The net structure of the lightfront algebra is defined in terms of this intersected slice algebras. As modular inclusions of wedges are inexorably linked to dilation-translation symmetries, modular intersections of wedges are related to Wigner's little group $E(2)$ [64][65][66]. For more on the operator algebraic aspects of modular intersections we refer to the literature [67].

. The holographic projection method confirms that the vacuum polarization properties, which for free fields can be explicitly derived by the lightfront restriction method, continue to hold in the presence of

interactions. The most surprising aspect is certainly the total absence of transverse vacuum polarization which is a consequence of the following theorem on tensor factorization [68]

Theorem 3 *A von Neumann subalgebra $\mathcal{A}$ of $B(H)$ which admits a two-sided lightlike translation with positive generator is of type I, i.e. it tensor factorizes as $B(H) = \mathcal{A} \otimes \mathcal{A}'$ associated with $H = H_{\mathcal{A}} \otimes H_{\mathcal{A}'}$ and the factorization of the vacuum vector $\Omega = \Omega_{\mathcal{A}} \otimes \Omega_{\mathcal{A}'}$*

The transverse tensor factorization is corroborated by the application of the Takesaki theorem [29] which fits nicely into our modular based approach since it relates the existence of preservation of sub-algebras under the action of the modular group of the ambient algebra to the existence of conditional expectations.

Theorem 4 *The modular group of an operator algebra in standard position $(\mathcal{B}, \Omega)$ leaves a subalgebra $\mathcal{A} \subset \mathcal{B}$ invariant if and only if there exists a Ω-preserving conditional expectation $E : \mathcal{B} \to \mathcal{A}$. In that case the state $\omega(\cdot) = (\Omega, \cdot \Omega)$ is a factor state on $\mathcal{A} \times \mathcal{C}$ with $\mathcal{C} \equiv \mathcal{A}' \cap \mathcal{B}$ which leads a tensor factorization $H_{\mathcal{B}} = H_{\mathcal{A}} \otimes H_{\mathcal{C}}$ where the Hilbert spaces are cyclically generated from Ω by the application of the respective algebras.*

In the adaptation of this theorem to our problem we only have to set $\mathcal{B} = \mathcal{A}(LFB(W))$, $\mathcal{A} = \mathcal{A}(x_\perp \in Q, x_+ > 0)$, where Q is a compact region in the transverse coordinates.

These theorems clearly show that the holographic lightfront projection has a transverse quantum mechanical structure since tensor factorization upon subdivisions of spatial regions and factorization of the vacuum vector are the characteristic features of QM. This unexpected property of encountering quantum mechanical structures in relativistic QFT without having done any nonrelativistic approximation is a characteristic property of ALH. It is certainly related to the fact that the LF is not hyperbolic.

In addition to those symmetries inherited from the ambient theory there are new symmetries as the result of the "symmetry-improving" lightfront projection [70]. One of them is the vacuum-preserving conformal rotation (see later section for more comments).

It is interesting and useful to ask what kind of generating pointlike fields ψ could describe a holographic projection. The possibilities are severely limited by the transverse tensor factorization and the longitudinal chiral structure; they essentially amount to the following commutations structure (can be easily extended to include fermionic operators)

$$\left[\psi_i(x_\perp, x_+), \psi_j(x'_\perp, x'_+)\right] = \tag{62}$$

$$= \delta(x_\perp - x'_\perp) \left\{ \delta^{(n_{ij})}(x_+ - x'_+) + \sum_k \delta^{(n_{ijk})}(x_+ - x'_+)\psi_k(x_\perp, x_+) \right\}$$

where the common δ-function in the transverse direction takes care of the quantum mechanical property and the longitudinal structure parallels that known from the Lie-field structure of chiral observable algebras i.e. the ψ_i constitute a (finite or infinite) Lie-field basis and the sum extends over finitely many terms. As in the pure chiral case of W-algebras the number of the derivatives in the longitudinal δ-functions is controlled by the balance of scale dimensions on both sides of the equation.

The operators obtained by smearing with test functions $f(x_\perp, x_+)$ clearly produces the transverse quantum mechanical factorization as a result of the presence of just one δ-function without derivatives. Observables with nonoverlapping transverse extension factorize according to

$$\langle AA' \rangle = \langle A \rangle \langle A' \rangle \tag{63}$$

I believe that the existence of generating fields (62) for ALH can be similarly argued as in [69] where generating fields for ordinary chiral nets of algebras (without transverse extension) were constructed.

It is well-known that the wedge localization, and hence also the localization on its causal boundary $LFB(W)$, causes a thermal behavior; in more specific terms, the restriction of the vacuum to those localized algebras is indistinguishable from a thermal KMS state at a fixed temperature (the Unruh analog of the Hawking temperature) whose Hamiltonian is the generator of the W-affiliated Lorentz boost. The temperature for the thermal aspects caused by the quantum field theoretic vacuum polarization aspects

at the boundary of localization regions is related to the geometry of these regions; this is in marked contrast to the standard heat bath thermality which leads to freely variable temperatures and also exists in the classical setting. The absence of vacuum polarization in the transverse direction suggests that the localization-caused thermality leads to an entropy density i.e. an entropy per unit transverse volume which has the dimension of an area [65]. This is in marked contrast to the volume density of heat bath thermality and may well turn out to be the QFT prerequisite for the Bekenstein area law in the quasiclassical treatment of black holes.

In a constructive use of these ideas one would start with a classification of QFT on the lightfront in terms of extended chiral theories and aim at the reconstruction of the ambient theory as a kind of inverse ALH. The action of the 7-parametric invariance subgroup on the lightfront algebra is part of the ALH data. Their could be a restriction on the AHL data from the requirement that the three remaining transformation which together with the LF invariance group generate the ambient Poincaré symmetry act in a consistent way. In analogy with the many Hamiltonians one Certainly one has to expect many ways of Having arrived at the family of wedge algebras in terms of the ALH extended chiral algebra the remaining construction of the ambient algebraic net is then uniquely determined in terms of intersections.

Further inside can be gained by comparing the particle-based modular approach to factorizing models with ALH. The representation of the generators (28) in terms of on-shell Z-F creation/annihilation operators simplifies the calculation of the lightray limit $x_- = 0$. The method of lightfront restriction works exactly as in the case of d=1+1 free fields (18) except that corresponding formulas in terms of the Z-F operators only serve as generators of half-line algebras. The algebras of finite intervals have to be calculated as relative commutants by the modular inclusion formalism; the resulting infinite series in the Z-F operators are completely analogous to (35) in section 4. In terms of pointlike generating fields one has $(p_-(\theta) = me^\theta)$

$$A_{LF}(x_+) = \sum \frac{1}{n!} \int_C ... \int_C e^{ix_+(p_-(\theta_1)+...p_-(\theta_n))} a_n(\theta_1,...\theta_n) \cdot \tag{64}$$
$$: Z(\theta_1)...Z(\theta_n) : d\theta_1...d\theta_n$$

The corresponding ambient massive pointlike localized fields are of the form $(p_+(\theta) = me^{-\theta})$

$$A(x) = \sum \frac{1}{n!} \int_C .. \int_C e^{ix_+(p_-(\theta_1)+..+p_-(\theta_n))+ix_-(p_+(\theta_1+..+p_+(\theta_n))} a_n(\theta_1,...\theta_n) \tag{65}$$
$$: Z(\theta_1)...Z(\theta_n) : d\theta_1..d\theta_n$$
$$p_+ p_- = m^2$$

The additional exponents involving the total P_+ momentum can be thought of originating from a nonlocal "Hamiltonian" propagation law of the form $e^{iP_+ x_-}$

Apparently those chiral theories which arise as ALH projections[35] from factorizing models (and hence have PFGs in terms of Z-F variables) have a LF restriction which in terms of these variables is similar to that for free fields. In particular the covariance of the $Z's$ renders the extension into the ambient x_- direction unique. As mentioned before we do not expect such a uniqueness of the inverse lightfront holography beyond factorizing models, in particular for higher dimensional theories.

The calculation of the intersection spaces associated with intervals on the lightray is entirely analogous to that of the double cone intersections, in both cases one obtains infinite series (35) which applied to the vacuum lead to rich vacuum polarization clouds. This interplay between a massive 2-dimensional and chiral models is a new aspect of QFT since it does not depend on any approximation or scaling limit and is therefore somewhat surprising. It shows that at least certain chiral theories admit novel descriptions in terms of a 2-dimensional particle basis. Whereas the dilation-translation subgroup of the Moebius group leaves the vacuum as well as the holographic images of the massive one-particle states invariant, the Moebius rotation leaves the vacuum invariant but adds vacuum polarization clouds to the alias one-particle states. More investigations on this interesting point are required.

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[35]The reader should note that the relation between the holographic chiral projection and the ambient factorizing model is exact, whereas Zamolodchikov's working hypothesis is based on a construction of factorizing models from their chiral scaling limits by specific perturbations. Nevertheless there may be connection between holographic and scaling universality classes.

As a result of insufficient knowledge about higher dimensional models, there is presently no model illustration of the ALH in the presence of transverse directions.

7   Concluding remarks

In these notes we have been exploring nonperturbative ideas to access QFT without using the classical "crutches" inherent in Lagrangian quantization and without being subject to the severe short distance restrictions of the related canonical commutation relations or those of the euclidean functional integral representations[36] to make sense outside QM.

A common aspect is the important role which modular localization plays in these attempts. Without interactions, modular particle and field localizations are functorially related as expressed by the "commuting square" (8), but as a result of interaction-induced vacuum polarization the particle localization is lost apart from the existence of wedge-localized PFGs i.e. wedge-localized operators which applied to the vacuum create one-particle states free of vacuum polarization admixtures. If one in addition requires these operators to have reasonable domain properties with respect to translations (tempered PFGs), only the d=1+1 factorizing models remain. In the latter case it is also possible to formulate a quantum field theory of the system of formfactors of a distinguished field called the masterfield. Whether this masterfield idea has a higher dimensional generalization remains a matter of speculation.

An interesting link between the old S-matrix bootstrap program and the formfactor approach to QFT is the uniqueness of the inverse scattering problem in QFT. Although it says nothing about the existence of a QFT, it at least shows that if formfactors fulfill the crossing property, there is only one local off-shell extrapolation i.e. only one local net with a given S_{scat}. This is interesting in view of the historical relations of string theory to Veneziano's dual model in which crossing property was implemented with infinitely many particle states. Although this is quite distinct from how crossing is expected to be achieved in QFT where both the particle poles and the cuts from the scattering continuum enter the crossing relation (as can be exemplified by the S-matrices of factorizing models), the idea that the string prescriptions may turn out to be a just a differently formulated local quantum physics was never totally ruled out[37], despite many conflicting opinions.

One would feel more confident about this point, if crossing would have continued to play the same pivotal role in string theory as it did in the (genus zero) formulation of the dual model. But a glance at contemporary string theory indicates that it dropped out of sight; it is not even clear whether it holds at all. In this conceptually somewhat opaque situation it is interesting to note that very recently the local quantum physics interpretation of bosonic string field theory received some support from one of the protagonists of string theory [71] by indicating the possible construction of a (presumably infinite) set of local fields which interpolate the string field theory S-matrix. In the spirit of "intrinsicness" set forward in the present work, one might add the remark that by investigating the crossing property associated with such an S-matrix, the uniqueness of the inverse scattering problem based on crossing secures the uniqueness of the associated local quantum physics in a way which does not depend on the art (and luck) of finding local interpolating fields.

An alternative idea would be that the relevant objects of string field theory are really semiinfinite string-localized [2][74] in the sense of modular localization (which is the only relativistic quantum localization). Since, as already remarked before, such fields cannot be "interpolating" and their S-matrix could not even be crossing symmetric, to contemplate such a possibility would only make sense if the string field theory S-matrix turnes out to really violate crossing.

In this context it is interesting to mention that recent results on string localization lead to the apparent paradoxical conclusion that quantum (modular) string localization does not admit a Lagrangian quantization representation and classical Lagrangian string theories (e.g. Nambu-Goto) do not lead to quantum string-localized objects. The coalescence of these two different notions of localizations via quantization was a lucky circumstance without which Pascual Jordan could not have succeeded with his

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[36]Functional integral representations suffer the same limitations (for the same mathematical reasons) for interacting QFT as the previously explained limitations of ETCR.

[37]Actually the canonical second quantization of the classical Nambu-Goto string leads to pointlike local objects [72][73].

"Quantelung der Wellenfelder" and QFT would have begun with the 1939 representation theoretical approach of Eugene P. Wigner.

Whereas the construction of wedge algebras and their intersections based on PFG particle properties seems to be limited to factorizing models, the idea of getting to ambient wedge algebras and their intersections via ALH is completely general. The lightfront algebras turn out to be transverse quantum mechanical extensions of chiral QFTs and their classification does not appear to be much more difficult than that of standard chiral theories on which a lot of progress has been made. Among the ideas to construct QFTs in an intrinsic manner, I consider the holographic projection setting the most promising. Compared to the canonical ETCR setting it places the kinematics/dynamics cut in such a way that the kinematical side (chiral theories) becomes much richer and the dynamical side amounts to the reconstruction of the ambient theory (inverse holography). This resembles in some way the role which chiral theories are supposed to play in the dynamics of string theory.

Among the many unsolved conceptional problems of QFT there is the question of how particle-based concepts (S-matrix, formfactors crossing..) and the causality based algebraic lightfront holography (transverse extended chiral theories) fit together, e.g. questions like what is the holographic interpretation of the S_{scat} matrix? This is basically the old question concerning the particle-field relation in a new context.

The very fact that there are fundamental unanswered problems suggests that despite its almost 80 years of existence, QFT still remains a project and is still quite a distance from having reached maturity. It has remained young in the sense of not having accomplished an ultimate formulation in purely intrinsic terms, without the use of quasiclassical crutches with which Pascual Jordan introduced field quantization, but from which he wanted to get away[14].

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