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A new realization for two dimensional topological algebras

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Abstract

We derive topological algebras for 2-dimensional models admitting a $d = [\delta, b]$ decomposition (with b the BRST operator). In general, the topological algebra we obtain is not derived from the twisting of a N=2 supersymmetry algebra. We show how this situation arises for an abelian model described by a ladder of matter.

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1 Introduction

Topological algebras appeared originally in the context of two dimensional N=2 supersymmetry models which, after twisting, provide a description of topological theories with matter fields [1]. The main characteristic of such algebras lies on the existence of odd generators G_μ, Q ($Q^2 = 0$) satisfying $[Q, G_\mu] = P_\mu$, which is an essential ingredient of topological quantum field theories (TQFT) [2]. The generators Q and G_μ depend on the supersymmetry charge $Q_{\alpha a}$ ($\alpha, a \in \{1,2\}$; $(1,2) \equiv (+,-)$) through the combinations $Q := Q_{+-} + Q_{-+}$ and $G_\mu = (G_1, G_2) := (\frac{1}{2}(Q_{++} + Q_{--}), \frac{i}{2}(Q_{++} - Q_{--}))$. It is then clear that the topological algebras constructed in [1, 3] arise as a direct consequence of an underlying N=2 supersymmetry algebra.

Here, we want to construct topological algebras that are not necessarily related to supersymmetry. In a previous work [4], we have studied the properties of models described by generalized gauge and curvature ladders $\mathcal{A} \equiv \sum_{i=0}^{D} \varphi_i^{1-i} := c + A + \varphi_2^{-1} + \cdots + \varphi_D^{1-D}$, $\mathcal{F} \equiv \sum_{i=0}^{D} \eta_i^{2-i} := \phi + \psi + B + \cdots + \eta_D^{2-D}$ satisfying

$$\tilde{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] = \mathcal{F} \tag{1}$$

$$\tilde{d}\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0 \tag{2}$$

$$\mathcal{A} = e^{\delta} c, \quad \mathcal{F} = e^{\delta} \phi \tag{3}$$

$$\tilde{d} = e^{\delta} b e^{-\delta} \tag{4}$$

with $\tilde{d} := b + d$ and $\tilde{d}^2 = 0$. δ is a superderivation of bidegree (1,-1), i.e. it acts on a field increasing its form degree by 1 and reducing the ghost number by the same amount. From (4) we obtain

$$d = [\delta, b] \tag{5}$$

which is the starting point for our definition of topological algebras. In fact, if we can identify the odd generators G_μ, Q with δ, b then (5) becomes a natural realization of $[G_\mu, Q] = P_\mu$ in the space of fields and their derivatives $\mathcal{V} = \{\varphi_i^{1-i}, d\varphi_i^{1-i}, \eta_i^{2-i}, d\eta_i^{2-i}\}_{0 \le i \le D}$. We can then proceed to a full realization of the topological algebra by defining systematically on $\mathcal{V}$ the action of the other generators and ensuring their remaining algebraic relations are satisfied. In much the same way, we can introduce a matter ladder $\mathcal{H} \equiv \sum_{i=0}^{D} h_i^{-i}$ [7] with $\mathcal{H} = e^{\delta} h_0^0$ and extend the topological algebra to the space $\mathcal{V}$ defined by the component fields of $\mathcal{A}$, $\mathcal{F}$, $\mathcal{H}$ and their exterior derivatives. Depending on the way we define the relations among these ladders we will obtain distinct representations for the generators of the topological algebra.

In this work we will present a systematic procedure on how to obtain topological algebras for models admitting a $d = [\delta, b]$ decomposition. In order to relate our construction to the model described in [1] we will consider 2-dimensional models defined in an euclidean space $\mathcal{M}$. The fields are then considered as differential forms in $\mathcal{M}$ with values in a certain Lie algebra (that is not restricted to SO(2)). With respect to the isometries of $\mathcal{M}$, $SO(2)_{\mathcal{M}}$, [1] the ladders $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{H}$ carry no $SO(2)_{\mathcal{M}}$ spinor field, therefore all fields in $\mathcal{V}$ will transform as $SO(2)_{\mathcal{M}}$ tensors. We will see that the choice of a 2-dimensional ladder of matter $\mathcal{H}$ together with the gauge ladder $\mathcal{A}$ will result in a model that contains all fields of the topological matter of [1]. In this case, we will derive an action that is both BRST and M invariant but it is not included in the formalism of [1].

Our work is organized as follows. In Section **2** we introduce the topological algebra and set out our notation. In Section **3** we realize the topological algebra in the space $\mathcal{V} = \{\varphi_i^{1-i}, d\varphi_i^{1-i}, \eta_i^{2-i}, d\eta_i^{2-i}\}$ determined by component fields of gauge and curvature ladders $\mathcal{A} = c + A + \varphi_2^{-1}$, $\mathcal{F} = \phi + \psi + B$. In Section **4** we analyse a model defined by a ladder of matter $\mathcal{H} \equiv \sum_{i=0}^{2} \zeta_i^{-i} := h + \rho + \chi$ and a zero curvature condition. We restrict the fields to be abelian, a condition that will allow us later on to derive an N=2 supersymmetry algebra from a (un)twisting procedure. We exhibit then an invariant action S ($bS = 0 \leftrightarrow QS = 0$) by solving the descent equations associated to $bS = 0$. In Section **5** we redo the same analysis of section **4** but relaxing the zero curvature condition. In Section **6**, having in mind the particular cases of sections 4 and 5, we show how we can define supersymmetry generators from the odd generators of the topological algebra.

2 Topological algebras in d=2

In this work fields and operators carry a bidegree (i,j). As a field, X_i^j means a i-form with ghost number j; as an operator, X_i^j means a superderivation which acts on a field with bidegree (m,n) producing another field with bidegree (i+m,j+n). The total degree of X_i^j is $i + j$. Products of objects like $X_i^j X_k^l$ result in an object with bidegree $(i + k, j + l)$. We define $[X_i^j, X_k^l] := X_i^j X_k^l - (-1)^{(i+j)(k+l)} X_k^l X_i^j$.

[1] Even though $\mathcal{M}$ is an euclidean space, we will use the terminology of [1] and sometimes we will refer $SO(2)_{\mathcal{M}}$ as "Lorentz" SO(2).

We deal with a two-dimensional Euclidean space with metric $g_{\mu\nu} = \delta_{\mu\nu}$. The antisymetric symbol $\epsilon_{\mu\nu}$ has $\epsilon_{12} = 1$. The gamma matrices are defined as in [1].

By a topological algebra we understand the algebra generated by the superderivations $\{P_\mu, \tilde{J}, b, G_\mu, M\}$ with bidegrees (0,0), (0,0), (0,1), (0,-1), (0,1) and satisfying

$$[P_\mu, P_\nu] = [P_\mu, b] = [P_\mu, G_\nu] = [P_\mu, M] = b^2 = M^2 = 0 \tag{6}$$

$$[\tilde{J}, b] = [\tilde{J}, M] = [b, M] = [G_\mu, G_\nu] = 0 \tag{7}$$

$$[\tilde{J}, P_\mu] = -i\epsilon_\mu{}^\nu P_\nu, \;\; [\tilde{J}, G_\mu] = -i\epsilon_\mu{}^\nu G_\nu \tag{8}$$

$$[G_\mu, b] = -P_\mu \tag{9}$$

$$[M, G_\mu] = -i\epsilon_\mu{}^\nu P_\nu \; . \tag{10}$$

Our $\tilde{J}$ generator corresponds in [1] to the twisting of internal and Lorentz SO(2) generators. Here, since we are not restricting the fields to be SO(2) valued, $\tilde{J}$ is related only to the generator of Lorentz SO(2) transformations and we write it as $\tilde{J}\Omega_\mu := -i\epsilon_\mu{}^\nu\Omega_\nu$. The generator of internal symmetry, at this point, doesn't enter the topological algebra. In what follows we will write $P_\mu = \partial_\mu$.

3 A model with $\mathcal{A}$ and $\mathcal{F}$

Let us consider a model defined by ladders

$$\mathcal{A} = c + A + \varphi, \;\; \mathcal{F} = \phi + \psi + B \tag{11}$$

where $c \equiv c_0^0$, $A \equiv A_1^0$, $\varphi \equiv \varphi_2^{-1}$, $\phi \equiv \phi_0^2$, $\psi \equiv \psi_1^1$, $B \equiv B_2^0$. The field B is an arbitrary two form independent of A.

3.1 Defining G_μ, b, M

From (3,4) we obtain the following δ transformations

$$\delta c = A, \; \delta A = 2\varphi, \;\; \delta\varphi = 0; \;\; \delta\phi = \psi, \;\; \delta\psi = 2B, \;\; \delta B = 0, \;\; [\delta, d] = 0 \; . \tag{12}$$

$\delta \equiv \delta_1^{-1}$ being a superderivation of bidegree (1,-1) can be written as $\delta = G_\mu \otimes dx^\mu$. This together with $[\delta, d] = 0$ determines G_μ as

$$G_\mu c = -A_\mu, \;\; G_\mu A_\nu = \varphi_{\mu\nu}, \;\; G_\mu \varphi_{\alpha\beta} = 0, \;\; G_\mu \partial_\nu = \partial_\nu G_\mu \tag{13}$$

$$G_\mu \phi = \psi_\mu, \;\; G_\mu \psi_\nu = -B_{\mu\nu}, \;\; G_\mu B_{\alpha\beta} = 0 \; . \tag{14}$$

The BRST transformations arise from (1,2) and assume the form

$$
\begin{aligned}
bc &= -c^2 + \phi && (15)\\
bA_\mu &= \partial_\mu c - [c, A_\mu] + \psi_\mu && (16)\\
b\varphi_{\mu\nu} &= -F_{\mu\nu} - [c, \varphi_{\mu\nu}] + B_{\mu\nu} && (17)\\
b\phi &= -[c, \phi] && (18)\\
b\psi_\mu &= -\partial_\mu \phi - [c, \psi_\mu] - [A_\mu, \phi] && (19)\\
bB_{\mu\nu} &= \partial_\mu \psi_\nu - \partial_\nu \psi_\mu - [c, B_{\mu\nu}] + [A_\mu, \psi_\nu] - [A_\nu, \psi_\mu] - [\varphi_{\mu\nu}, \phi] && (20)
\end{aligned}
$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and $b^2 = 0$.

$M \equiv M_0^1$ is a generator with bidegree (0,1), therefore we start defining it on c and ϕ as $Mc = a_1 c^2 + a_2 \phi$ and $M\phi = a_3 c^3 + a_4 c\phi + a_5 \phi c$. With $a_1, \cdots, a_5$ arbitrary constants. Imposing (10), $[M, \partial_\mu] = 0$ and $[M, b] = 0$ we obtain $a_1 = a_4 = -a_5,\ a_3 = 0$ and

$$
\begin{aligned}
Mc &= a_1 c^2 + a_2 \phi && (21)\\
MA_\mu &= a_1 [c, A_\mu] + a_2 \psi_\mu + i\epsilon_\mu{}^\nu \partial_\nu c && (22)\\
M\varphi_{\mu\nu} &= a_1([A_\mu, A_\nu] + [c, \varphi_{\mu\nu}]) + a_2 B_{\mu\nu} + i\epsilon_\nu{}^\alpha \partial_\alpha A_\mu - i\epsilon_\mu{}^\alpha \partial_\alpha A_\nu && (23)\\
M\phi &= a_1 [c, \phi] && (24)\\
M\psi_\mu &= a_1([A_\mu, \phi] + [c, \psi_\mu]) - i\epsilon_\mu{}^\nu \partial_\nu \phi && (25)\\
MB_{\mu\nu} &= a_1([\varphi_{\mu\nu}, \phi] - [A_\mu, \psi_\nu] + [A_\nu, \psi_\mu] + [c, B_{\mu\nu}]) - i\epsilon_\nu{}^\alpha \partial_\alpha \psi_\mu + i\epsilon_\mu{}^\alpha \partial_\alpha \psi_\nu\ . && (26)
\end{aligned}
$$

It is straightforward to verify that the set of generators $\{\partial_\mu, \tilde{J}, G_\mu, b, M\}$ satisfy the topological algebra (6-10). Since the constants appearing in the definition of M are not determined by the topological algebra, we have then established a family of topological algebras indexed by the values of (a_1, a_2).

4 A model with a matter ladder and a zero curvature condition

Let us consider an abelian model defined by ladders

$$\mathcal{A} = c + A + \varphi,\ \ \mathcal{H} = h + \rho + \chi \tag{27}$$

with $h \equiv h_0^0,\ \rho \equiv \rho_1^{-1},\ \chi \equiv \chi_2^{-2}$. We impose they obey equations

$$\tilde{d}\mathcal{H} = \mathcal{A},\ \ \tilde{d}\mathcal{A} = 0 \tag{28}$$

$$\mathcal{A} = e^{\delta} c, \;\; \mathcal{H} = e^{\delta} h \tag{29}$$

$$\tilde{d} = e^{\delta} b e^{-\delta} \; . \tag{30}$$

From (29) we obtain the δ-transformation as

$$\delta c = A, \; \delta A = 2\varphi, \; \delta\varphi = 0, \; \delta h = \rho, \; \delta\rho = 2\chi, \; \delta\chi = 0 \tag{31}$$

which, in addition to (13), determines

$$G_\mu h = \rho_\mu, \;\; G_\mu \rho_\nu = -\chi_{\mu\nu}, \;\; G_\mu \chi_{\alpha\beta} = 0 \; . \tag{32}$$

From (28) we have the following BRST transformations

$$\begin{aligned} & bc = 0, \;\; bA_\mu = \partial_\mu c, \;\; b\varphi_{\mu\nu} = -F_{\mu\nu}, \;\; bh = c, \;\; b\rho_\mu = -\partial_\mu h + A_\mu \\ & b\chi_{\mu\nu} = \partial_\mu \rho_\nu - \partial_\nu \rho_\mu + \varphi_{\mu\nu} \; . \end{aligned} \tag{33}$$

Since we are dealing with an abelian model subjected to a zero curvature condition we write $Mc = 0$ and $Mh = a\,c$. Then (9) determines the M transformations as

$$\begin{aligned} & Mc = 0, \;\; MA_\mu = i\epsilon_\mu{}^\nu \partial_\nu c, \;\; M\varphi_{\mu\nu} = i\epsilon_\nu{}^\alpha \partial_\alpha A_\mu - i\epsilon_\mu{}^\alpha \partial_\alpha A_\nu, \;\; Mh = ac \\ & M\rho_\mu = aA_\mu - i\epsilon_\mu{}^\nu \partial_\nu h, \;\; M\chi_{\mu\nu} = a\varphi_{\mu\nu} - i\epsilon_\nu{}^\alpha \partial_\alpha \rho_\mu + i\epsilon_\mu{}^\alpha \partial_\alpha \rho_\nu \; . \end{aligned} \tag{34}$$

G_μ, b, M given in (32,33,34) satisfy the topological algebra (6-10).

4.1 An invariant action

Let us consider $\mathcal{S} = \int \omega_2^0$ satisfying $b\mathcal{S} = \int b\omega_2^0$. This is equivalent to the system of descent equations

$$b\omega_2^0 + d\omega_1^1 = 0, \;\; b\omega_1^1 + d\omega_0^2 = 0, \;\; b\omega_0^2 = 0 \; . \tag{35}$$

In order to solve (35) let us consider $b\omega_0^2 = 0$ with $\omega_0^2 \in \mathcal{V} = \{c, A, \varphi, h, \rho, \chi, dc, dA, d\varphi, dh, d\rho, d\chi\}$. Here, since the fields are abelian we have $\omega_0^2(c,h) = 0$. One possibility for obtaining a non-trivial solution is to consider a set of ladders $\mathcal{A}^I, \; \mathcal{H}^I, \; I = 1, ..., 2N$, where the index I splits as $I = (i, \hat{i}), \; i = 1, ..., N$. For each value of I we have the same set of equations as before, (32,33,34), satisfied by the I-th component of the ladders. It is possible to consider the splitting of I in such a way that the corresponding ladders are complex conjugates, i.e $\mathcal{A}^{\hat{i}} := \mathcal{A}^{*i}$, $\mathcal{H}^{\hat{i}} := \mathcal{H}^{*i}$, however, our discussion is not restricted by this choice. [2]

Now we can write ω_0^2 as [3]

$$\omega_0^2 = f_{ij}(h,\hat{h})c^i c^j + f_{i\hat{j}}(h,\hat{h})c^i\hat{c}^j + f_{\hat{i}\hat{j}}(h,\hat{h})\hat{c}^i\hat{c}^j \tag{36}$$

with $f_{ij} = -f_{ji}$, $f_{\hat{i}\hat{j}} = -f_{\hat{j}\hat{i}}$. Then, $b\omega_0^2 = 0$ determines the following conditions on the functions f_{ij}, $f_{i\hat{j}}$, $f_{\hat{i}\hat{j}}$,

$$\begin{array}{ll} \dfrac{\partial f_{ij}}{\partial \hat{h}^k} - \dfrac{1}{2}\dfrac{\partial f_{i\hat{k}}}{\partial h^j} + \dfrac{1}{2}\dfrac{\partial f_{j\hat{k}}}{\partial h^i} = 0, & \dfrac{\partial f_{ij}}{\partial h^k} + \dfrac{\partial f_{jk}}{\partial h^i} + \dfrac{\partial f_{ki}}{\partial h^j} = 0 \\ \dfrac{\partial f_{\hat{i}\hat{j}}}{\partial h^k} - \dfrac{1}{2}\dfrac{\partial f_{k\hat{j}}}{\partial \hat{h}^i} + \dfrac{1}{2}\dfrac{\partial f_{k\hat{i}}}{\partial \hat{h}^j} = 0, & \dfrac{\partial f_{\hat{i}\hat{j}}}{\partial \hat{h}^k} + \dfrac{\partial f_{\hat{j}\hat{k}}}{\partial \hat{h}^i} + \dfrac{\partial f_{\hat{k}\hat{i}}}{\partial \hat{h}^j} = 0 \end{array} \tag{37}$$

which is solved by

$$f_{ij} = \frac{\partial K}{\partial h^i \partial \hat{h}^j} - \frac{\partial K}{\partial h^j \partial \hat{h}^i}, \quad f_{i\hat{j}} = 2\lambda \frac{\partial K}{\partial h^i \partial h^j} - 2\frac{\partial K}{\partial \hat{h}^i \partial \hat{h}^j}, \quad f_{\hat{i}\hat{j}} = -\lambda f_{ij} \tag{38}$$

with K an arbitrary function of $(h,\hat{h})$ and λ an arbitrary constant that should be made equal to 1 in case we describe our model by a pair of complex fields and their conjugates. Replacing (38) into (36) we obtain

$$\omega_0^2 = 2K_{i\hat{j}}c^i c^j + 2K_{\hat{i}\hat{j}}\hat{c}^i c^j + 2\lambda K_{ij}c^i\hat{c}^j + 2\lambda K_{\hat{i}j}\hat{c}^i\hat{c}^j \tag{39}$$

with $K_{i\hat{j}} \equiv \frac{\partial K}{\partial h^i \hat{h}^j}$ etc. The use of the δ operator allow us to exhibit a particular solution to the descent equations [4, 5, 6, 7, 8, 9]. In fact, in the case of $[\delta, d] = 0$ we can write (35) in the form $\tilde{d}\tilde{\omega} = 0$ for $\tilde{\omega} \equiv \omega_0^2 + \omega_1^1 + \omega_2^0 := e^\delta \omega_0^2$. Then from (39) and the definition of δ we obtain,

$$\tilde{\omega} = 2\widetilde{K}_{i\hat{j}}\mathcal{A}^i\mathcal{A}^j + 2\widetilde{K}_{\hat{i}\hat{j}}\widehat{\mathcal{A}}^i\mathcal{A}^j + 2\lambda\widetilde{K}_{ij}\mathcal{A}^i\widehat{\mathcal{A}}^j + 2\lambda\widetilde{K}_{\hat{i}j}\widehat{\mathcal{A}}^i\widehat{\mathcal{A}}^j \tag{40}$$

with $\widetilde{K} \equiv K(\mathcal{H},\widehat{\mathcal{H}}) := e^\delta K(h,\hat{h})$ and $\widetilde{K}_{i\hat{j}} \equiv \frac{\partial \widetilde{K}}{\partial \mathcal{H}^i \partial \widehat{\mathcal{H}}^j}$ etc. Writing $\mathcal{H} = h + \Theta$, $\widehat{\mathcal{H}} = \hat{h} + \widehat{\Theta}$ with $\Theta \equiv \rho + \chi$ and $\widehat{\Theta} \equiv \hat{\rho} + \hat{\chi}$, we expand $\widetilde{K}(\mathcal{H},\widehat{\mathcal{H}})$ in a Taylor series around $(h,\hat{h})$

$$\begin{aligned} \widetilde{K}(\mathcal{H},\widehat{\mathcal{H}}) \;=\;& K(h,\hat{h}) + \Theta^m K_m(h,\hat{h}) + \widehat{\Theta}^m K_{\hat{m}}(h,\hat{h}) + \frac{1}{2}\Theta^m\Theta^n K_{mn}(h,\hat{h}) + \\ & + \Theta^m\widehat{\Theta}^n K_{m\hat{n}}(h,\hat{h}) + \frac{1}{2}\widehat{\Theta}^m\widehat{\Theta}^n K_{\hat{m}\hat{n}}(h,\hat{h}) \end{aligned}$$

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[2]This condition should be assumed in [1] since there it is necessary to have a set of chiral and antichiral superfields which consequently generate pairs of complex conjugate fields.

[3]We denote $c^{\hat{i}} \equiv \hat{c}^i$ etc.

which gives the decompositions

$$\widetilde{K}_{i\hat{j}}|_2^{-2} = \chi^m K_{mi\hat{j}} + \hat{\chi}^m K_{\hat{m}i\hat{j}} + \frac{1}{2}\rho^m\rho^n K_{mni\hat{j}} + \rho^m\hat{\rho}^n K_{m\hat{n}i\hat{j}} + \frac{1}{2}\hat{\rho}^m\hat{\rho}^n K_{\hat{m}\hat{n}i\hat{j}} \quad (41)$$

$$\widetilde{K}_{i\hat{j}}|_1^{-1} = \rho^m K_{mi\hat{j}} + \hat{\rho}^m K_{\hat{m}i\hat{j}} \quad (42)$$

$$\widetilde{K}_{i\hat{j}}|_0^{0} = K_{i\hat{j}} \; . \quad (43)$$

ω_2^0 is obtained by taking terms with bidegree (2,0) in $\tilde{\omega}$. Therefore, replacing (41,42,43) in (40) we obtain

$$\begin{aligned}
\omega_2^0 &= \tilde{\omega}|_2^0 \\
&= K_{mni\hat{j}}\rho^m\rho^n c^i c^j + 2K_{m\hat{n}i\hat{j}}\rho^m\hat{\rho}^n c^i c^j + K_{\hat{m}\hat{n}i\hat{j}}\hat{\rho}^m\hat{\rho}^n c^i c^j + K_{mn\hat{i}\hat{j}}\rho^m\rho^n\hat{c}^i c^j + \\
&\quad + 2K_{m\hat{n}\hat{i}\hat{j}}\rho^m\hat{\rho}^n\hat{c}^i c^j + K_{\hat{m}\hat{n}\hat{i}\hat{j}}\hat{\rho}^m\hat{\rho}^n\hat{c}^i c^j + 2K_{mi\hat{j}}\chi^m c^i c^j + 2K_{\hat{m}i\hat{j}}\hat{\chi}^m c^i c^j + 2K_{m\hat{i}\hat{j}}\chi^m\hat{c}^i c^j + \\
&\quad + 2K_{\hat{m}\hat{i}\hat{j}}\hat{\chi}^m\hat{c}^i c^j + 2K_{mi\hat{j}}\rho^m c^i A^j + 2K_{\hat{m}i\hat{j}}\hat{\rho}^m c^i A^j - 2K_{mi\hat{j}}\rho^m c^j A^i - 2K_{\hat{m}i\hat{j}}\hat{\rho}^m c^j A^i + \\
&\quad + 2K_{m\hat{i}\hat{j}}\rho^m\hat{c}^i A^j + 2K_{\hat{m}\hat{i}\hat{j}}\hat{\rho}^m\hat{c}^i A^j - 2K_{m\hat{i}\hat{j}}\rho^m c^j\hat{A}^i - 2K_{\hat{m}\hat{i}\hat{j}}\hat{\rho}^m c^j\hat{A}^i + 2K_{i\hat{j}}c^i\varphi^j - \\
&\quad - 2K_{i\hat{j}}c^j\varphi^i + 2K_{\hat{i}\hat{j}}\hat{c}^i\varphi^j - 2K_{\hat{i}\hat{j}}c^j\hat{\varphi}^i + 2K_{i\hat{j}}A^iA^j + 2K_{\hat{i}\hat{j}}\hat{A}^iA^j + \\
&\quad + \lambda\Big(K_{mnij}\rho^m\rho^n c^i\hat{c}^j + 2K_{m\hat{n}ij}\rho^m\hat{\rho}^n c^i\hat{c}^j + K_{\hat{m}\hat{n}ij}\hat{\rho}^m\hat{\rho}^n c^i\hat{c}^j + K_{mn\hat{i}j}\rho^m\rho^n\hat{c}^i\hat{c}^j + \\
&\quad + 2K_{m\hat{n}\hat{i}j}\rho^m\hat{\rho}^n\hat{c}^i\hat{c}^j + K_{\hat{m}\hat{n}\hat{i}j}\hat{\rho}^m\hat{\rho}^n\hat{c}^i\hat{c}^j + 2K_{mij}\chi^m c^i\hat{c}^j + 2K_{\hat{m}ij}\hat{\chi}^m c^i\hat{c}^j + 2K_{m\hat{i}j}\chi^m\hat{c}^i\hat{c}^j + \\
&\quad + 2K_{\hat{m}\hat{i}j}\hat{\chi}^m\hat{c}^i\hat{c}^j + 2K_{mij}\rho^m c^i\hat{A}^j + 2K_{\hat{m}ij}\hat{\rho}^m c^i\hat{A}^j - 2K_{mij}\rho^m\hat{c}^j A^i - 2K_{\hat{m}ij}\hat{\rho}^m\hat{c}^j A^i + \\
&\quad + 2K_{m\hat{i}j}\rho^m\hat{c}^i\hat{A}^j + 2K_{\hat{m}\hat{i}j}\hat{\rho}^m\hat{c}^i\hat{A}^j - 2K_{m\hat{i}j}\rho^m\hat{c}^j\hat{A}^i - 2K_{\hat{m}\hat{i}j}\hat{\rho}^m\hat{c}^j\hat{A}^i + 2K_{ij}c^i\hat{\varphi}^j - \\
&\quad - 2K_{ij}\hat{c}^j\varphi^i + 2K_{\hat{i}j}\hat{c}^i\hat{\varphi}^j - 2K_{\hat{i}j}\hat{c}^j\hat{\varphi}^i + 2K_{ij}A^i\hat{A}^j + 2K_{\hat{i}j}\hat{A}^i\hat{A}^j\Big) \; .
\end{aligned} \quad (44)$$

From (28) we conclude that $\tilde{d}$ has trivial cohomology on $\mathcal{V} = \{\mathcal{A}, \mathcal{H}\}$. Therefore any solution of $\tilde{d}\tilde{\omega} = 0$ implies it exists $\hat{\omega}$ such that $\tilde{\omega} = \tilde{d}\hat{\omega}$. In particular $\omega_2^0 = \tilde{\omega}|_2^0 = (\tilde{d}\hat{\omega})|_2^0 \equiv b\,\hat{\omega}|_2^{-1} + d\,\hat{\omega}|_1^0$. Explicitly

$$\begin{aligned}
\omega_2^0 &= b\Big\{K_{mn\hat{j}}\rho^m\rho^n c^j + 2K_{m\hat{n}\hat{j}}\rho^m\hat{\rho}^n c^j + K_{\hat{m}\hat{n}\hat{j}}\hat{\rho}^m\hat{\rho}^n c^j + 2K_{m\hat{j}}\rho^m A^j + 2K_{\hat{m}\hat{j}}\hat{\rho}^m A^j + \\
&\quad + 2K_{m\hat{j}}\chi^m c^j + 2K_{\hat{m}\hat{j}}\hat{\chi}^m c^j + 2K_{\hat{j}}\varphi^j + \lambda(K_{mnj}\rho^m\rho^n\hat{c}^j + 2K_{m\hat{n}j}\rho^m\hat{\rho}^n\hat{c}^j + \\
&\quad + K_{\hat{m}\hat{n}j}\hat{\rho}^m\hat{\rho}^n\hat{c}^j + 2K_{mj}\rho^m\hat{A}^j + 2K_{\hat{m}j}\hat{\rho}^m\hat{A}^j + 2K_{mj}\chi^m\hat{c}^j + 2K_{\hat{m}j}\hat{\chi}^m\hat{c}^j + 2K_j\hat{\varphi}^j)\Big\} + \\
&\quad + d\Big\{2K_{m\hat{j}}\rho^m c^j + 2K_{\hat{m}\hat{j}}\hat{\rho}^m c^j + 2K_{\hat{j}}A^j + \lambda(2K_{mj}\rho^m\hat{c}^j + 2K_{\hat{m}j}\hat{\rho}^m\hat{c}^j + 2K_j\hat{A}^j)\Big\}
\end{aligned} \quad (45)$$

then our action writes simply as a BRST variation $\mathcal{S} = b\int\hat{\omega}|_2^{-1}$.

We also have $[M, G_\mu] = -i\epsilon_\mu{}^\nu\partial_\nu \Rightarrow Me^\delta = e^\delta M - i\epsilon_\mu{}^\nu\partial_\nu\otimes dx^\mu e^\delta$. Therefore $M\omega_2^0 = (M\tilde{\omega})|_2^1 = (Me^\delta\omega_0^2)|_2^1 = \frac{1}{2}\delta^2 M\omega_0^2 - (i\epsilon_\mu{}^\nu\partial_\nu\otimes dx^\mu)\omega_1^1 = i\epsilon_\mu{}^\nu\partial_\nu\omega_\alpha^1 dx^\mu dx^\alpha$ (we have $M\omega_0^2 = 0$, $\omega_1^1 \equiv \omega_\alpha^1 dx^\alpha$). Thus $M\mathcal{S} = \int dx^2(-i\partial^\mu\omega_\mu^1) = 0$, i.e the action is M-invariant.

5 A model with matter, gauge and curvature ladders

Let us consider the previous abelian model without the zero curvature condition. We have ladders $\mathcal{A}$, $\mathcal{F}$, $\mathcal{H}$ which satisfy

$$\tilde{d}\mathcal{A} = \mathcal{F},\ \tilde{d}\mathcal{F} = 0 \text{ and } \tilde{d}\mathcal{H} = 0\ . \tag{46}$$

The G_μ transformations are given by (13,14,32) and the BRST transformations assume the form

$$\begin{aligned} &bc = \phi,\ bA_\mu = \partial_\mu c + \psi_\mu,\ b\varphi_{\mu\nu} = -F_{\mu\nu} + B_{\mu\nu},\ b\phi = 0,\ b\psi_\mu = -\partial_\mu\phi \\ &bB_{\mu\nu} = \partial_\mu\psi_\nu - \partial_\nu\psi_\mu,\ bh = 0,\ b\rho_\mu = -\partial_\mu h,\ b\chi_{\mu\nu} = \partial_\mu\rho_\nu - \partial_\nu\rho_\mu\ . \end{aligned} \tag{47}$$

In order to determine the M transformation we assume that $Mc = a_1\phi$, $M\phi = a_2\, c\,\phi$ and $Mh = a_3\, c$. The algebraic relation (10) together with $[b, M] = 0$ and $M^2 = 0$ impose $a_2 = 0,\ a_3 = 0$ and fix the M transformations as

$$\begin{aligned} &Mc = a_1\phi,\ MA_\mu = a_1\psi_\mu + i\epsilon_\mu{}^\nu\partial_\nu c,\ M\varphi_{\mu\nu} = a_1 B_{\mu\nu} + i\epsilon_\nu{}^\alpha\partial_\alpha A_\mu - i\epsilon_\mu{}^\alpha\partial_\alpha A_\nu \\ &M\phi = 0,\ M\psi_\mu = -i\epsilon_\mu{}^\nu\partial_\nu\phi,\ MB_{\mu\nu} = -i\epsilon_\nu{}^\alpha\partial_\alpha\psi_\mu + i\epsilon_\mu{}^\alpha\partial_\alpha\psi_\nu,\ Mh = 0 \\ &M\rho_\mu = -i\epsilon_\mu{}^\nu\partial_\nu h,\ M\chi_{\mu\nu} = -i\epsilon_\nu{}^\alpha\partial_\alpha\rho_\mu + i\epsilon_\mu{}^\alpha\partial_\alpha\rho_\nu\ . \end{aligned} \tag{48}$$

Let us take ladders $\mathcal{A}^i$, $\mathcal{F}^i$, $\mathcal{H}^i$, $\quad i = 1, ..., N$. Here, we write $\omega_0^2 = F_{ij}(h,\hat{h})c^i c^j + F_{i\hat{j}}(h,\hat{h})c^i\hat{c}^j + F_{\hat{i}\hat{j}}(h,\hat{h})\hat{c}^i\hat{c}^j + G_i(h,\hat{h})\phi^i + G_{\hat{i}}(h,\hat{h})\hat{\phi}^i$. Then $b\omega_0^2 = 0$ gives

$$\omega_0^2 = G_i(h,\hat{h})\phi^i + G_{\hat{i}}(h,\hat{h})\hat{\phi}^i\ . \tag{49}$$

An invariant action is given by $\mathcal{S} = \int \omega_2^0$ with

$$\begin{aligned} \omega_2^0 &= (e^\delta\omega_0^2)|_2^0 \\ &= \frac{1}{2}G_{i,jk}\phi^i\rho^j\rho^k + G_{i,j\hat{k}}\phi^i\rho^j\hat{\rho}^k + \frac{1}{2}G_{i,\hat{j}\hat{k}}\phi^i\hat{\rho}^j\hat{\rho}^k + \frac{1}{2}G_{\hat{i},jk}\hat{\phi}^i\rho^j\rho^k + G_{\hat{i},j\hat{k}}\hat{\phi}^i\rho^j\hat{\rho}^k + \\ &\quad + \frac{1}{2}G_{\hat{i},\hat{j}\hat{k}}\hat{\phi}^i\hat{\rho}^j\hat{\rho}^k + G_{i,j}\phi^i\chi^j + G_{i,\hat{j}}\phi^i\hat{\chi}^j + G_{\hat{i},j}\hat{\phi}^i\chi^j + G_{\hat{i},\hat{j}}\hat{\phi}^i\hat{\chi}^j + G_{i,j}\psi^i\rho^j + \\ &\quad + G_{i,\hat{j}}\psi^i\hat{\rho}^j + G_{\hat{i},j}\hat{\psi}^i\rho^j + G_{\hat{i},\hat{j}}\hat{\psi}^i\hat{\rho}^j + G_i B^i + G_{\hat{i}}\hat{B}^i \end{aligned} \tag{50}$$

$$\begin{aligned} &= b\Big(\frac{1}{2}G_{i,jk}c^i\rho^j\rho^k + G_{i,j\hat{k}}c^i\rho^j\hat{\rho}^k + \frac{1}{2}G_{i,\hat{j}\hat{k}}c^i\hat{\rho}^j\hat{\rho}^k + \frac{1}{2}G_{\hat{i},jk}\hat{c}^i\rho^j\rho^k + G_{\hat{i},j\hat{k}}\hat{c}^i\rho^j\hat{\rho}^k + \\ &\quad + \frac{1}{2}G_{\hat{i},\hat{j}\hat{k}}\hat{c}^i\hat{\rho}^j\hat{\rho}^k + G_{i.j}c^i\chi^j + G_{i.\hat{j}}c^i\hat{\chi}^j + G_{\hat{i}.j}\hat{c}^i\chi^j + G_{\hat{i}.\hat{j}}\hat{c}^i\hat{\chi}^j + G_{i,j}A^i\rho^j \\ &\quad + G_{i,\hat{j}}A^i\hat{\rho}^j + G_{\hat{i},j}\hat{A}^i\rho^j + G_{\hat{i},\hat{j}}\hat{A}^i\hat{\rho}^j + G_i\varphi^i + G_{\hat{i}}\hat{\varphi}^i\Big) + \\ &\quad + d\left(G_{i,j}c^i\rho^j + G_{i,\hat{j}}c^i\hat{\rho}^j + G_{\hat{i},j}\hat{c}^i\rho^j + G_{\hat{i},\hat{j}}\hat{c}^i\hat{\rho}^j + G_i A^i + G_{\hat{i}}\hat{A}^i\right) \end{aligned} \tag{51}$$

with G_i, $G_{\hat{i}}$ arbitrary functions of h, $\hat{h}$ and $G_{i,j} := \frac{\partial G_i}{\partial h^j}$ etc. The expression given in (51) is essentially the same one given in (45) if we identify $G_i \leftrightarrow 2K_{\hat{i}}$, $G_{\hat{i}} \leftrightarrow 2K_i$ and take $\lambda = 1$. Nonethless, these models differ due to their different BRST transformations.

In addition to the solution given in (51) we can also include BRST-invariant terms involving derivates, for example

$$\int d^2x G_{ij}(\partial_\mu h^i \partial^\mu h^j + \partial_\mu h^i A^{\mu j} + \rho^i_\mu \partial^\mu c^j + \rho^i_\mu \psi^{\mu i}) = b \int d^2x(-\rho^i_\mu \partial^\mu h^j - \rho^i_\mu A^{\mu j}) \ . \quad (52)$$

These terms are not generated by the expansion of $e^\delta \omega_0^2$. This shows explicitly the particular character of our solution.

Note that ω_0^2 given in (49) is M-invariant, then adopting the same procedure of the last section we also obtain $M\mathcal{S} = 0$.

6 Deriving a supersymmetry from the topological algebra

Let us now derive a realization of the $N = 2$ supersymmetry generators in the space of abelian fields $\mathcal{V} = \{c, A, \varphi, h, \rho, \chi, dc, dA, d\varphi, dh, d\rho, d\chi\}$ by following the procedure of [1]. We define supersymmetry generators $Q_{\alpha a} \equiv (Q_{++}, Q_{+-}, Q_{-+}, Q_{--})$ as $Q_{++} := \gamma^\mu_{++} G_\mu$, $Q_{--} := \gamma^\mu_{--} G_\mu$, $Q_{+-} := \frac{1}{2}(-b + M)$, $Q_{-+} := \frac{1}{2}(-b - M)$.[4] Then the topological algebra (6-10) together with these definitions determine

$$\left.\begin{array}{rcl} [Q_{\alpha+}, Q_{\beta+}] &=& 0 \\ {[Q_{\alpha-}, Q_{\beta-}]} &=& 0 \\ {[Q_{\alpha+}, Q_{\beta-}]} &=& \gamma^\mu_{\alpha\beta}\partial_\mu \end{array}\right\} \Rightarrow [Q_{\alpha a}, Q_{\beta b}] = C_{ab}\gamma^\mu_{\alpha\beta}\partial_\mu \ . \quad (53)$$

that corresponds to the algebra of the generators of N=2 supersymmetry. It should be noticed though that our model differs from the description of topological matter of [1], first, because we have two extra fields φ_2^{-1} and χ_2^{-2} that are necessary to garantee $d = [\delta, b]$ (see [4]), and second, because our fields are not components of a pair of chiral and antichiral superfields.

Here, both actions with lagrangian densities given by (45,51) are invariant under b and M transformations, but are not necessarily invariant by G_μ. G-invariance may be obtained

[4]Note that (9) differs by a minus sign to the corresponding relation $[Q, G_\mu] = \partial_\mu$ of [1]. Then we have to consider here the association $-b \leftrightarrow Q$.

by introducing additional ladders or by imposing restrictions on the fields. These actions depend on arbitrary functions K, G_i, $G_{\hat{i}}$ of $(h, \hat{h})$ which, contrarily to the topological matter of [1], don't define a Kähler metric. In fact, the kinetic term given in (52) for the action (51) doesn't bring any restriction to G_i, $G_{\hat{i}}$ (the same applies to K if we introduce a kinetic term for (45)).

7 Conclusion

All models exhibited here admit the decomposition $d = [\delta, b]$ which translates into the fundamental relation $[G_\mu, b] = \partial_\mu$ of topological algebras. In addition, we also have $[d, \delta] = 0$, which is equivalent to $[G_\mu, \partial_\nu] = 0$. As it was shown in [8, 9], there are models where this relation doesn't hold and, as a result, a new operator $\Delta^{-1}_{\mu\nu}$ of bidegree (0,-1) arises, i.e $[G_\mu, \partial_\nu] = \Delta^{-1}_{\mu\nu}$. In these cases there is no natural way to introduce the generator M in order to reproduce some of the relations of the topological algebra.

The same ideas presented here in the context of two dimensions also apply to 4 dimensions. However, what seems more significant is that they apply to any dimension and to any set of Lie algebra valued fields as far as they are components of ladders satisfying $d = [\delta, b]$. The special cases of two and four dimensions can be used to formulate a N=2 supersymmetric model provided we restrict the fields to be respectively SO(2) and SU(2) valued [1, 3]. It becomes clear that the δ operator is not only a usefull tool in the analysis of the descent equations [4, 5, 6, 7, 8, 9], or in the study of some aspects of topological Yang-Mills theories [10, 11] but it also allows us to represent topological algebras for a broader class of models.

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