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Some Aspects of Geometrical Confinement

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Abstract

In this paper we present a toy model for the dynamics of a gauge field theory in such way that spin-one particles can be confined in a compact domain. We show that the property of confinement can be associated to the formation of a null surface identified to a horizon. This is due to the presence of an effective geometry generated by the self-interaction of the gauge field that guides the wave propagation of the field.

This phenomenon has a striking analogy to the gravitational black hole in Einstein general theory of relativity, separating two domains of spacetime that can be trespassed only into one direction.

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1 Introduction

It is a common knowledge that the necessary condition for a field theory to produce confinement is to be non-linear. Although this seems to be a good requirement, it is not enough. Since such a confinement, for instance, in the SU(3) non-Abelian gauge theory containing an octet of massless vector gluons, is still not available [1], a question appears: is there a natural way to generates confined structures in a non linear theory? The proposal of this paper is to exhibit in a simple version a pattern that answers affirmatively to this question.

The crucial point of our interest is concentrated on the way the field discontinuity evolves. Thus, our main concern here is the evolution of the wave propagation for certain classes of dynamics. We shall prove that, in a very general context, massless gauge fields do not propagate through the null-cones of the underlying Minkowski spacetime metric but instead, they follow paths of geodesics in another effective geometry that depends on the properties of the unperturbed background field. We show here that this property is not restricted to the case of an Abelian group (as, for instance, in the theory of a non linear photon), but instead it is a very general behavior for non linear gauge invariant theories. We propose to use such mechanism to produce confinement for spin-one gauge fields on the gluonic world. Finally let us remark that the reader should be warned of the following. We will deal here only with the classical property of the gauge field. This is not due to a naive belief that the confinement phenomenon does not belong to the quantum world, but just because we think that T.D. Lee [2] is correct when he argues that "...Quark confinement is a large-scale phenomenon. Therefore, at least on the phenomenological level, it should be understandable through a quasi-classical macroscopic theory, much like the London-Landau theory of superfluidity...". This is precisely what we intend to present in this paper.

2 Classical Version of the Confinement

We identify a classical version of the confinement of massless spin-one particles with the interpretation of the deformation of their surfaces of propagation, the corresponding *light cones*, in such a way that the gluons encounter an unsurmountable barrier that forbids them to get outside the confinement region. In other terms, it appears, for the external world, as a situation that can be described equivalently in terms of the formation of a horizon. Such a deformation does not occur in the standard theory of gauge field in the Minkowskian structure of spacetime. How to create a similar effect in a scenario containing just a set of spin-one self-interacting fields?

Our aim is then reduced to obtain the required property that the information carried on by the gluons operates in a different way from the Minkowski null cones. In this vein, our first step is to change the dynamics in an adequate way. The reason for this is related to the fact that massless Yang-Mills (YM) particles, like the linear photon in Maxwell's theory, travel along the Minkowski null cone[1]. This property is the same as

[1]The need for this modification comes from the property of the structure of geodesics in Minkowski spacetime, that imposes that any particle that follows null cones cannot be bounded in a compact region.

in the Abelian case and has its origin as a direct consequence on the construction of the YM model that makes all non-linearity of this theory to be restricted uniquely to the algebraical dependence of the field $F^a_{\mu\nu}$ on the potential A^a_μ.

To change this in the realm of the gluon interaction, conserving the color covariance, one should modify the Lagrangian conveniently. This is precisely the case which we will consider here and constitutes the basis for the actual new ingredient of our model. We shall see that a slight modification of the YM theory[2] concerning its Lagrangian seems to be the key to the understanding of the confinement problem.

However, before this and in order to gain some insight on the generic properties of such modified action let us concentrate our analysis on the simpler case of the dynamics of a single vector field. In the next section we will make a short overview of a class of nonlinear Electrodynamics.

3 The General Framework of Spin-1 Theory

The nonlinear electrodynamic theory[3] is described by a Lagrangian L given uniquely in terms of the invariant $F \equiv F_{\mu\nu} F^{\mu\nu}$. We set[4]

$$L = L(F). \tag{1}$$

The corresponding equation of motion is given by

$$\partial_\nu \{L_F F^{\mu\nu}\} = J^\mu_{ext} \tag{2}$$

in which L_F represents the functional derivative of the Lagrangian $(\delta L/\delta F)$ with respect to invariant F; L_{FF} is the second derivative.

This equation can be written in another, more appealing form, by just isolating the linear Maxwell term and taking all remaining non linear parts as an additional *internal* current to be added to the external one:

$$\partial_\nu F^{\mu\nu} = J^\mu_{int} + J^\mu_{ext} \tag{3}$$

in which the associated *internal* current, the self-term is given by

$$J^\mu_{int} \equiv -\frac{L_{FF}}{L_F} F^{\mu\nu} \partial_\nu F. \tag{4}$$

Written under this form it can be thought as nothing but a modeling of the response in a self-interacting way of some special plasma medium. Indeed, let us consider the quantity χ_ν

$$\chi_\nu \equiv \frac{L_{FF}}{L_F} \partial_\nu F. \tag{5}$$

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[2]We remark that the major part, and by far the most important one, of Yang-Mills theory is maintained.

[3]We note that these remarks concern any spin-1 theory.

[4]We do not consider here the invariant constructed with the dual $F^*_{\mu\nu}$.

We define a normalized frame $n^\mu \equiv \chi^\mu/\chi^2$, which in the case χ^μ is time-like in the Minkowski background, could be identified to a real observer that co-moves with χ^μ. The extra term of the current assumes then the form

$$J^{int}{}_\mu = \sigma E_\mu \tag{6}$$

in which E_μ is the electric part of the field as seen in the frame n^μ and $\sigma\,(F)$ may depend on the field variables in a complicated way. Under the form of Eq. (6) the analogy with situations treated within Maxwell electrodynamics in material media is transparent.

From the definition of the energy-momentum tensor we obtain from the non-linear Lagrangian:

$$T_{\mu\nu} = -L\gamma_{\mu\nu} - 4L_F\,F_{\mu\alpha}\,F^\alpha{}_\nu. \tag{7}$$

Using the equation of motion and after some manipulation, one obtains the expression that contains all information on the balance of forces through the exchange of energy of the field and the currents independently of the particular form of the Lagrangian. Indeed, we obtain

$$\partial_\nu\,T^{\mu\nu} = -F^{\mu\nu}\,J^{ext}_\nu. \tag{8}$$

3.1 Propagation of the Discontinuities in Non-Linear Electrodynamics

From the standard geometrical optic approximation, front waves can be characterized by discontinuities of certain derivatives of the field. The evolution of these discontinuities is determined by the field equations of motion. A rather simple and elegant method to deal with such discontinuities was provided by Hadamard [4], and will be used in this section.

Let Σ be a surface of discontinuity for the electromagnetic field and k_λ is the normal 4-vector on Σ. Following Hadamard's condition let us assume that the field is continuous through Σ but its first derivative is discontinuous. We set

$$[F_{\mu\nu}]_\Sigma = 0, \tag{9}$$

and

$$[\partial_\lambda\,F_{\mu\nu}]_\Sigma = f_{\mu\nu}k_\lambda, \tag{10}$$

in which the symbol $[J]_\Sigma$ represents the discontinuity of the function J through the surface Σ.

Applying these conditions into the equation of motion (2) we obtain

$$L_F f^{\mu\nu}\,k_\nu + 2L_{FF}\,\xi F^{\mu\nu}k_\nu = 0, \tag{11}$$

where ξ is defined by

$$\xi \equiv F^{\alpha\beta}\,f_{\alpha\beta}. \tag{12}$$

The cyclic identity yields

$$f_{\mu\nu}k_\lambda + f_{\nu\lambda}k_\mu + f_{\lambda\mu}k_\nu = 0. \tag{13}$$

Multiplying this equation by $k^\lambda\,F^{\mu\nu}$ yields

$$\xi k_\nu\,k_\mu\gamma^{\mu\nu} + 2\,F^{\mu\nu}f_{\nu\lambda}k^\lambda\,k_\mu = 0. \tag{14}$$

From the Eq. (11) it results:

$$f_{\mu\nu}\, k^{\nu} = -\, 2\, \frac{L_{FF}}{L_F}\, \xi F_{\mu\nu}\, k^{\nu}. \tag{15}$$

After some algebraic manipulations, the equation of propagation of the disturbances is obtained:

$$\{\gamma^{\mu\nu} + \Lambda^{\mu\nu}\}\, k_{\mu} k_{\nu} = 0 \tag{16}$$

in which $\gamma^{\mu\nu}$ is the Minkowski metric written in an arbitrary coordinate system and the quantity $\Lambda^{\mu\nu}$ is provided by:

$$\Lambda^{\mu\nu} \equiv -4\, \frac{L_{FF}}{L_F}\, F^{\mu\alpha}\, F_{\alpha}{}^{\nu}. \tag{17}$$

The main lesson we learn from this is that, in the non-linear electrodynamics, the disturbances propagate not in the Minkowskian background, but in an effective geometry which depends on the energy distribution of the field. The net effect of the nonlinearity can thus be summarized in the following property.

- The disturbances of nonlinear electrodynamics are null geodesics that propagate in the modified geometry:

$$g^{\mu\nu} = \gamma^{\mu\nu} - 4\, \frac{L_{FF}}{L_F}\, F^{\mu\alpha}\, F_{\alpha}{}^{\nu}. \tag{18}$$

A simple inspection on this formula shows that only in the particular linear case of Maxwell electrodynamics does the discontinuity of the electromagnetic field propagates in a Minkowski background[5]. From equation (18), we obtain the specific form of the components of the metric tensor:

$$g^{00} = 1 - 4\, \frac{L_{FF}}{L_F} E^2, \tag{19}$$

$$g^{ij} = \gamma^{ij} + 4\, \frac{L_{FF}}{L_F}\, \left(E^i\, E^j + B^i\, B^j - \gamma^{ij}\, B^k\, B_k\right), \tag{20}$$

$$g^{ol} = -4\, \frac{L_{FF}}{L_F} \gamma^{li} \epsilon_{ijk}\, E^j\, B^k, \tag{21}$$

in which we have set $E^2 = -\, E_{\alpha}\, E^{\alpha}$.

Before going into a specific model, let us make here a comment. Linear photons propagate in a Minkowskian underlying background. Non linear photons propagate in an effective geometry given by Eq. (18). Note, however, that this situation is not competitive to gravity processes. The reason for this is easy to understand: the above modified geometry (in case of non-linear electrodynamics) is not a universal one. Indeed, other kinds of particles and radiations behave as if the background metric were that dealt with in special relativity: the charged electrons follow time-like paths with respect to Minkowski metric.

[5]However, it is possible to present the wave propagation of Maxwell electrodynamics in a dieletric medium in terms of a modified geometry of the spacetime — see Appendix A.

Let us make a final comment on these expressions of the effective geometry. Although we will not examine the properties associated to the metric tensor $g^{\mu\nu}$, we would like to add the following. The background geometry, the Minkowski metric tensor, is the responsible for lowering and raising all coordinate indices. However, there may exist circumstances, if one is dealing with the associated curvature for instance, where one should know the inverse metric $g_{\mu\nu}$ defined by

$$g_{\mu\lambda}\, g^{\lambda\nu} = \delta_{\mu}{}^{\nu}. \tag{22}$$

In this case, the inverse is not obtained by means of the Minkowski tensor. Nevertheless there is no difficulty to obtain such expression in a compact form — see the Appendix B.

4 Strong Interaction

So much for an Abelian theory. Let us analyse now the case of the gauge theory of strong interactions. We start by considering a set A^a_μ of color multiplet that constitutes a Yang-Mills field of a non-Abelian theory[6] with $F^a_{\mu\nu}$ as the corresponding field. Let $\mathbf{F}$ be the invariant under space-time and internal color coordinates, defined by:

$$\mathbf{F} \equiv \vec{F}^{\mu\nu}.\,\vec{F}_{\mu\nu} = F^{a\,\mu\nu}\,F_{a\,\mu\nu}. \tag{23}$$

The dynamics set up in the Yang-Mills approach recover Maxwell electrodynamics by the identification of the Lagrangian to such a quantity, that is, $L_{\rm YM} = \mathbf{F}$. Should this be taken as an irretrievable paradigm? Does a change on this hypothesis, in the hadron world, yield the desirable consequences? Before answering this question, let us make a small comment on the classical description.

From a broad principle the Lagrangian should have the general non-linear form

$$L = L(\mathbf{F}). \tag{24}$$

Although one can go further without being necessary to specify the form of such a functional, in order to have a definite model that exhibits in a simple manner the main aspects of our ideas, we limit all our considerations here to a specific toy model provided by[7]:

$$L_{\rm NDE} = -\frac{1}{4}\mathbf{F}\left(1 - \frac{\mathbf{F}}{\epsilon_s}\right)^{-1}, \tag{25}$$

in which ϵ_s is a constant. The corresponding equation of motion is given by

$$D^{ac}_{\nu}\left[\left(1 - \frac{\mathbf{F}}{\epsilon_s}\right)^{-2} F^{\mu\nu}_{c}\right] = 0 \tag{26}$$

that is

$$[\delta_{cd}\partial_\nu + g\, c_{acd}A^a_\nu]\left\{\left(1 - \frac{\mathbf{F}}{\epsilon_s}\right)^{-2} F^{d\mu\nu}\right\} = 0, \tag{27}$$

[6]We have in mind, for instance, the standard $SU(3)$ non-Abelian QCD model.

[7]We would like to note that all of the following conclusions are qualitatively independent of such a form.

where c_{abc} are the constants of the structure of the gauge group and g is the strong interaction coupling constant. To proceed with the examination of the corresponding behavior of the classical gluons in such non-linear theory there is no better way than to consider the very high energy case through the analysis of the eikonal. In the standard Yang-Mills dynamics, the eikonal is nothing but null-cones of the Minkowski background spacetime, as in Maxwell theory. This is not the case for our Lagrangian. Indeed, there exist examples of spin-one theories in which the eikonal follows null geodesics in an effective geometry which depends not only on the background metric, but also on the field properties. This has been shown in the case of pure non-linear electrodynamics [3]. This result is still valid in the non-Abelian gauge theory, as we will now show[8].

Let Σ be a surface of discontinuity for the gauge field. Following Hadamard's [4] condition, we take the potential and the field as being continuous through Σ but having its first derivative discontinuous, that is:

$$[F^a_{\mu\nu}]_\Sigma = 0, \tag{28}$$

and

$$[\partial_\lambda F^a_{\mu\nu}]_\Sigma = f^a_{\mu\nu} k_\lambda, \tag{29}$$

in which the symbol $[J]_\Sigma$ represents the discontinuity of the function J through the surface Σ and k_λ is the normal 4-vector on Σ.

Applying these conditions into the equation of motion (26), we obtain

$$f_a^{\mu\nu}\, k_\nu + \frac{4}{\epsilon_s - \mathbf{F}}\, \xi F_a^{\mu\nu}\, k_\nu = 0, \tag{30}$$

where ξ is defined by

$$\xi \equiv F_a^{\alpha\beta}\, f^a_{\alpha\beta}. \tag{31}$$

From the cyclic identity,

$$D^{bc}_\lambda\, F^a_{\mu\nu} + D^{bc}_\mu\, F^a_{\nu\lambda} + D^{bc}_\nu\, F^a_{\lambda\mu} = 0 \tag{32}$$

and using the above continuity conditions of the potential and the fields, we obtain

$$f^a_{\mu\nu} k_\lambda + f^a_{\nu\lambda} k_\mu + f^a_{\lambda\mu} k_\nu = 0. \tag{33}$$

Multiplying this equation by $k^\lambda\, F_a^{\mu\nu}$ yields

$$\xi k_\nu\, k_\mu \gamma^{\mu\nu} + 2\, F_a^{\mu\nu}\, f^a_{\nu\lambda}\, k^\lambda\, k_\mu = 0. \tag{34}$$

Using Eq. (30) in this expression and after some algebraic manipulations, the equation of propagation of the disturbances is obtained:

$$\{\gamma^{\mu\nu} + \Lambda^{\mu\nu}\}\, k_\mu k_\nu = 0 \tag{35}$$

[8] At the basis of this property rests the fact that the dependence of the group connection on the potential does not contain derivatives, but only an algebraic form. Standard conditions for the wave disturbances, like Hadamard's structure, imply that this sector of the non-linearity does not affect the velocity of propagation.

in which the quantity $\Lambda^{\mu\nu}$ is

$$\Lambda^{\mu\nu} \equiv -\frac{8}{\epsilon_s - \mathbf{F}}\, F^{a\,\mu\lambda}\, F_{a\,\lambda}{}^{\nu}. \tag{36}$$

The net effect of this modification of the non-linearity of the Yang-Mills theory can thus be summarized in the following property: *The disturbances of the gauge field controlled by the non-linear Lagrangian $L_{\rm NDE}$ propagate through null geodesics of the modified effective geometry given by:*

$$g^{\mu\nu} = \eta^{\mu\nu} - \frac{8}{\epsilon_s - \mathbf{F}}\, F^{a\,\mu\lambda}\, F_{a\,\lambda}{}^{\nu}. \tag{37}$$

Let us emphasize that this property stands only from the structural form of the dynamics of our theory. To avoid misunderstanding[9] we state:

- **This geometry modification is a pure spin-one non-linear phenomenon.**

Thus, we conclude from the above statement that gluon dynamics can be examined through the properties of null geodesics in the modified geometry. The fact that in this theory massless spin-one particles do not follow Minkowski null cone occurs as a direct consequence of the particular non-linear dynamics used in our model, which is distinct from the one contained in standard Yang-Mills theory.

In order to show the confinement induced by such non-linear model, there is no better way than to investigate the behavior of the null geodesics in this geometry. For our purposes, we restrict ourselves here to the analysis of a spherically symmetric and static solution. A direct computation shows that, in the spherical coordinate system, such a particular solution can be found uniquely in terms of a radial component given by:

$$F^a_{01} \quad = \quad f(r)\, n^a, \tag{38}$$

in which n^a is a constant vector in the color space and $f(r)$ is given by the relation:

$$\frac{f(r)}{[\epsilon_s + 2f(r)^2]^2} = \frac{e^{\frac{4C_o}{\epsilon_s^2}}}{r^2}, \tag{39}$$

where C_o is the constant of integration. We can determine its value by imposing the Maxwell asymptotic limit for large r, i.e.,

$$f(r) = \frac{Q}{r^2}. \tag{40}$$

The parameter Q is related to distribution of the charge $q(r)$:

$$Q = \int \mathrm{d}^3x q(r). \tag{41}$$

[9]The reason for this additional assertion is due to the fact that modifications on the underlying geometry are traditionally supposed to be connected to gravitational forces. We would like to stress that this **is not** the case here.

Introducing this result in the exact solution we obtain:

$$\frac{f(r)}{[\epsilon_s + 2f(r)^2]^2} = \frac{Q}{\epsilon_s^2 r^2}. \tag{42}$$

From the standard definition of the energy momentum tensor we obtain:

$$T_{\mu\nu} = \frac{1}{4}\frac{\epsilon_s \mathbf{F}}{\epsilon_s - \mathbf{F}}\gamma_{\mu\nu} + \frac{\epsilon_s^2}{(\epsilon_s - \mathbf{F})^2}\vec{F}_{\mu\alpha}.\vec{F}^{\alpha}{}_{\nu}, \tag{43}$$

and for the density of energy T_{00} results:

$$T^0{}_0 = \frac{Qf(r)}{2\epsilon_s r^2}\left[\epsilon_s - 2f(r)^2\right]. \tag{44}$$

A remarkable characteristic of the effective geometry induced by this non linear spin-1 field theory may be made explicit by looking into the line element of the corresponding effective geometry in which the massless spin-1 particles travel. Before this, however, it seems worth to make the following remark. The method of Hadamard used in the previous section enables us to obtain the propagation of the disturbances in terms of $g^{\mu\nu}$, a modified metric of the underlining Minkowski one. In order to analyse this propagation as a null geodesic we need the covariant form of the effective geometry — $g_{\mu\nu}$ — defined by the relation (22).

In this particular case where the metric is diagonal, the inverse is obtained trivially. For completeness and future reference we present the general expression of the inverse metric in a compact form in the Appendix B.

From Eq. (37) and the spherically symmetric solution of the background, we obtain the following non-vanishing contravariant components of the metric:

$$g^{00} = \frac{\epsilon_s - 6f(r)^2}{\epsilon_s + 2f(r)^2} \tag{45}$$

$$g^{11} = -g^{00}, \tag{46}$$

$$g^{22} = -\frac{1}{r^2} = \sin^2\theta g^{33}. \tag{47}$$

Thus the line element associated to the effective metric seen by the perturbations of the gauge field, that is,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{48}$$

is provided by

$$ds^2 = \left[\frac{\epsilon_s + 2f(r)^2}{\epsilon_s - 6f(r)^2}\right]dt^2 - \left[\frac{\epsilon_s + 2f(r)^2}{\epsilon_s - 6f(r)^2}\right]dr^2 - r^2\,d\theta^2 - r^2\,\sin^2\theta\,d\phi^2, \tag{49}$$

in which $f(r)$ is given by Eq. (42). A direct inspection of the line element shows that there is a value of the function $f(r)$ in which the g_{00} and g_{11} metric components are singular. This critical point (r_c) corresponds to the solution given by

$$f(r_c) = \sqrt{\frac{\epsilon_s}{6}} \tag{50}$$

Note, however, that this is not a true physical singularity. Indeed, let us look into the effective potential in order to prove this[10].

To obtain the form of the potential, it is enough to look for the radial equation of motion of the geodesics in this solution, as a function of the proper time. The simplest way to arrive at this result is by means of the variational principle

$$\delta \int \left\{ \left[\frac{\epsilon_s + 2f(r)^2}{\epsilon_s - 6f(r)^2} \right] \dot{t}^2 - \left[\frac{\epsilon_s + 2f(r)^2}{\epsilon_s - 6f(r)^2} \right] \dot{r}^2 - r^2\, \dot{\theta}^2 - r^2 \, \sin^2\theta \, \dot{\phi}^2 \right\} ds = 0 \qquad (51)$$

in which we have used the effective geometry — as it appears in Eq. (49). A dot means proper time derivative. The radial dependence yields:

$$\dot{r}^2 + V_{eff} = l_0^2 \qquad (52)$$

in which the potential V_{eff} has the form:

$$V_{eff} = \frac{\epsilon_s - 6f(r)^2}{\left[\epsilon_s + 2f(r)^2\right]^2} \left\{ \frac{\epsilon_s^2 h_0^2 f(r)}{Q} - l_0^2 \left[\epsilon_s - 6f(r)^2\right] \right\} + l_0^2, \qquad (53)$$

and h_0 and l_0 are constants of motion.

A direct inspection on the form of this potential shows that the gluons in the $L_{\rm NDE}$ non-linear theory behave as particles endowed with energy l_0^2, immersed in a central field of forces characterized by the potential V_{eff}. To present a specific form of this potential in terms of the radial coordinate r, let us consider an approximated solution for the function $f(r)$. Expanding the function $f(r)$ in a series we can write:

$$f(r) = \frac{Q}{r^2} + O\left(r^{-6}\right). \qquad (54)$$

Which yields the potential:

$$V_{eff} = \frac{\left[\epsilon_s - \frac{6Q}{r^4}^2\right] h_0^2}{r^2} - \frac{\left[\epsilon_s - \frac{6Q}{r^4}^2\right]^2 l_0^2}{\epsilon_s^2} + l_0^2. \qquad (55)$$

The behavior of $V_{eff}(r)$ is very similar to the behavior of photons in a Schwarzschild gravitational field. We note that coordinates t and r interchange their role at the critical radius r_c. The region $r = r_c$ defines a null surface for the effective geometry [11], which means that gluons path (i.e., the null cones in the effective geometry) have their concavity turned to the inside domain, in a very similar way as it happens in a gravitational black hole. This allows us to claim that gluons are hidden in a compact domain, limited by the critical radius.

[10] It seems worthwhile to quote here an analogous situation occurring in Einstein's theory of general relativity in the case of the Schwarzschild solution. In both situations we are dealing with a horizon and not with a true singularity.

[11] See Appendix C for details.

5 Conclusion

Let us summarize what we have achieved. Massless spin-one particles (gluons) obeying Yang-Mills dynamics travel along null cones. In a Minkowski spacetime there is no way to confine such particles in a compact region, once it could be associated with the presence of a singular horizon. We are then led to a modification of the self interaction properties of the gluons. We present here a model that can be equivalently described in terms of an effective change of the background geometry. We analyse a particular example of a static, spherically symmetric solution and proceed to the exam of the corresponding null geodesics, the gluon paths, in the associated geometry. It then follows that the behavior of gluons can be examined in terms of the potential given in Eq. (53) showing, through the appearance of a horizon, the required confining feature. This result allows us to argue that the solution of the confinement of the gluons could well be found along these lines.

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Appendix

A – Dielectric Constant and the Effective Geometry

In order to show the non-familiar reader the treatment that involves dealing with the propagation of the non-linear theory as a modification of the background geometry, we will present here the simplest possible case of the standard Maxwell theory in a dielectric medium. We will show how it is possible to present the wave propagation of linear electrodynamics in a medium in terms of a modified geometry of the spacetime.

In this section we take the Maxwell theory in a medium such that the electromagnetic field is represented by two anti-symmetric tensors $F_{\mu\nu}$ and $P_{\mu\nu}$ given in terms of the electric and magnetic vectors, as seen by an arbitrary observer endowed with a four-velocity v^{μ}, by the standard expressions:

$$F_{\mu\nu} = E_{\mu}\, v_{\nu} - E_{\nu}\, v_{\mu} + \eta_{\mu\nu}{}^{\rho\sigma}\, v_{\rho}\, H_{\sigma}, \tag{A.1}$$

and

$$P_{\mu\nu} = D_{\mu}\, v_{\nu} - D_{\nu}\, v_{\mu} + \eta_{\mu\nu}{}^{\rho\sigma}\, v_{\rho}\, B_{\sigma}. \tag{A.2}$$

Maxwell equations are:

$$\partial^{\nu}\, F^{*}_{\mu\nu} = 0, \tag{A.3}$$

$$\partial^{\nu}\, P_{\mu\nu} = 0. \tag{A.4}$$

Following Hadamard, we consider the discontinuities on the fields as given by:

$$[\partial_\lambda E_\mu]_\Sigma = k_\lambda e_\mu, \tag{A.5}$$

$$[\partial_\lambda D_\mu]_\Sigma = k_\lambda d_\mu, \tag{A.6}$$

$$[\partial_\lambda H_\mu]_\Sigma = k_\lambda h_\mu, \tag{A.7}$$

$$[\partial_\lambda B_\mu]_\Sigma = k_\lambda b_\mu. \tag{A.8}$$

Using the constitutive relations[12]:

$$d_\mu = \epsilon\, e_\mu, \tag{A.9}$$

$$b_\mu = \frac{h_\mu}{\mu}, \tag{A.10}$$

one obtains after a straightforward calculation:

$$k_\mu\, k_\nu\, [\eta^{\mu\nu} + (\epsilon\,\mu - 1) v^\mu\, v^\nu] = 0. \tag{A.11}$$

This shows that even the simple case of the evolution of the wave front in standard Maxwell equation in a medium can be interpreted in terms of an effective geometry $g^{\mu\nu}$ that depends not only on the medium properties ϵ and μ, but also on the observer's velocity, given by:

$$g^{\mu\nu} \equiv \eta^{\mu\nu} + (\epsilon\,\mu - 1) v^\mu\, v^\nu. \tag{A.12}$$

This ends our proof.

B – Effective Geometry

In order to obtain the general form of the inverse geometry, one must use some well-known properties of the $F_{\mu\nu}$ tensor. Let us set the geometry as a non-linear perturbation of the Minkowski metric

$$g^{\mu\nu} = \gamma^{\mu\nu} + \phi^{\mu\nu} \tag{B.1}$$

For the case we are interested here we have

$$\phi^{\mu\nu} = F^{\mu\alpha}\, F_\alpha{}^\nu. \tag{B.2}$$

The inverse metric tensor could be obtained in the usual form as an infinite series[13]

$$g_{\mu\nu} = \gamma_{\mu\nu} - \phi^\alpha{}_\mu \phi_{\alpha\nu} + \ldots \tag{B.3}$$

However, in the particular case we are considering in this paper, such a procedure can be considerably simplified. This can be done by using the following relations:

$$F^*_{\mu\nu}\, F^{\nu\lambda} = -\frac{1}{4}\, F^* \delta_\mu{}^\lambda, \tag{B.4}$$

[12] We deal here with the simplest case of linear isotropic relations, just for didactic reasons.

[13] We would like to thank R. Rodrigues and R. K. Barcellos for the suggestion that this series can be written in a compact way under general circumstances.

and

$$F^*_{\mu\lambda}\, F^{*\,\lambda\nu} - F_{\mu\lambda}\, F^{\lambda\nu} = \frac{1}{2}\, F\, \delta_\mu{}^\nu, \tag{B.5}$$

in which

$$F^* = F^{\mu\alpha} F^*_{\mu\alpha}. \tag{B.6}$$

From these identities we obtain

$$\phi_\mu{}^\nu\, \phi_{\nu\lambda} = \frac{1}{16}\, F^{*2}\gamma_{\mu\lambda} - \frac{1}{2} F\, \phi_{\mu\lambda}. \tag{B.7}$$

Thus the covariant form of the metric yields

$$g_{\mu\nu} = \alpha\gamma_{\mu\nu} + \beta\phi_{\mu\nu} \tag{B.8}$$

where,

$$\beta = \frac{16}{F^{*2} + 8F - 16}, \tag{B.9}$$

and

$$\alpha = \beta\left(\frac{F}{2} - 1\right). \tag{B.10}$$

C – The Null Surface

Let us consider the surface $\psi = r = const$ in the case of the solution examined in the previous section. We are interested here in the analysis of the characteristics of the equation of motion of the non linear electromagnetic field in the neighborhood of the critical radius defined by relation,

$$f(r = r_c) = \sqrt{\frac{\epsilon_s}{6}}, \tag{C.1}$$

that results the value for this radial coordinate:

$$r_c = \frac{4}{3}\sqrt{Q\sqrt{\frac{6}{\epsilon_s}}}. \tag{C.2}$$

Using the metric $g^{\mu\nu}$, we have

$$\psi_\mu\psi_\nu g^{\mu\nu} = \psi_1\psi_1 g^{11} = -(\psi_1)^2\left[\frac{\epsilon_s - 6f(r)^2}{\epsilon_s + 2f(r)^2}\right], \tag{C.3}$$

where we have set $\psi_\mu = \partial_\mu\psi$. At the value $r = r_c$, this relation vanishes showing that the surface ψ is a null surface at the critical radius, for the non-linear photon.

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