The Symmetry Algebra of Open Spin Chains in a Magnetic Field

M. Rego-Monteiro *
INFN-Sezione di Torino,
Dipartimento di Fisica dell'Universita' di Torino
Via Pietro Giuria, 1
10125, Torino – Italy

and

Centro Brasileiro de Pesquisas Fisicas-CBPF
Rua Dr. Xavier Sigaud 150
22290-180, Rio de Janeiro – Brazil

Abstract

We show that the one-dimensional Ising and XXZ open spin chains in a magnetic field and with surface fields are invariant under a two-parametric generalization of the $sl_q(2)$ algebra with deformation parameters being a root of unit.

Key-words: quantum algebras; Yang-Baxter equation; one-dimensional spin chains; Ising chain; XXZ chain

*e-mail: monteiro@to.infn.it

Since the discovery of quantum (or deformed) algebras there has been an intense activity on the study of their connections to physical systems [1]. Very recently, an old discrepancy between theory and experiment concerning the stability of phonon spectrum in ^4He was overcome [2] by treating the gas of phonon excitations in ^4He as an ideal deformed bosonic gas. Moreover, this deformed model [2] reproduces within 5% of accuracy the experimental ^4He molar specific heat for $T < 1^0K$. In brief, the low energy excitations in superfluid ^4He seem to be an example of the presence and relevance of deformed algebras in nature.

The quantum algebra $sl_q(2)$ was first introduced by Kulish and Reshetikhin [3] and independently by Sklyanin [4] within the approach of the quantum inverse scattering method (QISM) [5]. Later, quantum algebras were introduced by Jimbo [6] as q-analogues of classical Lie algebras and presented as quasitriangular Hopf algebras by Drinfeld [7]. Of particular interest is the relation of this algebraic structure to one-dimensional spin chains [8]; for example, the XXZ and XYZ open chains are connected to the $sl_q(2)$ [9] and Sklyanin [10] algebras respectively. It is the purpose of this letter to give a further contribution to the above connection: we show that the one-dimensional Ising and XXZ open chains in an external magnetic field and with surface fields are both invariant under a two-parametric (q, λ) generalization of the $sl_q(2)$ algebra at q and λ being a root of unit. This generalization, firstly introduced by Bazhanov and Stroganov [11], is a six-generator bi-algebra defined from the Fundamental Commutation Relations of the QISM for a two-state trigonometric R-matrix with two modulo one parameters.

Let $R(x)$ be an invertible matrix acting in $\mathbf{C}^2 \otimes \mathbf{C}^2$, given explicitly as [12]

$$R(x) = \begin{pmatrix} qx - q^{-1}x^{-1} & 0 & 0 & 0 \\ 0 & \lambda(x - x^{-1}) & \Omega x & 0 \\ 0 & \Omega x^{-1} & \lambda^{-1}(x - x^{-1}) & 0 \\ 0 & 0 & 0 & qx - q^{-1}x^{-1} \end{pmatrix}, \qquad (1)$$

where x is a variable, $\Omega \equiv q - q^{-1}$, λ and q are non-zero complex parameters. This R-matrix (eq.(1)) satisfies the Yang-Baxter equation [13]

$$R_{12}(x/y)R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(x/y), \qquad (2)$$

where $R_{12} \equiv \sum_i a_i \otimes b_i \otimes \mathbf{1}$, $R_{13} \equiv \sum_i a_i \otimes \mathbf{1} \otimes b_i$ etc., with the R-matrix written in this notation as $R \equiv \sum_i a_i \otimes b_i$.

Define an operator $L(x)$ in $\mathbf{C}^2 \otimes \mathbf{C}^N$, $N \geq 2$, satisfying the Fundamental Commutation Relations (FCR) [5]

$$R_{12}(x/y)L_1(x)L_2(y) = L_2(y)L_1(x)R_{12}(x/y), \tag{3}$$

where $L_1 \equiv L \otimes \mathbf{1}$, $L_2 \equiv \mathbf{1} \otimes L$ and $\mathbf{1}$ is the unit matrix in $\mathbf{C}^2$.

Using for $L(x)$ a similar dependence in x that we have for $R(x)$ (eq.(1)) one chooses an L-operator of the form

$$L(x) = xL^+ + x^{-1}L^-, \tag{4}$$

where $L^+(L^-)$ is independent of x and has an upper(lower) triangular form in $\mathbf{C}^2$.

Substituting eq.(4) into the FCR one gets seven equations but only three of them are independent. Considering the R-matrix given by eq.(1) the three independent equations reduce to

$$\begin{aligned}
&[L^\pm_{ii}, L^\pm_{jj}] = [L^+_{ii}, L^-_{jj}] = 0 \ , \ \ i,j = 1,2, \\
&L^\pm_{11}L^+_{12} = \omega^{-s_{\sigma(\pm)}}L^+_{12}L^\pm_{11} \ , \ L^\pm_{11}L^-_{21} = \omega^{s_{\sigma(\pm)}}L^-_{21}L^\pm_{11}, \\
&L^\pm_{22}L^+_{12} = \omega^{-s_{\sigma(\mp)}}L^+_{12}L^\pm_{22} \ , \ L^\pm_{22}L^-_{21} = \omega^{s_{\sigma(\mp)}}L^-_{21}L^\pm_{22}, \\
&L^-_{21}L^+_{12} = \omega^{s_3}L^+_{12}L^-_{21} - (\omega^{-s_1} - \omega^{-s_2})(L^+_{11}L^-_{22} - L^+_{22}L^-_{11}),
\end{aligned} \tag{5}$$

where $\sigma(+) = 2$, $\sigma(-) = 1$, $\omega = q^2$ and [14]

$$(q\lambda)^{-1} = \omega^{s_1} \ , \ q\lambda^{-1} = \omega^{s_2} \ , \ \lambda^2 = \omega^{s_3}. \tag{6}$$

The relations in eq.(5) can be considered [11] as the definition of a bi-algebra with six generating elements $L^\pm_{ii}$ and L^+_{12}, L^-_{21} and co-product

$$\Delta L^\pm_{ij} \equiv \sum_k L^\pm_{ik} \otimes L^\pm_{kj}. \tag{7}$$

This algebra will be called the two-state fundamental commutation algebra and will be denoted as $FC_q(2)$ since it is defined through the relations (eq.(5)) which comes from the FCR (eq.(3)) using the two-state R-matrix defined in eq.(1). The $sl_q(2)$ algebra is obtained by setting

$$\lambda = 1 \ , \ L^\epsilon_{11} = L^{-\epsilon}_{22} \ , \ \epsilon = \pm. \tag{8}$$

Moreover, if one requires that

$$det_{\mathbf{C}^2}(L^{\epsilon}_{ij}) \neq 0, \tag{9}$$

for all values of ϵ, i, j, from eq.(5), it follows that this happens provided that $\omega = \exp(2\pi i/N)$ and in this case from eq.(6) one easily obtains

$$s_1 + s_2 + s_3 = 0 \ (mod.N) \quad , \quad s_2 - s_1 = 1 \ (mod.N). \tag{10}$$

It is convenient to introduce a matrix S^+ defined by [14]

$$\sum_k S^+_{ik} L^+_{kj} = \sum_k L^+_{ik} S^+_{kj} = \delta_{ij} 1_{(N\times N)}, \tag{11}$$

where $1_{(N\times N)}$ is the $N \times N$ unit matrix. Let $\wp_2$ be the algebra generated by S^+_{ij} and L^-_{ij}. One can show that Q_M given by [14]

$$Q_M = Tr\left\{\Theta(L^- S^+)^M\right\} \quad , \quad M = 1, 2, \tag{12}$$

where $\Theta \equiv diag(q^2, 1)$ and the trace is taken in $\mathbf{C}^2$, are Casimir operators of $\wp_2$.

When $M = 1$, we have

$$Q_1 = \omega A_1 + A_2 + L^-_{21} S^+_{12} = A_1 + \omega A_2 + S^+_{12} L^-_{21}, \tag{13}$$

where $A_i \equiv L^-_{ii} S^+_{ii}$ and the second form of Q_1 in the above equation is obtained from the first one through the use of

$$[L^-_{21}, S^+_{12}] = (1 - \omega)(A_1 - A_2). \tag{14}$$

From the commutation relations among A_i, L^-_{ij} and S^+_{ij}

$$\begin{aligned} A_1 L^-_{21} &= \omega^{-1} L^-_{21} A_1 \quad , \quad A_1 S^+_{12} = \omega S^+_{12} A_1, \\ A_2 L^-_{21} &= \omega L^-_{21} A_1 \quad , \quad A_2 S^+_{12} = \omega^{-1} S^+_{12} A_2, \end{aligned} \tag{15}$$

the commutation properties of Q_1 are readily seen. Since in eq.(13) the operators S^+_{12} and L^-_{21} always appear in pairs, using eq.(15) one sees that

$$[Q_1, A_i] = [Q_1, L^{\pm}_{ii}] = 0. \tag{16}$$

Moreover,

$$\begin{aligned} L_{21}^- Q_1 &= L_{21}^- A_1 + \omega L_{21}^- A_2 + L_{21}^- S_{12}^+ L_{21}^- \\ &= \left(\omega A_1 + A_2 + L_{21}^- S_{12}^+\right) L_{21}^- = Q_1 L_{21}^-, \\ S_{12}^+ Q_1 &= \omega S_{12}^+ A_1 + S_{12}^+ A_2 + S_{12}^+ L_{21}^- S_{12}^+ \\ &= \left(A_1 + \omega A_2 + S_{12}^+ L_{21}^-\right) S_{12}^+ = Q_1 S_{12}^+, \end{aligned} \tag{17}$$

where it was made use of eqs.(13, 15). Finally, from eqs.(11, 16) one also has $[Q_1, L_{12}^+] = 0$.

In the same way one analyses the case where $M = 2$. From eq.(12) and by a repeated use of eq.(14) two useful forms of Q_2 are obtained

$$\begin{aligned} Q_2 &= \omega^2 A_1^2 + A_2^2 + \omega(1-\omega) A_1 A_2 + 2\left(\omega A_1 + A_2\right) L_{21}^- S_{12}^+ \\ &+ L_{21}^- S_{12}^+ L_{21}^- S_{12}^+ \\ &= A_1^2 + \omega^2 A_2^2 + \omega(1-\omega) A_1 A_2 + 2\left(A_1 + \omega A_2\right) S_{12}^+ L_{21}^- \\ &+ S_{12}^+ L_{21}^- S_{12}^+ L_{21}^-. \end{aligned} \tag{18}$$

Following the steps described in the previous case one gets the commutation properties of Q_2. It is interesting to note that if one considers the two-parameter solution of eq.(5) referred to in the literature as $sl_{q,\lambda}(2)$ (for instance, see [15]) one easily obtains that Q_2 is dependent of Q_1. In what follows, we shall see for the generalization we are considering here that Q_i, for $i = 1, 2$, are independent.

Let us now analyse the role played by Q_M in the algebra $FC_q(2)$. First we rewrite them in terms of the generators of $FC_q(2)$; performing appropriate substitutions we obtain

$$\begin{aligned} Q_1 &= \omega L_{11}^- (L_{11}^+)^{-1} + L_{22}^- (L_{22}^+)^{-1} - \omega^{s_2} (L_{11}^+)^{-1} (L_{22}^+)^{-1} L_{21}^- L_{12}^+ \\ &= L_{11}^- (L_{11}^+)^{-1} + \omega L_{22}^- (L_{22}^+)^{-1} - \omega^{-s_1} (L_{11}^+)^{-1} (L_{22}^+)^{-1} L_{12}^+ L_{21}^- \end{aligned} \tag{19}$$

and

$$\begin{aligned} Q_2 &= \omega^2 (L_{11}^-)^2 (L_{11}^+)^{-2} + \omega(1-\omega) L_{11}^- L_{22}^- (L_{11}^+)^{-1} (L_{22}^+)^{-1} \\ &- 2\omega^{s_2} \left(\omega L_{11}^- (L_{11}^+)^{-1} + L_{22}^- (L_{22}^+)^{-1}\right) (L_{11}^+)^{-1} (L_{22}^+)^{-1} L_{21}^- L_{12}^+ \\ &+ \omega^{2s_2} (L_{11}^+)^{-2} (L_{22}^+)^{-2} L_{21}^- L_{12}^+ L_{21}^- L_{12}^+ + (L_{22}^-)^2 (L_{22}^+)^{-2}. \end{aligned} \tag{20}$$

For the case where the parameters are roots of unit we see from eq.(5) that $(L_{ii}^{\pm})^N$ belongs to the center of $FC_q(2)$ i.e,

$$(L_{ii}^{\pm})^N = (\mu_i^{\pm})^N 1_{(N\times N)}, \tag{21}$$

where $\mu_i^{\pm}$ are constants. Thus, we can rewrite Q_M as

$$\begin{aligned} Q_1 &= (\mu_1^+)^{-N}\omega L_{11}^-(L_{11}^+)^{N-1} + (\mu_2^+)^{-N}L_{22}^-(L_{22}^+)^{N-1} \\ &- (\mu_1^+\mu_2^+)^{-N}\omega^{s_2}(L_{11}^+)^{N-1}(L_{22}^+)^{N-1}L_{21}^-L_{12}^+ \end{aligned} \tag{22}$$

and

$$\begin{aligned} &Q_2 = \omega^2(\mu_1^+)^{-N}(L_{11}^-)^2(L_{11}^+)^{N-2} + (\mu_2^+)^{-N}(L_{22}^-)^2(L_{22}^+)^{N-2} \\ &+\omega(1-\omega)(\mu_1^+\mu_2^+)^{-N}L_{11}^-L_{22}^-(L_{11}^+)^{N-1}(L_{22}^+)^{N-1} \\ &-2\omega^{s_2}(\mu_1^+\mu_2^+)^{-N}\left(\omega(\mu_1^+)^{-N}L_{11}^-(L_{11}^+)^{N-1}\right. \\ &\left.+(\mu_2^+)^{-N}L_{22}^-(L_{22}^+)^{N-1}\right)(L_{11}^+)^{N-1}(L_{22}^+)^{N-1}L_{21}^-L_{12}^+ \\ &+\omega^{2s_2}(\mu_1^+\mu_2^+)^{-N}(L_{11}^+)^{N-2}(L_{22}^+)^{N-2}L_{21}^-L_{12}^+L_{21}^-L_{12}^+. \end{aligned} \tag{23}$$

From the above expressions we see that depending on which representation of $FC_q(2)$ is considered, one has a different form of eqs.(22-23); thus Q_1 and Q_2 are more properly referred to as invariants of $FC_q(2)$.

We shall now compute Hamiltonians of spin chains, invariant under the $FC_q(2)$ algebra for $N = 2$. In the tensor product space of two $N = 2$ representations of $FC_q(2)$ the invariants take the form

$$\begin{aligned} \Delta Q_1 &= -(\mu_1^+)^{-2}\Delta L_{11}^-\Delta L_{11}^+ + (\mu_2^+)^{-2}\Delta L_{22}^-\Delta L_{22}^+ \\ &- (\mu_1^+\mu_2^+)^{-2}\Delta L_{11}^+\Delta L_{22}^+\Delta L_{21}^-\Delta L_{12}^+ \end{aligned} \tag{24}$$

and

$$\begin{aligned} &\Delta Q_2 = (\mu_1^+)^{-2}(\Delta L_{11}^-)^2 + (\mu_2^+)^{-2}(\Delta L_{22}^-)^2 - 2(\mu_1^+\mu_2^+)^{-2}\Delta L_{11}^-\Delta L_{22}^-\Delta L_{11}^+\Delta L_{22}^+ \\ &+2(\mu_1^+\mu_2^+)^{-2}\Delta L_{11}^-\Delta L_{22}^+\Delta L_{21}^-\Delta L_{12}^+ - 2(\mu_1^+\mu_2^+)^{-2}\Delta L_{22}^-\Delta L_{11}^+\Delta L_{21}^-\Delta L_{12}^+ \\ &+(\mu_1^+\mu_2^+)^{-2}\Delta L_{21}^-\Delta L_{12}^+\Delta L_{21}^-\Delta L_{12}^+, \end{aligned} \tag{25}$$

where $\Delta L_{ij}^{\pm}$ are given in eq.(7) and we have used $s_2 = 2l_2$ and $s_1 = 2l_1 + 1$, l_i integers, which is a solution of eq.(10) for $N = 2$.

The L-matrices in the cyclic representation are $N \times N$ matrices given by [11, 14]

$$
\begin{aligned}
&L_{11}^{+} = \mu_1^{+} X^{-s_2} \quad , \quad L_{11}^{-} = \mu_1^{-} X^{-s_1}, \\
&L_{22}^{+} = \mu_2^{+} X^{-s_1} \quad , \quad L_{22}^{-} = \mu_2^{-} X^{-s_2}, \\
&L_{12}^{+} = t Z^{-1} \left(\mu_2^{+} X^{-s_1} + \xi^{-2} \mu_1^{-} X^{-s_2} \right), \\
&L_{21}^{-} = t^{-1} Z \left(\xi^{2} \mu_1^{+} X^{-s_2} + \mu_2^{-} X^{-s_1} \right),
\end{aligned} \tag{26}
$$

where Z and X are $N \times N$ matrices satisfying $ZX = \omega XZ$ and t, ξ are constants. For $N = 2$ one immediately has

$$
\begin{aligned}
&L_{11}^{+} = \mu_1^{+} \mathbf{1} \quad , \quad L_{11}^{-} = \mu_1^{-} \sigma^{x}, \\
&L_{22}^{+} = \mu_2^{+} \sigma^{x} \quad , \quad L_{22}^{-} = \mu_2^{-} \mathbf{1}, \\
&L_{12}^{+} = t \left(i \mu_2^{+} \sigma^{y} + \xi^{-2} \mu_1^{-} \sigma^{z} \right), \\
&L_{21}^{-} = t^{-1} \left(\xi^{2} \mu_1^{+} \sigma^{z} + i \mu_2^{-} \sigma^{y} \right),
\end{aligned} \tag{27}
$$

with $\mathbf{1}$ the 2×2 unit matrix and $(\sigma^x, \sigma^y, \sigma^z)$ the Pauli matrices.

Using eqs.(7, 27) in eq.(24) we obtain, after a long but straightforward calculation,

$$
\begin{aligned}
\Delta Q_1 \;&=\; c_0 \mathbf{1} \otimes \mathbf{1} + c_1 \sigma^x \otimes \sigma^x + c_2 \sigma^y \otimes \sigma^y + c_3 \sigma^z \otimes \sigma^z \\
&+\; c_4 \mathbf{1} \otimes \sigma^x + c_5 \sigma^x \otimes \mathbf{1} + c_6 \sigma^y \otimes \sigma^z + c_7 \sigma^z \otimes \sigma^y,
\end{aligned} \tag{28}
$$

where

$$
\begin{aligned}
c_1 \;&=\; -(\mu_1^-)^2 + (\mu_2^-)^2 + \mu_1^+ \mu_2^+ \left(-(\mu_1^-)^2 + (\mu_2^-)^2 \right) \\
&+\; \mu_1^- \mu_2^- \left(-(\mu_1^+)^2 + (\mu_2^+)^2 \right), \\
c_2 \;&=\; \mu_1^- \mu_2^+ \left(-\xi^2 (\mu_1^+)^2 + \xi^{-2} (\mu_2^-)^2 \right), \\
c_3 \;&=\; \mu_1^+ \mu_2^- \left(-\xi^2 (\mu_2^+)^2 + \xi^{-2} (\mu_1^-)^2 \right), \\
c_4 \;&=\; \mu_1^- \mu_2^+ \left(-\xi^2 \mu_1^+ \mu_2^+ + \xi^{-2} \mu_1^- \mu_2^- \right), \\
c_5 \;&=\; \mu_1^+ \mu_2^- \left(-\xi^2 \mu_1^+ \mu_2^+ + \xi^{-2} \mu_1^- \mu_2^- \right), \\
c_6 \;&=\; i \mu_1^+ \mu_1^- \left(\mu_1^+ \mu_1^- - \mu_2^+ \mu_2^- \right), \\
c_7 \;&=\; i \mu_2^+ \mu_2^- \left(\mu_1^+ \mu_1^- - \mu_2^+ \mu_2^- \right).
\end{aligned} \tag{29}
$$

Considering real, non-zero $\mu_i^\pm$, in order to have ΔQ_1 Hermitean we choose

$$\mu_1^+\mu_1^- = \mu_2^+\mu_2^-; \tag{30}$$

for this choice eq.(29) reduces to

$$\begin{aligned}
&c_6 = c_7 = 0 \ , \quad c_1 = -(\mu_1^-)^2 + (\mu_2^-)^2, \\
&c_2 = c_3 = \mu_2^+\mu_2^- \left(-\xi^2\mu_1^+\mu_2^+ + \xi^{-2}\mu_1^-\mu_2^-\right), \\
&c_4 = \mu_2^+\mu_2^- \left(-\xi^2(\mu_2^+)^2 + \xi^{-2}(\mu_1^-)^2\right), \\
&c_5 = \mu_2^+\mu_2^- \left(-\xi^2(\mu_1^+)^2 + \xi^{-2}(\mu_1^-)^2\right).
\end{aligned} \tag{31}$$

The condition of non-zero determinant (eq.(9)) can be expressed in the case we are analysing as

$$(\mu_1^-)^2 \neq \xi^4(\mu_2^+)^2 \ , \ (\mu_2^-)^2 \neq \xi^4(\mu_1^+)^2, \tag{32}$$

which implies, for finite, non-zero ξ, $\mu_i^\pm$, a non-zero c_4 and c_5 coefficients. Moreover, c_2(or c_3) is zero iff

$$\mu_1^-\mu_2^- = \xi^4\mu_1^+\mu_2^+, \tag{33}$$

which contradicts eq.(9). This conflict can be seen by multiplying the above equation by μ_2^+ and using eq.(30). Finally, we note that the coefficient c_1 can be zero.

For the second invariant, ΔQ_2, we obtain the same expansion as we have for ΔQ_1, eq.(28), but with different coefficients

$$\begin{aligned}
&c_2 = c_3 = c_5 = c_6 = 0, \\
&c_1 = (\mu_1^+)^{-2}(\mu_2^+)^2(\mu_1^-)^2\mu_2^- \left(2\mu_1^- + \mu_2^-\right), \\
&c_4 = \xi^4(\mu_1^+)^2(\mu_1^-)^2 + (\mu_1^-)^4 + (\mu_2^-)^4 + 4(\mu_1^-)^2(\mu_2^-)^2 \\
&+\xi^{-4}(\mu_1^+)^{-2}(\mu_1^-)^2(\mu_2^-)^4, \\
&c_7 = 2i\xi^{-2}(\mu_1^+)^{-1}\left(\xi^4(\mu_1^+)^2 + (\mu_2^-)^2\right)\left((\mu_1^-)^2 + (\mu_2^-)^2\right).
\end{aligned} \tag{34}$$

The Hermiticity condition in this case is

$$(\mu_1^+)^2 = -\xi^{-4}(\mu_2^-)^2 \tag{35}$$

and under this condition the coefficients in eq.(34) reduce to

$$\begin{aligned} & c_2 = c_3 = c_5 = c_6 = c_7 = 0, \\ & c_1 = -\xi^4(\mu_2^+)^2(\mu_1^-)^2(\mu_2^-)^{-1}\left(2\mu_1^- + \mu_2^-\right), \\ & c_4 = \left((\mu_1^-)^2 + (\mu_2^-)^2\right)^2. \end{aligned} \tag{36}$$

Let $h_i \equiv \Delta Q_i - c_0 \mathbf{1} \otimes \mathbf{1}$, for $i = 1, 2$, and consider the Hamiltonian

$$H_i = J_i \sum_{j=1}^{N-1} \mathbf{1} \otimes ... \otimes h_i^{j,j+1} \otimes \mathbf{1} \otimes ... \otimes \mathbf{1}, \tag{37}$$

where J_i are constants and $h_i^{j,j+1}$ is h_i acting in the $(j, j+1)$ slot of $(\mathbf{C}^2)^{\otimes N}$. Then we have

$$\begin{aligned} H_1 = \quad & \sum_{i=1}^{N-1} \left(J_x \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z\right) \\ & +(B + \bar{B}) \sum_{i=1}^{N} \sigma_i^x - \left(B \sigma_1^x + \bar{B} \sigma_N^x\right), \end{aligned} \tag{38}$$

where

$$\begin{aligned} & J_x = \frac{-(\mu_1^-)^2 + (\mu_2^-)^2}{\mu_2^+ \mu_2^-}, \\ & B = -\xi^2(\mu_2^+)^2 + \xi^{-2}(\mu_1^-)^2, \\ & \bar{B} = -\xi^2(\mu_1^+)^2 + \xi^{-2}(\mu_2^-)^2, \\ & J_1 = -\xi^2 \mu_1^+ \mu_2^+ + \xi^{-2} \mu_1^- \mu_2^-. \end{aligned} \tag{39}$$

The Hamiltonian H_1 describes a one-dimensional XXZ chain with open boundary conditions in a magnetic field in the direction of the anisotropy and with surface fields.

Eq.(37) for $i = 2$ results in

$$H_2 = J_x \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x + B \sum_{i=1}^{N} \sigma_i^x - B \sigma_1^x, \tag{40}$$

where

$$\begin{aligned} J_x \quad &= \quad -\xi^4(\mu_1^+)^2(\mu_1^-)^2(\mu_2^-)^{-1}\left(2\mu_1^- + \mu_2^-\right), \\ B \quad &= \quad (\mu_1^-)^2 + (\mu_2^-)^2, \\ J_2 \quad &= \quad 1. \end{aligned} \tag{41}$$

This is the Hamiltonian of a one-dimensional Ising chain with open boundary conditions in a longitudinal magnetic field and with a surface term which cancels the magnetic field for site 1.

By construction H_i, $i = 1, 2$, are invariant under the algebra $FC_q(2)$, i.e,

$$[H_i, \mathcal{L}^{\pm}_{jj}] = [H_i, \mathcal{L}^{+}_{12}] = [H_i, \mathcal{L}^{-}_{21}] = 0 \ , \ i,j = 1,2, \tag{42}$$

where

$$\begin{aligned} \mathcal{L}^{\pm}_{jj} &= \prod^{N} {}^{\otimes} L^{\pm}_{jj}, \\ \mathcal{L}^{+}_{12} &= \sum_{k=1}^{N} L^{+}_{11} \otimes ... \otimes L^{+}_{11} \otimes (L^{+}_{12})_k \otimes L^{+}_{22} \otimes ... \otimes L^{+}_{22}, \\ \mathcal{L}^{-}_{21} &= \sum_{k=1}^{N} L^{-}_{22} \otimes ... \otimes L^{-}_{22} \otimes (L^{-}_{21})_k \otimes L^{-}_{11} \otimes ... \otimes L^{-}_{11}, \end{aligned} \tag{43}$$

with the subscript k of L^{+}_{12} and L^{-}_{21} in the above equations indicating that the referred operators are in the k-th slot of $(\mathbf{C}^2)^{\otimes N}$. The operators $\mathcal{L}^{\pm}_{ii}$, $\mathcal{L}^{+}_{12}$ and $\mathcal{L}^{-}_{21}$ are the generators of $FC_q(2)$ on $(\mathbf{C}^2)^{\otimes N}$.

We have shown that the quantum one-dimensional Ising and XXZ open chains in a external magnetic field and with special surface fields are both invariant under a two-parametric (q, λ) generalization of $sl_q(2)$ at q and λ being a root of unit. This generalizes some results of ref.[9] where systems in an external magnetic field were not considered. Finally, it is interesting to stress that our derivation relies strongly on the representation theory of the quadratic relations eq.(3) when the parameters are roots of unit.

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