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The quantum harmonic oscillator on a circle and a deformed quantum field theory

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Abstract

We construct a deformed free quantum field theory with an standard Hilbert space based on a deformed Heisenberg algebra. This deformed algebra is a Heisenberg-type algebra describing the first levels of the quantum harmonic oscillator on a circle of large length L. The successive energy levels of this quantum harmonic oscillator on a circle of large length L are interpreted, similarly to the standard quantum one-dimensional harmonic oscillator on an infinite line, as being obtained by the creation of a quantum particle of frequency w at very high energies.

Key-words: deformed Heisenberg algebra; deformed quantum field theory.

1 Introduction

The perspective of experimental results coming from a new generation of particle accelerators has enhanced the interest on the question of whether it is needed to introduce new concepts at very high energies. There has been a long time interest [1] on this question connected to the fact that it seems to be necessary to take new routes in order to overcome the incompatibility between general relativity and quantum mechanics at Planck scale, where gravitational forces become relevant in quantum processes.

Among the present attempts to describe physics at Planck scale, superstring theory [2] seems the most promising one. However, there is a vast range of energy from the present accelerators energies ($\approx 10^3$Gev) to the Planck energy (10^{19} Gev) where it is believed that there is room for surprises [3],[2]. It is also believed that field theories based on deformed algebras can play an important role to describe physics in this vast energy range [4]. These algebras have a parameter, known as deformation parameter, that it is expected to regularize the ultraviolet divergences in deformed field theories [5].

Physicists have always considered deformations since nature seems to choose this route when it is possible. For example one can interpret special relativity and quantum mechanics as very successful deformations, recovering the undeformed classical theories for $c \to \infty$ and $\hbar \to 0$ respectively. In the last years, there has been an increasing interest in generalized statistical mechanics which is a deformation of the Boltzmann-Gibbs statistics [6]. There are several physical systems, mainly those with long-range interactions, that are more appropriately treated by generalized statistical mechanics [7].

Recently, it was constructed [8] a generalization of the Heisenberg algebra describing the algebraic structure of a class of one-dimensional quantum systems characterized by having an spectrum where any successive energy levels are related by $\epsilon_{n+1} = f(\epsilon_n)$ [9]. The generators of this algebraic structure is given by ladder operators and the Hamiltonian operator of the system under consideration. Among the systems belonging to this class we find also the well-known q-oscillators [10]. A similar generalization was also constructed for the $su(2)$ algebra giving rise to a non-linear generalization of the $su(2)$ algebra containing the $su_q(2)$ algebra as a particular case [11].

In this paper, we construct a deformed free quantum field theory with an standard Hilbert space based on a deformed Heisenberg algebra. This deformed Heisenberg algebra, that belongs to the class of generalized Heisenberg algebras we mentioned before, comes from an interpretation of quantum particles at very high energies we suggest here. What we do is to consider the first levels of the one-dimensional quantum harmonic oscillator on a compact interval of the real line with asymptotic length L where the successive energy levels are interpreted, as in the standard quantum one-dimensional harmonic oscillator on the infinite line, as being obtained by the creation of a quantum particle of frequency w at very high energies. The deformation parameter of the Heisenberg-type algebra describing the first levels of the one-dimensional harmonic oscillator on the interval of the real line of length L is given by π/L, such that when $L \to \infty$ we recover the standard Heisenberg algebra.

In section 2, we review the generalized Heisenberg algebra. In section 3, we discuss a model of a quantum particle at very high energies and also a particular asymptotic solution of the Mathieu's equation that it is shown to correspond to the first energy levels of the quantum mechanical equation of the one-dimensional quantum harmonic oscillator defined on a real interval of length L with $L \gg 1$. In section 4, we construct the Heisenberg-type algebra that describes the particular asymptotic solution of the one-dimensional quantum harmonic oscillator on L and present a physical realization of the ladder operators of this Heisenberg-type algebra. Using the physical realization of the ladder operators presented in the previous section, we construct in section 5 a free quantum field theory based on the deformed Heisenberg algebra under consideration. Section 6 is devoted to our final comments.

2 Generalized Heisenberg algebras

Let us consider an algebra generated by J_0, A and $A^\dagger$ described by the relations [8]

$$\begin{aligned} J_0\, A^\dagger &= A^\dagger\, f(J_0), & (1)\\ A\, J_0 &= f(J_0)\, A, & (2)\\ \left[A, A^\dagger\right] &= f(J_0) - J_0, & (3)\end{aligned}$$

where † is the Hermitian conjugate and, by hypothesis, $J_0^\dagger = J_0$ and $f(J_0)$ is a general analytic function of J_0. We can see that the generators of the algebra satisfy trivially the Jacobi identity

$$\left[\left[J_l, J_m\right], J_n\right] + \text{cyclic permutations} = 0 \;, \qquad (4)$$

where $l,\ m,\ n = 0,\ \pm$, with $J_+ = A^\dagger$ and $J_- = A$. In order to prove it first note that the only non-trivial part of this identity is obtained when the sub indices are all different giving

$$\left[\left[J_0, A\right], A^\dagger\right] + \left[\left[A^\dagger, J_0\right], A\right] = 0 \;. \qquad (5)$$

We now rewrite the algebraic relations in eqs. (1-3) as commutators

$$\begin{aligned} \left[J_0, A^\dagger\right] &= A^\dagger(f(J_0) - J_0) = A^\dagger\left[A, A^\dagger\right], & (6)\\ \left[A, J_0\right] &= (f(J_0) - J_0)\, A = \left[A, A^\dagger\right] A\,. & (7)\end{aligned}$$

Substituting eqs. (6-7) into eq. (5) we get

$$\begin{aligned} \left[A^\dagger, \left[J_0, A\right]\right] &+ \left[A, \left[A^\dagger, J_0\right]\right] = -\left[A^\dagger, \left[A, A^\dagger\right]\right] A - A^\dagger\left[A, \left[A, A^\dagger\right]\right] \\ &= -\left[A^\dagger A, \left[A, A^\dagger\right]\right] = 0\,, & (8)\end{aligned}$$

since, from eqs. (1-2) we have

$$f(J_0) A^\dagger A = A^\dagger A f(J_0)\,. \qquad (9)$$

Using the algebraic relations in eqs. (1-3) we see that the operator

$$C = A^\dagger A - J_0 = A A^\dagger - f(J_0) \tag{10}$$

satisfies

$$[C, J_0] = [C, A] = \left[C, A^\dagger\right] = 0, \tag{11}$$

being thus a Casimir operator of the algebra.

We analyze now the representation theory of the algebra when the function $f(J_0)$ is a general analytic function of J_0. We assume we have an n-dimensional irreducible representation of the algebra given in eqs. (1-3). We also suppose that there is a state $|0\rangle$ with the lowest eigenvalue of the Hermitian operator J_0

$$J_0 |0\rangle = \alpha_0 |0\rangle. \tag{12}$$

For each value of α_0 we have a different vacuum thus, a better notation could be $|0\rangle_{\alpha_0}$ but for simplicity will shall omit the subscript α_0.

Let $|m\rangle$ be a normalized eigenstate of J_0,

$$J_0|m\rangle = \alpha_m |m\rangle\,. \tag{13}$$

Applying eq. (1) to $|m\rangle$ we have

$$J_0(A^\dagger|m\rangle) = A^\dagger f(J_0)|m\rangle = f(\alpha_m)(A^\dagger|m\rangle)\,. \tag{14}$$

Thus, we see that $A^\dagger|m\rangle$ is a J_0 eigenvector with eigenvalue $f(\alpha_m)$. Starting from $|0\rangle$ and applying $A^\dagger$ successively to $|0\rangle$ we create different states with J_0 eigenvalue given by

$$J_0\left((A^\dagger)^m|0\rangle\right) = f^m(\alpha_0)\left((A^\dagger)^m|0\rangle\right)\,, \tag{15}$$

where $f^m(\alpha_0)$ denotes the m-th iterate of f. Since the application of $A^\dagger$ creates a new vector, whose respective J_0 eigenvalue has iterations of α_0 through f augmented by one unit, it is convenient to define the new vectors $(A^\dagger)^m|0\rangle$ as proportional to $|m\rangle$ and we then call $A^\dagger$ a raising operator. Note that

$$\alpha_m = f^m(\alpha_0) = f(\alpha_{m-1})\,, \tag{16}$$

where m denotes the number of iterations of α_0 through f.

Following the same procedure for A, applying eq. (2) to $|m+1\rangle$, we have

$$A\, J_0|m+1\rangle = f(J_0)\,(A|m+1\rangle) = \alpha_{m+1}\,(A|m+1\rangle)\,, \tag{17}$$

showing that $A\,|m+1\rangle$ is also a J_0 eigenvector with eigenvalue α_m. Then, $A\,|m+1\rangle$ is proportional to $|m\rangle$ being A a lowering operator.

Since we consider α_0 the lowest J_0 eigenvalue, we require that

$$A\,|0\rangle = 0. \tag{18}$$

As was shown in [12], depending on the function f and its initial value α_0, it may happen that the J_0 eigenvalue of state $|m+1\rangle$ is lower than that of state $|m\rangle$. Then, as shown in [8], given an arbitrary analytical function f (and its associated algebra in eqs. (1-3)) in order to satisfy eq. (18), the allowed values of α_0 are chosen in such a way that the iterations $f^m(\alpha_0)$ $(m \geq 1)$ are always bigger than α_0; in other words, eq. (18) must be checked for every function f, giving consistent vacua for specific values of α_0.

As was proven in [8], under the hypothesis stated previously [1], for a general function f we obtain

$$J_0\,|m\rangle = f^m(\alpha_0)\,|m\rangle, \quad m = 0,1,2,\cdots\,, \tag{19}$$

$$A^\dagger\,|m-1\rangle = N_{m-1}\,|m\rangle, \tag{20}$$

$$A\,|m\rangle = N_{m-1}\,|m-1\rangle, \tag{21}$$

where $N_{m-1}^2 = f^m(\alpha_0) - \alpha_0$. We note that for each function $f(x)$ the representations are discussed by the analysis of the above equations as was done in [8] for the linear and quadratic $f(x)$.

When the functional $f(J_0)$ is linear in J_0, i.e., $f(J_0) = q^2 J_0 + s$, it was shown in [8] that the algebra in eqs. (1-3) recovers the q-oscillators for $\alpha_0 = 0$. Moreover, it was shown in [8], where the representation theory was constructed in detail for the linear and quadratic functions $f(x)$, that the essential tool in order to construct representations of the algebra in (1-3) for a general analytic function $f(x)$ is the analysis of the stability of the fixed points of $f(x)$ and their composed functions.

We showed in [8] and [9] that there is a class of one-dimensional quantum systems that are described by these generalized Heisenberg algebras. This class is characterized by those quantum systems having energy eigenvalues that can be written as

$$\epsilon_{n+1} = f(\epsilon_n)\,, \tag{22}$$

where ϵ_{n+1} and ϵ_n are successive energy levels and $f(x)$ is a different function for each physical system. This function $f(x)$ is exactly the same function that appears in the construction of the algebra in eqs. (1-3)! In this algebraic description of the class of quantum systems, J_0 is the Hamiltonian operator of the system, $A^\dagger$ and A are the creation and annihilation operators that are related as in eq. (10) where C is the Casimir operator of the representation associated to the quantum system.

3 Working definition of a quantum particle at very high energies

The quantum harmonic oscillator is very important in physics. In particular, because their energy eigenvalues are given by $E_n = (n+1/2)\hbar w$, their successive energy levels

[1] J_0 is Hermitian and the existence of a vacuum state.

are interpreted as being obtained by the creation of a quantum particle of frequence w. One can say that associated to the one-dimensional quantum harmonic oscillator there is a "working" definition of a quantum particle. We recall that this working definition is used in the standard construction of a quantum field theory through the ladder operators of the one-dimensional quantum harmonic oscillator [13].

Now, suppose we want to describe the interaction of quantum particles at very high energies (energies higher than 10^3 Gev). We know that this very high energy interaction will simulate the circumstances of an early universe. Thus, the definition of a quantum particle used at this very high energy interaction must be consistent with the definition of a quantum particle in an early universe. But, since the universe at this circumstance has a scale factor [14] much less than the today's factor it can be inappropriate to use the standard harmonic oscillator, that supposes an infinite line, as a working definition of a quantum particle at this very high energy scale.

We are going to discuss an equation defined on an interval of length L that under certain circumstances reproduces the ordinary harmonic oscillator in the limit $L \to \infty$. It is convenient to know how to describe quantum mechanics on a periodic line and we shall follow here the Ohnuki-Kitakado formalism [15]. According to this formalism there are inequivalent quantum mechanics on S^1 (periodic line) depending on a parameter α $(0 \leq \alpha < 1)$. The momentum operator G on S^1 in the coordinate representation is given in this formalism as [15],[16]

$$G \longrightarrow \frac{1}{i}\frac{d}{d\theta} + \alpha\,, \quad 0 \leq \alpha < 1 \tag{23}$$

and the coordinate operator is given in terms of the unitary operator W

$$W \longrightarrow e^{i\theta}\,. \tag{24}$$

Let us construct the following equation on S^1:

$$G^2\Psi + K\left[W + W^\dagger\right]\Psi = \epsilon\Psi\,, \tag{25}$$

where G and W are defined above [2]. In order to have the above equation in the coordinate representation we substitute eqs. (23-24) into eq. (25) for $\alpha = 0$, we obtain

$$\frac{d^2\Psi(\theta)}{d\theta^2} + (\epsilon - 2K\cos\theta)\Psi(\theta) = 0\,, \tag{26}$$

with $\Psi(\theta = 0) = \Psi(\theta = 2\pi)$. This equation can be rewritten as

$$\frac{d^2\Psi(\bar\theta)}{d\bar\theta^2} + (4\epsilon - 8K\cos 2\bar\theta)\Psi(\bar\theta) = 0\,, \tag{27}$$

[2]It would be also possible to define an equation with quadratic powers of W and $W^\dagger$, the above equation is the simplest one.

where $\Psi(\bar{\theta} = 0) = \Psi(\bar{\theta} = \pi)$, which is the well-known Mathieu's equation that first appeared in 1868 in the study of the vibrations of a stretched membrane of elliptic cross-section [17]. Mathieu's equation is an important equation in physics arising in the study of a variety of physical problems, from ordered crystals with the potential $\cos 2x$ [18] to the wave equation of scalar fields in the background of a D-brane metric [19]. We note that, this is one possible equation on a periodic line since we chose for simplicity $\alpha = 0$ in eq. (26). According to Ohnuki-Kitakado's formalism [15] there are inequivalent quantum mechanics on S^1 for each value of the parameter α $(0 \leq \alpha < 1)$.

To the end of considering the limit of eq. (26) when the radius of the circle goes to infinity we perform the change of variables

$$\theta = \frac{\pi}{L}x + \pi\,, \quad -L \leq x \leq L\,. \tag{28}$$

Using eq. (28), eq. (26) becomes

$$\frac{d^2\Psi}{dx^2} + \left(E + \frac{2\pi^2}{L^2}K\cos\frac{\pi}{L}x\right)\Psi = 0\,, \tag{29}$$

where $E = \pi^2\epsilon/L^2$. Now, using a trivial trigonometric identity and calling $\lambda \equiv E + 2\pi^2K/L^2$ we obtain

$$\frac{d^2\Psi}{dx^2} + \left[\lambda - \frac{\pi^4}{L^4}Kx^2\left(\frac{\sin\pi x/2L}{\pi x/2L}\right)^2\right]\Psi = 0\,. \tag{30}$$

Then, we see that for $K = L^4/\pi^4$ we obtain for $L \to \infty$, apart from a trivial energy renormalization, the well-known Schröedinger's equation for the harmonic oscillator

$$\frac{d^2\Psi}{dx^2} + \left(\lambda - x^2\right)\Psi = 0\,, \tag{31}$$

when x is not of order of L that is infinite. Suppose now we consider the Mathieu's equation for $K = L^4/\pi^4$ and L asymptotic. In this case the first levels are concentrated in values $x \ll L$ thus, according to the previous discussion, these energy levels, that we call ϵ_n^L, will correspond to those energy levels of the standard harmonic oscillator when $L \to \infty$. Now, analogously to the definition of a quantum particle through the ordinary quantum harmonic oscillator, we define n quantum particles at very high energies as having energy ϵ_n^L. By consistency, $\epsilon_n^{L\to\infty} - \epsilon_0^{L\to\infty} = n(\epsilon_1^{L\to\infty} - \epsilon_0^{L\to\infty})$.

As a matter of fact, there is a solution by Ince and Goldstein [20],[21],[17] of the Mathieu's equation, eq. (26), for asymptotic values of K. Their expansion for ϵ, the characteristic values of the equation, for our case of interest, i.e, $K = L^4/\pi^4$, gives [21]

$$\lambda_n = \nu_n - \frac{a^2}{32}\left(\nu_n^2 + 1\right) + \cdots\,, \tag{32}$$

where $\nu_n = 2n + 1$ and $a = \pi/L$. Note that dots in the above equation means higher orders in a. For $L \to \infty$, $(a \to 0)$ we recognize the energy eigenvalues of the harmonic oscillator.

The above solution corresponds to the energy levels of the Mathieu's equation when the height of the potential is very large, i.e, when $a^4(2n+1)^2/16$ is very small [20],[21]. Note that, even if L is large leading to a localization of the solution, this one is periodic with period $2L$.

4 Deformed Heisenberg algebra and its physical realization

The asymptotic solution [20],[21],[17] of the characteristic values of Mathieu's equation we presented in the last section, eq. (32), can be interpreted as a deformation of the harmonic oscillator with deformation parameter equal to $a = \pi/L$. Moreover, as shown in the last section, the solutions of the Mathieu's equation, eq. (26) with $K = L^4/\pi^4$ and for $x \ll L$, correspond to the solutions of the harmonic oscillator on S^1.

In section 2, we presented a class of algebras that describes Heisenberg-type algebras for a class of one-dimensional quantum systems. We are going to show in this section that the asymptotic solution of the Mathieu's equation we presented in the last section belongs to this class of algebras. In other words, we shall construct a Heisenberg-type algebra, an algebra with creation and destruction operators, for the Ince-Goldstein solution (eq. (32)) of the harmonic oscillator on S^1 and we shall find the characteristic function $f(x)$ (see eqs. (1-3)) for this algebra. Moreover, we shall also propose a realization, as in the case of the standard harmonic oscillator, of the ladder operators in terms of the physical operators of the system.

As described in [9] and [22] the first thing we have to do in order to describe the Heisenberg-type structure of a one-dimensional quantum system is to relate the energy of the system for two arbitrary successive levels (see eq. (22)). For the energy spectrum given in eq. (32), i.e,

$$\epsilon_n^L = n + \frac{1}{2} - \frac{a^2}{64}\left((2n+1)^2 + 1\right) + \cdots , \tag{33}$$

(dots, here and in what follows, means higher orders in a) we obtain

$$\epsilon_{n+1}^L = \epsilon_n^L + 1 - \frac{a^2}{8}(n+1) + \cdots . \tag{34}$$

Thus, we have to invert eq. (33) in order to obtain n in terms of ϵ_n^L. Taking n from eq. (33) we get

$$\epsilon_{n+1}^L \equiv f(\epsilon_n^L) = \epsilon_n^L + 1 - \frac{a^2}{16}\left(2\epsilon_n^L + 1\right) + \cdots . \tag{35}$$

According to refs. [20] and [21], this solution is valid when $a^4(2n+1)^2/16$ is small. Thus, since $a = \pi/L$ is considered very small, n cannot be very large.

Now, if we assume that ϵ_n^L is the eigenvalue of the operator J_0 on state $|n\rangle$ we identify the $f(x)$ appearing in eqs. (19-21) with that one in eq. (35) for the quantum system under consideration. Then, the algebraic structure describing the quantum system under consideration is obtained using the $f(x)$ defined in eq. (35) into eqs. (1-3) and can be written as

$$\left[J_0, A^\dagger\right] = A^\dagger - \frac{a^2}{16} A^\dagger (2J_0 + 1) + \cdots \,, \tag{36}$$

$$[J_0, A] = -A + \frac{a^2}{16} (2J_0 + 1) A + \cdots \,, \tag{37}$$

$$\left[A, A^\dagger\right] = 1 - \frac{a^2}{16} (2J_0 + 1) + \cdots \,, \tag{38}$$

where, according to eqs. (19-21), A and $A^\dagger$ are the ladder operators for the system under consideration, i.e, $A^\dagger$ when applied to the state $|m\rangle$, that has J_0 eigenvalue ϵ_m^L, gives, apart a multiplicative factor depending on m, the state $|m+1\rangle$ that has energy eigenvalue ϵ_{m+1}^L; with a similar role played by A.

Note that, when $a \to 0$ ($L \to \infty$) we reobtain the well-known Heisenberg algebra, as it should be since we showed in the previous section that the Mathieu's equation, eq. (26), for $K = L^4/\pi^4 = a^{-4}$ gives the well-known Schröedinger's equation for the harmonic oscillator, eq. (31), in this limit.

The next step we have to do is to realize the operators A, $A^\dagger$ and J_0 in terms of physical operators as in the case of the one-dimensional harmonic oscillator and as it was done in [9] and in [22] for the square-well potential. To this end, we briefly review the formalism of non-commutative differential and integral calculus on a one-dimensional lattice developed in [23] and [24]. Let us consider an one dimensional lattice in a momentum space where the momenta are allowed only to take discrete values, say p_0, $p_0 + a$, $p_0 + 2a$, $p_0 + 3a$ etc, with $a > 0$.

The non-commutative differential calculus is based on the expression[23],[24]

$$[p, dp] = dp\, a \,, \tag{39}$$

implying that

$$f(p)\, dg(p) = dg(p)\, f(p + a) \,, \tag{40}$$

for all functions f and g. We introduce partial derivatives as

$$d\, f(p) = dp\, (\partial_p\, f)\, (p) = (\bar{\partial}_p\, f)\, (p)\, dp \,, \tag{41}$$

where the left and right discrete derivatives are given by

$$(\partial_p\, f)\, (p) = \frac{1}{a} \left[f(p + a) - f(p)\right] , \tag{42}$$

$$(\bar{\partial}_p\, f)\, (p) = \frac{1}{a} \left[f(p) - f(p - a)\right] , \tag{43}$$

that are the two possible definitions of derivatives on a lattice. The Leibniz rule for the left discrete derivative can be written as,

$$(\partial_p\, fg)\, (p) = (\partial_p f)\, (p) g\, (p) + f(p + a)(\partial_p g)\, (p) \,, \tag{44}$$

with a similar formula for the right derivative[23].

Let us now introduce the momentum shift operators

$$T = 1 + a\,\partial_p \tag{45}$$
$$\bar{T} = 1 - a\,\bar{\partial}_p\,, \tag{46}$$

that shift the momentum value by a

$$(Tf)\,(p) = f(p+a) \tag{47}$$
$$(\bar{T}f)\,(p) = f(p-a) \tag{48}$$

and satisfies

$$T\,\bar{T} = \bar{T}T = \hat{1}\,, \tag{49}$$

where $\hat{1}$ means the identity on the algebra of functions of p.

Introducing the momentum operator P[23]

$$(Pf)\,(p) = p\,f(p)\,, \tag{50}$$

we have

$$TP = (P+a)T \tag{51}$$
$$\bar{T}P = (P-a)\bar{T}\;. \tag{52}$$

Integrals can also be defined in this formalism. It is shown in ref. [23] that the property of an indefinite integral

$$\int df = f + \text{periodic function in } a\;, \tag{53}$$

suffices to calculate the indefinite integral of an arbitrary one form. It can be shown that[23] for an arbitrary function f

$$\int d\bar{p}\,f(\bar{p}) = \begin{cases} a\sum_{k=1}^{[p/a]} f(p-ka)\,, & \text{if } p \geq a \\ 0\;, & \text{if } 0 \leq p < a \\ -a\sum_{k=0}^{-[p/a]-1} f(p+ka)\,, & \text{if } p < 0 \end{cases} \tag{54}$$

where $[p/a]$ is by definition the highest integer $\leq p/a$.

All equalities involving indefinite integrals are understood modulo the addition of an arbitrary function periodic in a. The corresponding definite integral is well-defined when the length of the interval is multiple of a. Consider the integral of a function f from p_d to p_u ($p_u = p_d + Ma$, where M is a positive integer) as

$$\int_{p_d}^{p_u} dpf(p) = a\sum_{k=0}^{M} f(p_d + ka). \tag{55}$$

Using eq. (55), an inner product of two (complex) functions f and g can be defined as

$$\langle f\,,\,g\rangle = \int_{p_d}^{p_u} dp\, f(p)^*\, g(p)\,, \tag{56}$$

where * indicates the complex conjugation of the function f. The norm $\langle f\,,\,f\rangle \geq 0$ is zero only when f is identically null. The set of equivalence classes [3] of normalizable functions f ($\langle f\,,\,f\rangle$ is finite) is a Hilbert space. It can be shown that[23]

$$\langle f, Tg\rangle = \langle \bar{T} f, g\rangle\,, \tag{57}$$

so that

$$\bar{T} = T^\dagger\,, \tag{58}$$

where $T^\dagger$ is the adjoint operator of T. Eqs. (49) and (58) show that T is a unitary operator. Moreover, it is easy to see that P defined in eq. (50) is an Hermitian operator and from (58) one has

$$(i\partial_p)^\dagger = i\bar{\partial}_p\,. \tag{59}$$

Now, we go back to the realization of the deformed Heisenberg algebra eqs. (36-38) in terms of physical operators. We can associate to the crystalline structure of the Mathieu's equation discussed in the previous section the one dimensional lattice we have just presented.

Observe that we can write J_0 for the asymptotic Ince-Goldstein solution of the Mathieu's equation, eq. (34), as

$$J_0 = \frac{P}{a} + \frac{1}{2} - \frac{a^2}{64}\left[\left(\frac{2P}{a}+1\right)^2 + 1\right] + \cdots\,, \tag{60}$$

where P is given in eq. (50) and its application to the vector states $|m\rangle$ appearing in (19-21) gives

$$P\,|m\rangle = m\,a\,|m\rangle\ , m = 0, 1, \cdots\,, \tag{61}$$

and

$$\bar{T}\,|m\rangle = |m+1\rangle\ , m = 0, 1, \cdots\,, \tag{62}$$

where $\bar{T}$ and $T = \bar{T}^\dagger$ are defined in eqs. (45-49).

With the definition of J_0 given in eq. (60) we see that ϵ_n^L given in eq. (34) is the J_0 eigenvalue of state $|n\rangle$ as we wanted. Let us now define

$$A^\dagger = S\,\bar{T}\,, \tag{63}$$

$$A = T\,S\,, \tag{64}$$

where,

$$S^2 = \frac{P}{a} - \frac{a^2}{64}\left[\left(\frac{2P}{a}+1\right)^2 - 1\right] + \cdots\,, \tag{65}$$

[3]Two functions are in the same equivalence class if their values coincide on all lattice sites.

such that $S^2 = J_0 - \alpha_0$ where α_0, defined in eq. (12), is ϵ_0^L.

Define an operator N as

$$N \equiv \frac{P}{a} \; , \tag{66}$$

such that,

$$N\,|m\rangle = m\,|m\rangle \; , m = 0, 1, 2, \cdots \; . \tag{67}$$

In terms of this operator, J_0 can be written as

$$J_0 = N + \frac{1}{2} - \frac{a^2}{64}\left[(2N+1)^2 + 1\right] + \cdots \; , \tag{68}$$

that can be interpreted as $J_0 = \epsilon_N^L$ where ϵ_N^L is ϵ_n^L in eq. (33) with the operator N in the place of the variable n. Yet, note that eqs. (51-52) can also be rewritten as

$$TN = (N+1)T \tag{69}$$

$$\bar{T}N = (N-1)\bar{T} \; . \tag{70}$$

It is easy to realize that A, $A^\dagger$ and J_0 defined in eqs. (60, 63-65) satisfy the a-deformed algebra given in eqs. (36-38). Consider firstly the relation between J_0 and $A^\dagger$,

$$J_0\,A^\dagger = \epsilon_N^L \;\; S\,\bar{T} = A^\dagger\,\epsilon_{N+1}^L \; , \tag{71}$$

where we have used the realizations in the first equality of the above equation and in the second one eq. (70). But, from eq. (35) $\epsilon_{N+1}^L = f(\epsilon_N^L) = f(J_0)$ thus,

$$J_0\,A^\dagger = A^\dagger\,f(J_0) \; , \tag{72}$$

that is, eq. (36) for $f(x)$ given in eq. (35). Eq. (37) is the Hermitian conjugate of eq. (36), then its proof using eq. (64) and (68) is similar to the previous one. Now, using that

$$A^\dagger\,A = S^2 = J_0 - \alpha_0 \; , \tag{73}$$

$$A\,A^\dagger = T\,S^2\bar{T} = f(J_0) - \alpha_0 \; , \tag{74}$$

for $f(x)$ defined in eq. (35) we arrive at eq. (38) and the proof is complete.

Note that the realization we have found in eqs. (63, 64 and 68) is qualitatively different from the the realization of the standard harmonic oscillator. This is sensible, since we have two physically different systems. Even if the standard harmonic oscillator defined on $-\infty \leq x \leq \infty$ is a limiting case of the periodic one, it is not periodic having no lattice associated to it. On the other hand, once L is finite, $-L \leq x \leq L$, the periodic structure is explicitly manifest and the realization in the finite case, given in eqs. (63, 64 and 68), shows it clearly.

5 A deformed free quantum field theory at very high energies

We are going to discuss in this section a free quantum field theory based on the asymptotic solution of the Mathieu's equation (an a-deformed harmonic oscillator) discussed in the previous sections. The vector states of this quantum field theory, eigenvectors of the Hamiltonian, are obtained by the application of the creation operators $A^\dagger$, satisfying the algebra defined in eqs. (36-38), to a vacuum state. Since the algebra defined in eqs. (36-38) describes a Heisenberg-type algebra for a deformed harmonic oscillator that we interpreted as being the appropriate framework to describe a quantum particle at very high energies (energies higher than 10^3 Gev), the energy scale of this quantum field theory would be very high.

We recall, we showed in section 3 that the first levels of the Mathieu's equation given in eq. (27) for $K = L^4/\pi^4$ correspond to the first energy levels of the Schröedinger's equation for the harmonic oscillator when $L \to \infty$ and that in the previous section we discussed an asymptotic solution of the Mathieu's equation, L finite but very large, that can be seen as a deformed harmonic oscillator with $a = \pi/L$ being the deformation parameter.

As shown in the previous section, associated to the periodic structure of the asymptotic solution of the Mathieu's equation for $K = L^4/\pi^4$, there is an one-dimensional lattice with $a = \pi/L$ being the lattice spacing. Using the momentum operator P defined on a lattice, eq. (50), and the associated lattice derivatives we can define two type coordinate operators as

$$\chi = i(\bar{\partial}_p + \partial_p) \ , \tag{75}$$
$$Q = \bar{\partial}_p - \partial_p \ , \tag{76}$$

where ∂_p and $\bar{\partial}_p$ are the left and right discrete derivatives defined in eqs. (42, 43). Of course, in the continuous limit ($a \to 0$) the operator Q is identically null since ∂_p and $\bar{\partial}_p$ represent, in this limit, the same derivative.

It can be checked that the operators P, χ and Q generate an algebra on the momentum lattice [23]

$$[\chi, P] = 2i\left(1 - \frac{a}{2}Q\right), \tag{77}$$
$$[P, Q] = -ia\chi, \tag{78}$$
$$[\chi, Q] = 0. \tag{79}$$

Note that, in the continuous limit $a \to 0$ we recover the standard Heisenberg algebra, $[x, p] = i$.

For the asymptotic solution of the Mathieu's equation presented in the last two previous sections, with the help of eqs. (45-46 and 63-64) we can rewrite χ and Q in terms of the ladder operators of the a-deformed Heisenberg algebra as

$$\chi = \frac{-i}{a}\left(S^{-1}A^\dagger - AS^{-1}\right), \tag{80}$$

$$Q = \frac{1}{a}\left(-2 + S^{-1}A^\dagger + A\,S^{-1}\right), \tag{81}$$

where S is defined in eq. (65). We stress that $A^\dagger$ and A are the creation and annihilation operators respectively, of the asymptotic solution of the Mathieu's equation presented in the last two previous sections that satisfy the a-deformed Heisenberg algebra in eqs. (36-38).

There is an important point to be explained here. As was already stressed, Mathieu's equation is a periodic equation. Thus, the asymptotic solution we are considering takes into account this periodicity as well, even if the interaction among the cells of the crystal is very small. That the correlations are very small can be inferred observing that the energy of the asymptotic solution, given in eq. (33), does not depend on what happens in other lattice cells.

Let us now introduce a three-dimensional discrete $\vec{k}$-space,

$$k_i = \frac{2\pi l_i}{L_i}, \quad i = 1, 2, 3 \quad , \tag{82}$$

with $l_i = 0, \pm 1, \pm 2, \cdots$ and L_i, the lengths of the three sides of a rectangular box Ω. We introduce for each point of this $\vec{k}$-space an independent a-deformed harmonic oscillator constructed in the last two previous sections such that the deformed operators commute for different three-dimensional lattice points. We also introduce an independent copy of the one-dimensional momentum lattice defined in the previous section for each point of this $\vec{k}$-lattice such that $P^\dagger_{\vec{k}} = P_{\vec{k}}$ and $T_{\vec{k}}$, $\bar{T}_{\vec{k}}$ and $S_{\vec{k}}$ are defined by means of the previous definitions, eqs. (45-46 and 65), through the substitution $P \to P_{\vec{k}}$.

It is not difficult to realize that

$$A^\dagger_{\vec{k}} = S_{\vec{k}}\,\bar{T}_{\vec{k}}\,, \tag{83}$$

$$A_{\vec{k}} = T_{\vec{k}}\,S_{\vec{k}}\,, \tag{84}$$

$$J_0(\vec{k}) = \frac{P_{\vec{k}}}{a} + \frac{1}{2} - \frac{a^2}{64}\left[\left(2\frac{P_{\vec{k}}}{a} + 1\right)^2 + 1\right] + \cdots, \tag{85}$$

satisfy the algebra in eqs. (36-38) for each point of this $\vec{k}$-lattice and the operators $A^\dagger_{\vec{k}}$, $A_{\vec{k}}$ and $J_0(\vec{k})$ commute among them for different points of this $\vec{k}$-lattice.

Now, we define the type-coordinate operators for each point of the three-dimensional lattice as

$$\chi_{\vec{k}} = i(\bar{\partial}_{p_{\vec{k}}} + \partial_{p_{-\vec{k}}})\,, \tag{86}$$

$$Q_{\vec{k}} = \bar{\partial}_{p_{\vec{k}}} - \partial_{p_{-\vec{k}}}\,, \tag{87}$$

such that $\chi^\dagger_{\vec{k}} = \chi_{-\vec{k}}$ and $Q^\dagger_{\vec{k}} = Q_{-\vec{k}}$, exactly as it happens in the construction of a spin-0 field for the spin-0 quantum field theory [13]. With the previous definitions, eqs. (83-84 and 86-87), we can rewrite the type-coordinate operators in terms of the ladder operators of the a-deformed Heisenberg algebra

$$\chi_{\vec{k}} = \frac{i}{a}\left(-S^{-1}_{-\vec{k}}A^{\dagger}_{-\vec{k}} + A_{\vec{k}}S^{-1}_{\vec{k}}\right) , \tag{88}$$

$$Q_{\vec{k}} = \frac{1}{a}\left(-2 + S^{-1}_{-\vec{k}}A^{\dagger}_{-\vec{k}} + A_{\vec{k}}\,S^{-1}_{\vec{k}}\right) . \tag{89}$$

By means of $\chi_{\vec{k}}S_{\vec{k}}$ and $Q_{\vec{k}}S_{\vec{k}}$ we define two fields $\phi_1(\vec{r},t)$ and $\phi_2(\vec{r},t)$ as

$$\phi_1(\vec{r},t) = \sum_{\vec{k}} \frac{i}{\sqrt{2\Omega\omega(\vec{k})}}\left(-S^{-1}_{\vec{k}}A^{\dagger}_{\vec{k}}S_{-\vec{k}}\,e^{-i\vec{k}.\vec{r}} + A_{\vec{k}}\,e^{i\vec{k}.\vec{r}}\right), \tag{90}$$

$$\phi_2(\vec{r},t) = \sum_{\vec{k}} \frac{1}{\sqrt{2\Omega\omega(\vec{k})}}\left(-2S_{\vec{k}}\,e^{i\vec{k}.\vec{r}} + S^{-1}_{\vec{k}}A^{\dagger}_{\vec{k}}\,S_{-\vec{k}}\,e^{-i\vec{k}.\vec{r}} + A_{\vec{k}}\,e^{i\vec{k}.\vec{r}}\right), \tag{91}$$

where $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$, m a real parameter and Ω is the volume of a rectangular box. Two type-momentum fields $\Pi(\vec{r},t)$ and $\wp(\vec{r},t)$ can be defined as well as

$$\Pi(\vec{r},t) = \sum_{\vec{k}} \sqrt{\frac{\omega(\vec{k})}{\Omega}}\; S_{\vec{k}}\,e^{i\vec{k}.\vec{r}}, \tag{92}$$

$$\wp(\vec{r},t) = \sum_{\vec{k}} \sqrt{\frac{\omega(\vec{k})}{\Omega}}\left(-\frac{3}{2}\,S_{\vec{k}}\,e^{i\vec{k}.\vec{r}} + S^{-1}_{\vec{k}}A^{\dagger}_{\vec{k}}\,S_{-\vec{k}}\,e^{-i\vec{k}.\vec{r}} + A_{\vec{k}}\,e^{i\vec{k}.\vec{r}}\right) . \tag{93}$$

By a straightforward calculation, we can show that the Hamiltonian

$$\begin{aligned} H = \int\; & d^3r\,\Big(\Pi(\vec{r},t)^{\dagger}\;\wp(\vec{r},t) + \wp(\vec{r},t)^{\dagger}\;\Pi(\vec{r},t) + \\ & \phi_1(\vec{r},t)^{\dagger}(-\vec{\nabla}^2 + m^2)\phi_1(\vec{r},t) + \phi_2(\vec{r},t)^{\dagger}(-\vec{\nabla}^2 + m^2)\phi_2(\vec{r},t)\Big)\;, \end{aligned} \tag{94}$$

can be written as

$$\begin{aligned} H &= \sum_{\vec{k}}\omega(\vec{k})A^{\dagger}_{\vec{k}}A_{\vec{k}} = \sum_{\vec{k}}\omega(\vec{k})S^2_{\vec{k}} \\ &= \sum_{\vec{k}}\omega(\vec{k})\left(N_{\vec{k}} - \frac{a^2}{64}\left[\left(2N_{\vec{k}}+1\right)^2 - 1\right] + \cdots\right) . \end{aligned} \tag{95}$$

The eigenvectors of H form a complete set and span the Hilbert space of this system. The eigenvectors are

$$|0\rangle,\; A^{\dagger}_{\vec{k}}|0\rangle,\; A^{\dagger}_{\vec{k}}A^{\dagger}_{\vec{k}'}|0\rangle \text{ for } \vec{k} \neq \vec{k}',\; (A^{\dagger}_{\vec{k}})^2|0\rangle,\; \cdots\;, \tag{96}$$

where the state $|0\rangle$ satisfies as usual $A_{\vec{k}}|0\rangle = 0$ (see eq. (18)) for all $\vec{k}$ and $A_{\vec{k}}$, $A^{\dagger}_{\vec{k}}$ for each $\vec{k}$ satisfy the a-deformed Heisenberg algebra eqs. (36-38).

One should note that even if we are considering a free theory the energy of the system is non-extensive, i.e, the energy of n-particles is different from n-times the energy of one particle. The origin of this non-extensivity of the energy of the system comes from the solution, eq. (33), of the Mathieu's equation we are considering and it is connected to our choice of describing the system on a finite line.

As was stressed before, the deformed solutions we are considering correspond to the first levels of the Mathieu's equation for asymptotic $K = L^4/\pi^4$. Moreover, this equation is periodic while for $L \to \infty$ these solutions correspond to the first energy levels of the ordinary quantum harmonic oscillator. This periodicity for L finite permits a special realization of the a-deformed algebra in terms of physical operators that are defined on a lattice of lattice spacing $a = \pi/L$ (see eqs. (60, 63 and 64)). Since the realization for L finite is different from the $L = \infty$ case it is sensible to have different quantum field theories for these two cases. However, it is very interesting to observe that even if the QFT in the periodic and non-periodic regimes are different, the limit $a \to 0$ ($L \to \infty$) in eq. (95) gives the standard spin-0 quantum Hamiltonian .

6 Final comments

We constructed in this paper a deformed free quantum field theory based on a deformed Heisenberg algebra. This deformed Heisenberg algebra is the type-Heisenberg structure of the first levels, ϵ_n^L, of the Mathieu's equation for asymptotic values of $K = L^4/\pi^4$.

We showed that the first levels of the Mathieu's equation with amplitude $K = L^4/\pi^4$ for $L \to \infty$ correspond to the first energy levels of the ordinary Schröedinger's equation for the one-dimensional harmonic oscillator. Then, analogously to the definition of a quantum particle through the standard quantum one-dimensional harmonic oscillator we defined n quantum particles at very high energies as having energy ϵ_n^L, that is the n-th energy eigenvalue of the Mathieu's equation for asymptotic values of $K = L^4/\pi^4$.

Using the first energy eigenvalues of the Mathieu's equation for asymptotic values of $K = L^4/\pi^4$ we constructed the associated type-Heisenberg algebra which is a deformed Heisenberg algebra with deformation parameter given by $a = \pi/L$. The ladder operators of this deformed algebra were realized in terms of physical operators belonging to a lattice and using this realization we constructed a free deformed quantum field theory based on this deformed algebra.

The energy scale of validity of this quantum field theory is supposed to be very high, since we used a definition of quantum particles we suppose to be valid for very high energies. The interesting point is that, since this deformed free quantum field theory has an standard Hilbert space, it is in principle possible to apply to this deformed theory the standard methods of quantum field theory to compute matrix elements of different operators.

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