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Feynman Path Integral Representation for the Quantum Harmonic Oscillator with Time-Dependent Stochastic Frequency

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Abstract

We propose a Feynman path-integral solution for the Feynman propagator associated to an isotropic quantum harmonic oscillator with time-depedent stochastic frequency and time-variable mass.

1 Introduction

One of the most interesting problem in non-relativistic quantum mechanics is the mathematical computations associated to quantum motions of particles in presence of random potentials ([1]).

Although there is a huge literature on the subject ([2]), it still worth the search for new mathematical methods to analyze the above cited problem and specially those possesing potentiality for numerical-computation analysis ([3]). In the subject of quantum motion in random potentials, one very important (formal) model is a system of a quantum harmonic oscillator with a time-variable mass of the form $m(t) = exp\left(\frac{\nu}{m}t\right)$ and a harmonic potential with a stochastic (random) frequency $w(t)$ satisfying a prescribed gaussian statistics and phenomenologically simulating a damped particle around a equilibrium position of the random potential. It is worth remark that outside the quantum physics, this model in its euclidean version (the diffusion equation with time-dependent parameters) has important applications on one-dimensional Burger turbulence and Paraxial wave-propagation in random medium ([5]) (see Appendix A for some pedagogical discussion on this subject).

In this letter to the Editor, we follow our previous studies ([3]) by proposing a path-integral solution for the above cited problem by considering in the first section a new explicit expression for the quantum system Feynman propagator, which by its turn, and opposite to others results on the literature ([4]) posseses the interesting property of allowing to perform exactly the stochastic frequency average on the associated Feynman Propagator. As a result it will lead to a suitable one-dimensional field theory path-integral representation for the quantum averaged propagator, the content of the second section of our letter. Finally, in the third section, we present a somewhat pedagogical discussion of application and modelling of the results previously exposed to Brownian quantum mechanics.

2 An integral representation for the quantum Feynman propagator of the anisotropic time-dependent harmonic oscillator

Let us start our analysis by considering the general case of the Schödinger equation associated to the Feynman propagator of an anisotropic harmonic oscillator with deterministic time-dependent frequency set $\{w_i(t)\}$ and a time dependent, mass parameter $m(t)$, namely:

$$-i\hbar \frac{\partial\psi(\vec{r},t)}{\partial t} = \left\{ -\frac{\hbar^2}{2m(t)}\Delta + \frac{1}{2}\ w_{ij}(t)x_i x_j \right\} \psi(\vec{r},t) \tag{1}$$

with the Plane-Wave initial condition

$$\psi(\vec{r},0) = exp(i\vec{k}(\vec{r}-\vec{r}\,')) \tag{2}$$

Note that due to the linearity of eq. (1), its Green function (the Feynman Propagator) is straightforwardly given by the simple integration in relation the Plane-Wave momentum $\vec{k}$

$$G(\vec{r},\vec{r}\,',t) = \int_{-\infty}^{+\infty} d^3\vec{k}.\psi(\vec{r},t,[\vec{r}\,',\vec{k}]) \tag{3}$$

where we have used the notation $\psi(\vec{r},t,[\vec{r}\,',\vec{k}])$ in order to emphasize the functional dependence of the Schrödinger wave function eq. (1) on initial-value parameters $(\vec{r}\,',\vec{k})$.

Let us try to solve eq. (1) in a elementary way by proposing an ansatz of the following form $(\vec{r}=(x_i), i=1,2,3)$

$$\psi(\vec{r},t) = exp\left[\frac{i}{2}\ x_i A_{ij}(t) x_j\right] exp[i\ x^s p_s(t)] exp[i\phi(t)] \tag{4}$$

where the symmetric 3×3 matrix $[A_{ij}(t)]$, the R^3 vector $[p_s(t)]$ and the (complex) scalar $\phi(t)$ are determined after substituting the ansatz eq. (4) into eq. (1) and leading, thus to the following system of *ordinary Riccati* differential equations after comparing the monomials on the spatial variables (x_i). We get, thus, the following result (a matrix ordinary differential equation) for the quadratic form coeficients $A_{ij}(t)$:

$$\frac{d}{dt}A_{ij}(t) - \frac{\hbar}{m(t)}\left(\sum_{k=1}^{3} A_{ik}(t)A_{kj}(t)\right) = \frac{1}{\hbar}(w_{ij}(t)) \tag{5}$$

with the initial condition

$$A_{ij}(0) = 0 \tag{6}$$

The $[p_s(t)]$ vector ordinary differential equation is given by

$$\frac{d}{dt}p_i(t) - \frac{\hbar}{m(t)}\sum_{k=1}^{3}(A_{ik}(t)\cdot p_k(t)) = 0 \tag{7}$$

with the initial constraint

$$p_i(0)_{=}k_i \tag{8}$$

and finally the scalar ordinary differential equation is written as

$$\hbar\frac{d\phi(t)}{dt} = \frac{\hbar^2}{2m(t)}\left[\sum_{k=1}^{3}(p_k p_k - iA_{kk})(t)\right] \tag{9}$$

with the initial value

$$\phi(0) = -(\vec{k}\cdot\vec{r}\,') \tag{10}$$

Let us firstly analyze the set of ordinary differential equations eq. (5) - eq. (10) in the simple case of an one-dimensional isotropic harmonic oscillator. In this case eq. (5) - eq. (6) reduces to the following second-order ordinary differential equation ($A_{ij}(t) = a(t); p_5(t) = p(t)$)

$$\frac{da(t)}{dt} - \frac{\hbar a^2(t)}{m(t)} = \frac{w(t)}{\hbar} \tag{11}$$

with

$$a(0) = 0 \tag{12}$$

Now it is straightforward to solve eq. (12) by means of the Riccati ansatz

$$a(t) = -\frac{m(t)}{\hbar}\left(\frac{\dot{u}(t)}{u(t)}\right) \tag{13}$$

where $u(t)$ satisfies the usual second order differential classical (damped) harmonic oscillator equation with the boundary condition $\dot{u}(t_0) = 0$

$$\frac{d^2u}{d^2t} + \left(\frac{\dot{m}}{m}\right)(t)\cdot\frac{du}{dt} + \frac{w(t)}{m(t)}u(t) = 0 \tag{14}$$

The remaining equations eq. (7) - eq. (10) are solved by trivial quadratures after solving eq. (11)-eq. (14). It yields the following results

$$p(t) = exp\left(\hbar\int_0^t \frac{a(\sigma)}{m(\sigma)}d\sigma\right)\cdot k \tag{15}$$

and

$$\phi(t) = -(\vec{k}\vec{r}\,') + \hbar \int_0^t d\sigma\, \frac{p^2(\sigma)}{2m(\sigma)} - \frac{i\hbar}{2} \int_0^t d\sigma \frac{a(\sigma)}{m(\sigma)} \tag{16}$$

At this point it is instructive to compare our elementary method of solution with the somewhat cumbersome technique of refs. [4] to solve time-dependent Schrödinger equation with time-dependent quadratic potentials by means of complicated space-time variable change.

It is worth remark that the case of the Caldirola-Kanai model of $m(t) = exp(\frac{\nu}{m}t)$, eq. (14) - eq. (16) are exactly soluble (Appendix A).

After we have displayed our elementary integral representation for eq. (1) - eq. (2) we, thus, pass to the important problem of random frequency $\{w_{ij}(t)\}$ in next section.

3 The random frequency averaged quantum propagator

In this section we intend to write a one-dimensional path-integral representation for the isotropic quantum harmonic oscillator where the isotropy is taken for simplicity of our exposition.

Let us, thus, consider the Schrödinger equation eq. (1) for this isotropic case in R^3

$$-i\hbar\, \frac{\partial \psi(\vec{r},t)}{\partial t} = \left\{ -\frac{\hbar^2}{2m(t)}\Delta + \frac{1}{2} w(t)(|\vec{r}\,^2|\right\} \psi(\vec{r},t) \tag{17}$$

and

$$\psi(\vec{r},0) = exp(i\vec{k}\cdot(\vec{r}-\vec{r}\,')) \tag{18}$$

Here the random frequency $w(t)$ satisfies the white-noise statistics $(0 \le t \le \infty)$

$$\langle w(t)\rangle_w = 0 \tag{19}$$

$$\langle w(t_1)w(t_2)\rangle_w = \lambda\delta^{(1)}(t_1-t_2) \tag{20}$$

The main object of this section will be the exact evaluation of the averaged quantum propagator associated to eq. (17) over the frequency ensemble namely

$$\langle G(\vec{r},\vec{r}\,',t)\rangle_w = \int_{\infty}^{+\infty} d^3\vec{k} \langle \psi(\vec{r},t,[\vec{r}\,',\vec{k}])\rangle_w \tag{21}$$

In order to evaluate exactly eq. (21) by means of a path-integral, we first consider our eq. (4) in the suitable path-integral form

$$G(\vec{r},\vec{r}\,',t) = \int_{-\infty}^{+\infty} d^3\vec{k}\left\{\int_{a(0)=0}^{a(\infty)=0} D^F[a(s)]exp\left(\frac{i}{2}a(t)|\vec{r}|^2\right)\right\} exp[i\vec{p}(t)\cdot\vec{r}]exp[i\phi(t)]$$
$$\delta^{(F)}\left[\frac{d}{dt}a(t) - \frac{\hbar}{m(t)}(a(t))^2 - \frac{1}{\hbar}w(t)\right]\delta^{(F)}\left[\frac{d}{dt}\vec{p}(t) - \frac{\hbar}{m(t)}a(t)\vec{p}(t)\right]$$
$$\delta^{(F)}\left[\frac{d\phi(t)}{dt} - \frac{\hbar}{2m(t)}(p^2(t) - ia(t))\right] \qquad (22)$$

where we have introduced as constraints (delta functionals) the ordinary differential equations determining the parameter $(a(t),\vec{p},\phi(t))$ on our proposed solution eq. (4). Note that after evaluating the $a(s)$-path integral one re-obtains the full set of equations eq. (4) - eq. (7) for the stochastic case.

Let us take the frequency average by means of a path-integral on the functional integrand eq. (22). We obtain, thus, the following double path-integral as a result

$$G(\vec{r},\vec{r}\,',t)\rangle_w = \int_{-\infty}^{+\infty} d^3k\left\{\int D^F[w(t)]e^{-\frac{1}{\lambda}\int_0^\infty ds\frac{w(s))^2}{2}}\right\}$$
$$\int_{a(0)=0}^{a(\infty)=0} D^F[a(s)]exp\left(\frac{i}{2}a(t)(\vec{r})^2\right)exp(i\vec{p}(t)\vec{r})exp(i\phi(t))$$
$$\delta^{(F)}\left[\frac{d}{dt}a(t) - \frac{\hbar}{m(t)}(a(t))^2 - \frac{1}{\hbar}w(t)\right]\delta^{(F)}\left[\vec{p}(t) - \left(exp\left[\hbar\int_0^t\frac{a(\sigma)}{m(\sigma)}d\sigma\right]\cdot\vec{k}\right)\right]$$
$$\delta^{(F)}\left[\phi(t) - \left(\vec{k}\cdot\vec{r}\,' - \hbar\int_0^t\frac{p^2(\sigma)}{2m(\sigma)}d\sigma + i\hbar\int_0^t d\sigma\frac{a(\sigma)}{2m(\sigma)}\right)\right] \qquad (23)$$

Now, it is straightforward to evaluate the $w(t)$-path integral by interchange of the integration order in eq. (23) and leading, thus, to our proposed one-dimensional $\lambda\varphi^4$ line path integral representation for our effective quantum propagator:

$$\langle G(\vec{r},\vec{r}\,',t)\rangle_w =$$
$$\int_{-\infty}^{+\infty} d^3\vec{k}\left\{\int_{a(0)=0}^{a(\infty)=0} D^F[a(s)]\times\ e^{-\frac{\hbar^2}{\lambda}\int_0^\infty ds\frac{1}{2}\left[(\dot{a}(s))^2+\frac{\hbar^2}{m^2(s)}(a(s))^4\right]}\right\}$$
$$\times\ exp\left[\left(\frac{i}{2}(\vec{r}^{\,2})a(t)\right)\right]\left[exp\left(i(\vec{k}\cdot\vec{r})exp\left(\hbar\int_0^t\frac{a(\sigma)}{m(\sigma)}d\sigma\right)\right)\right]$$
$$\left\{exp\ i\left[\vec{k}\cdot\vec{r}\,' + i\hbar\int_0^t\frac{a(\sigma)}{2m(\sigma)} - \hbar\int_0^t d\sigma\left(\frac{(\vec{k})^2}{2m(\sigma)}\times exp\left(2\hbar\int_0^\sigma\frac{a(s)}{m(s)}ds\right)\right)\right]\right\} \quad (24)$$

Note that $a(0) = 0$ from eq. (12) and $a(\infty) = 0$ as general property of Brownian trajectories $a(s)$ for $0 \le s \le \infty$.

Let us analyze *perturbativelly* eq. (24) at a formal one-loop order (a context with $\hbar$ very small but not zero). In this case one safely can make the following formal approximation for the exponential terms inside our proposed path-integral representation eq. (24)

$$exp\left(\hbar\int_0^t \frac{a(\sigma)}{m(\sigma)}d\sigma\right) \cong 1+\hbar\int_0^t \frac{a(\sigma)}{m(\sigma)}d\sigma + O(\hbar^2) \tag{25}$$

$$exp\left(2\hbar\int_0^\sigma \frac{a(s)}{m(s)}ds\right) \cong 1+2\hbar\int_0^\sigma \frac{a(s)}{m(s)}d\sigma + O(\hbar^2) \tag{26}$$

$$\left[(\dot{a}(s))^2+\frac{\hbar^2}{4(m(s))^2}(a(s))^2\right] \cong (\dot{a}(s))^2 + O(\hbar^2) \tag{27}$$

The above written one-loop approximation lead us to the following exactly soluble gaussian path-integral (in the simple case of $m(t)=m$: a constant mass parameter)

$$\langle G(\vec{r},\vec{r}\,',t)\rangle_w \sim \int_{-\infty}^{+\infty} d^3k \; exp[i(\vec{k}\cdot(\vec{r}+\vec{r}\,'))]exp\left[-i\frac{\hbar}{m}|\vec{k}|^2 t\right]$$
$$\times\int_{a(0)=0}^{a(\infty)=0} D^F[a(s)]exp\left(-\frac{\hbar^2}{\lambda}\int_0^\infty ds\frac{[\dot{a}(s)]^2}{2}\right)exp\left(\int_0^\infty ds\; a(s)j(s)\right) \tag{28}$$

here the current $j(s)$ is given explicitly by

$$j(s) = \left(\int_0^t d\sigma\delta(s-\sigma)\right)\frac{\hbar}{m}\left(i(\vec{k}\cdot\vec{r})-\frac{1}{2}\right) + -\left(\frac{i\hbar^2|\vec{k}|^2}{m^2}\right)\left(\int_0^t d\sigma\int_0^\sigma dw\;\delta(s-w)\right)$$
$$+(\delta(s-t))\frac{i}{2}|\vec{r}|^2 \tag{29}$$

The $a(s)$-path integral give us the following result

$$exp\left\{+\frac{\lambda}{\hbar^2}\int_0^\infty ds\int_0^\infty ds'\; j(s)\;(|s-s'|\theta(s-s'))j(s')\right\} =$$
$$exp\frac{\lambda}{\hbar^2}\left\{\left[+\frac{\hbar}{m}\left(i(\vec{k}\cdot\vec{r})-\frac{1}{2}\right)\right]^2 f_1(t)+\right.$$
$$\left.+\left[\frac{-i\hbar^2|\vec{k}|^2}{m^2}\right]^2 f_2(t)+\left[\frac{\hbar}{m}\left(i(\vec{k}\cdot\vec{r})-\frac{1}{2}\right)\left(\frac{-i\hbar^2|\vec{k}|^2}{m^2}\right)\right]f_3(t)\right\} \tag{30}$$

here the time-dependent factors are given by the integrals below

$$f_1(t) = \int_0^t d\sigma\int_0^t d\sigma'|\sigma-\sigma'|\theta(\sigma-\sigma') \tag{31}$$

$$f_2(t) = \int_0^t d\sigma\int_0^t d\sigma'\int_0^\sigma dw\int_0^{\sigma'} dw'|w-w'|\theta(w-w') \tag{32}$$

$$f_2(t) = \int_0^t d\sigma\int_0^t d\sigma'\int_0^{\sigma'} dw|\sigma-w| \tag{33}$$

At this point one should proceed by trying to evaluate at some physical limit or numerically eq. (28) and, thus, try to evaluate the quantum system effective (frequency averaged) state density function afterwards ([1])

$$n(E) = \int_0^\infty dE \; e^{i\frac{E}{\hbar}t} \left(\int_{-\infty}^{+\infty} d^3\vec{r} \langle G(\vec{r},\vec{r},t)\rangle_w \right) \tag{34}$$

which by its turn is a very important physical observable for practical applications in solid state physics. These studies will not be presented here and we hope to expose them in a extended paper since we have limited ourselves to the exposition of a method in mathematical physics of path integrals and not its applications in concrete problems in theoretical physics, which will appear elsewhere.

4 The Brownian Quantum Oscillator

In this section, somewhat of pedagogical purpose, we intend to expose the basis of the phenomenological quantum theory of an one-dimensional damped oscillator ([4]). The main point of generalizing the usual quantum mechanical Schrödinger equation to the case of damping is based in the following observation: if one uses the formal Caldirola-Kanai prescription on the second Newton law of motion

$$m \to m \; e^{\left(\frac{\nu}{m}\right)t} = m(t) \tag{35}$$

$$g \to g \; e^{\left(\frac{\nu}{m}\right)t} = g(t) \tag{36}$$

here m is the inertial mass of particle which becomes time-dependent by an exponential law and g the force coupling constant which appears on the force potential function $gV(x)$, one obtains straightforwardly the classical motion equation of a particle subject to damping, namely,

$$\frac{d}{dt}\left(m \; e^{\left(\frac{\nu}{m}\right)t} \frac{dx}{dt} \right) = g e^{\left(\frac{\nu}{m}\right)t} \left(-\frac{dV}{dx}(x(t),t) \right) \tag{37}$$

or equivalently

$$m\frac{d^2x}{d^2t} + \nu\frac{dx}{dt} = -\left(g \; \frac{dV}{dx} \right)(x(t),t) \tag{38}$$

Let us consider the Caldirola-Kanai prescription eq. (35) and eq. (36) formally into the usual Schrödinger equation and postulate that it is the generalization suitable to quantization of eq. (38), at least, in a phenomenological framework ([4])

$$\left[-\frac{\hbar^2}{2me^{(\frac{\nu}{m})t}}\frac{d^2}{d^2x}+\left(ge^{(\frac{\nu}{m})t}\right)V(x)\right]\psi(x,t)=-i\hbar\frac{\partial\psi(x,t)}{\partial t} \tag{39}$$

It is worth remark that the formal current conservation equation takes the form

$$\frac{\partial}{\partial t}(\psi^*\psi)(x,t)+\frac{i\hbar}{2me^{(\frac{\nu}{m})t}}\left(\frac{\partial}{\partial x}(\psi^*\frac{\partial}{\partial x}\psi-\psi\frac{\partial}{\partial x}\psi^*)\right)(x,t)=0 \tag{40}$$

Let us show the usefulness of eq. (39) by deducing the Ohm law by first principles. In order to implement such step, we consider a homogeneous electric field E and a initial electron plane wave function $\psi(x,0)=exp(i\bar{p}x)$ as an initial condition to be imposed on eq. (39). If one tries to solve eq. (39) by means of the Ansatz

$$\psi(x,t)=e^{ixp(t)}e^{i\phi(t)} \tag{41}$$

one gets the following result

$$p(t)=\overline{p}-\frac{eE}{\hbar}\frac{\left(e^{(\frac{\nu}{m})t}-1\right)}{\left(\frac{\nu}{m}\right)} \tag{42}$$

$$\phi(t)=\int_0^t d\sigma\left\{-\frac{\hbar^2}{2m}e^{-(\frac{\nu}{m}t)}\sigma\left[\bar{p}-\frac{eE}{\hbar}\left(\frac{e^{(\frac{\nu}{m})\sigma}-1}{\left(\frac{\nu}{m}\right)}\right)\right]^2\right\} \tag{43}$$

Now it is easy to evaluate the quantum mechanical damped electron velocity in the Caldirola-Kanai generalized quantum mechanics eq. (39). It yields the result below

$$\begin{aligned}v(t)&=\int_x^{x+\Delta x}d\xi\left[\psi^*\frac{i\hbar}{me^{\frac{\nu}{m}t}}\frac{\partial}{\partial\xi}\psi\right](\xi,t)=\\&=\frac{e^{-(\frac{\nu}{m})t}(i\hbar)}{m}\left[\frac{-ieE}{\hbar}\frac{\left(e^{(\frac{\nu}{m})t}-1\right)}{\left(\frac{\nu}{m}\right)}\right]\Delta x\end{aligned} \tag{44}$$

which at the steady limit lead us directly to the Ohm law

$$v(\infty)=\frac{e}{\nu}E\cdot\Delta x=\frac{e}{\nu}V \tag{45}$$

and by its turn give us the physical (observable) meaning for the damped mechanical parameter ν in eq. (39). It is the electrical resistance parameter of the Ohm's Law.

Let us consider another example of the usefulness of eq. (39) by considering a confined dampled particles in a box $0 \leq x \leq L$, namely

$$-\left[\frac{\hbar^2}{2me^{(\frac{\nu}{m})t}}\frac{d^2}{d^2x}\right]\psi(x,t) = -i\hbar\frac{\partial\psi(x,t)}{\partial t} \tag{46}$$

where

$$\psi(0,t) = \psi(L,t) = 0 \tag{47}$$

By considering again on ansatz of the form

$$\psi(x,t) = A(x)\ exp\left(\frac{i}{\hbar}E(t)\right) \tag{48}$$

one obtains the (damped) eigen-wave functions of the damped quantum particle confined in a box

$$A_n(x) = sin\left(\frac{n\pi}{L}x\right) \text{ with } n = 1,2,3,\cdots \tag{49}$$

$$E_n(t) = \frac{\hbar^2}{2n}\left(\frac{n^2\pi^2}{L^2}\right)\frac{\left(1-e^{-(\frac{\nu}{m})t}\right)}{\left(\frac{\nu}{m}\right)} \tag{50}$$

After displaying in simple examples the consistency of the Caldirola-Kanai prescription, let us analyze the quantization of a classical Brownian particle with motion equation ([5])

$$m\,\frac{d^2x}{d^2t} + \nu\,\frac{dx}{dt} = -gw(t)x \tag{51}$$

where we have introduce a stochastic harmonic potential responsible for the interaction of the environment in thermal equilibrium with temperature T with our classical particle (at least near a potential equilibrium position $V(x(t)) \cong -\frac{1}{2}w(t)x^2$ ([7])). Note that $w(t)$ belongs to a white-noise ensemble of random frequencies in order to produce the Brownian character of our classical system. The formal quantization of our Brownian-damped system eq. (52) in the three-dimensional case is given by eq. (1) of the text with $\lambda = kTm$ (see eq. (20) - eq. (21)).

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References

[1] P.W. Anderson, Phys. Rev. 109, 1492 (1958).
B. Halperin, Phys. Rev. A139, 104 (1965).
K. Von Klitzing, Rev. Mod. Phys. 58, 519 (1986).
Luiz C.L. Botelho, Modern Physics Letters **12B**, 301 (1998).
Modern Physics Letters, 12B, 1153 (1998).

[2] P. Sheng (ed) – Scattering and Localization of Classical Waves in Random Media, World Scientific Series on Directions in Condensed Matter Physics, vol. 8 (1990).
M. Kardar, G. Parisi and Y.C. Zhang, Phys. Rev. Lett. 15, 889 (1986).
I.E. Ozalloshinski, A.M. Polyakov and P.B. Wiegmann, Pbys. Lett. A127, 112 (1988).
F. Wegner, Z. Physik B51, 279 (1983).
D. Thouless, "Il Condensed Matter, Les Houches, ed. R. Balian et al., North-Holland (1979).

[3] Luiz C.L. Botelho, "Feynman path integral representation for the classical harmonic oscillator with stochastic frequency" (to appear in Journal of Physics, Math. & Gen.).

[4] Luiz C.L. Botelho, Phys. Rev. 58E, 1141 (1998).
A.B. Nassar, Luiz C.L. Botelho, J.M.F. Bassalo and P.T.S. Alencar, Phys. Scripta 42, 9 (1990).
Luiz C.L. Botelho, Modern Physics Letters 12B, 569 (1998).

[5] U. Weiss et al., Phys. Rev. 35B, 9535 (1987).
Luiz C.L. Botelho, International Journal of Modern Physics 12B, 2857 (1998).

[6] S.G. Mikhlim, "Mathematical Physics, and Advanced Course", North-Holland series in Applied Mathematics and Mechanics", ed. H.A. Lauwerien and W.T. Korter, vol. 11 (1970).

[7] A.J. Legget et al., Rev. Mod. Phys. 59, 1 (1987).
R.W. Zwanzig, J. Stat. Phys., 9, 215 (1973).

H. Fravenfelder, in "Tunneling in Biological Systems" (B. Chance et al.), Academic Press, New York (1970).

[8] U. Frisch, "Turbulence" (Cambridge Press, University Press, reprinted 1996), p. 112.
J. Bell and D. Marcus, Commum. Math. Phys., 371 (1992).
Luiz C.L. Botelho, Modern Physics Letters 13B, 317 (1999).

The Green Function for a Class of Parabolic-Diffusion Operator – Appendix A

In this somewhat pedagogical Appendix, we show how to write suitable integral representation for the following parabolic-diffusion operator Green function with time-variable coefficients [6]

$$\frac{\partial\phi(\vec{r},t)}{\partial t} = \{D^{ij}(t)\ \frac{\partial^2}{\partial x_i \partial x_j} + v_{ki}(t)x^i\ \frac{\partial}{\partial x_k} + \Omega_{ij}(t)x^i x^j + E_k(t)x^k + M(t)\}\phi(\vec{r},t) \tag{A.1}$$

with

$$\phi(\vec{r},0) = exp(i\vec{k}(\vec{r}-\vec{r}\,')) \tag{A.2}$$

It is important remark the usefulness to have an explicit integral representation for eq. (A-1) since it governs a great number of important physical phenomena like the advection of a scalar in a turbulent medium modeled phenomenologicaly by a variable (tensor) diffusion $D^{ij}(t)$, a vector field $\{v_{ki}(t)x^i\} = \{A_k(t,x)\}(i = 1,2,3)$ and time-dependent sources $\{\Omega_{ij}(t), E_k(t), M(t)\}$.

Let us, thus, follow the procedure exposed in the text by supposing an ansatz of the form

$$\phi(\vec{r},t) = exp\left\{-\frac{1}{2}x^i A_{ij}(t)x^j + p_i(t)x^i + \phi i(t)\right\} \tag{A.3}$$

After substituting eq. (A-3) into eq. (A-1) and taking into account the initial condition eq. (A-2), we get the following generalized Riccati system of ordinary differential equations similar to these exposed in the text

$$\dot{A}_{ik}(t) = -\{A^{is}(t)D_{sn}(t)A^{mk}(t) + \Omega_{ik}(t) - A_{ir}(t)v_{rk}(t)\} \tag{A.4}$$

$$\dot{p}_\ell(t) = -\{A_{\ell n}(t)D_{nk}(t)p_k(t) - p^i(t)D_{in}(t)A_{n\ell}(t)\} + p_k(t)v_{k\ell}(t) - E_\ell(t) \tag{A.5}$$

$$\dot{\phi}(t) = D_{ik}(t)\{A_{ik}(t) - p_i(t)p_k(t)\} - M(t) \tag{A.6}$$

with the following initial conditions

$$\dot{A}_{ik}(0) = 0 \quad ; \quad \dot{p}_\ell(0) = -ik_\ell \quad ; \quad \phi(0) - i\vec{k}\cdot\vec{r}\,' \tag{A.7}$$

On the basis of the unicity of the solution of eq. (A-1) on any suitable Sobelev space ([6]), we obtain the Green function written as an usual integral

$$G(\vec{r},\vec{r}',t) = \int_{-\infty}^{+\infty} d^3\vec{k}\ exp\left\{-\frac{1}{2}x^i A_{ij}(t)x^j + p_i(t;\vec{k})x^i + \phi(t,\vec{k},\vec{r}')\right\} \tag{A.8}$$

At this point we remark the usefulness of eq. (A-8) for numerical studies, since there are faster numerical schemes to handle the set eq. (A-4) - eq. (A-7) even if some objects there are random variables.

Let us write a integral representation proposed in this paper for a purely damped ($\nu^2 < 4m^2w_0^2$) quantum oscillator (in one-dimensional)

$$-\frac{i\hbar\partial\psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2me^{(\frac{\nu}{m})t}} - \frac{d^2}{d^2x} + \frac{1}{2}(mw_0^2)e^{(\frac{\nu}{m})t}x^2\right]\psi(x,t) \tag{A.9}$$

with

$$\psi(x,0) = exp(ik(x-y)) \tag{A.10}$$

By considering the text's ansatz

$$\psi(x,t) = exp\left(i\frac{x^2}{2}\alpha(t)\right)exp(ixp(t))exp(i\phi(t)) \tag{A.11}$$

one obtain that

$$\alpha(t) = \frac{2m^2w_0^2}{\hbar\sqrt{4m^2w_0^2 - v^2}}e^{(\frac{\nu}{m})t}$$
$$\left[\frac{\sin\left(\frac{\sqrt{4m^2w_0^2-\nu^2}}{2m}t\right)}{\cos\left(\frac{\sqrt{4m^2w_0^2-\nu^2}}{2m}t\right) + \frac{\nu}{\sqrt{4m^2w_0^2-\nu^2}}\sin\left(\frac{\sqrt{4m^2w_0^2-\nu^2}}{2m}t\right)}\right] \tag{A.12}$$

and

$$p(t) = k\ exp(A(\nu,m)t)\times$$
$$\left[\cos\left(\frac{\sqrt{4m^2w_0^2-\nu^2}}{2m}t\right) + \frac{\nu}{\sqrt{4m^2w_0^2-\nu^2}}\sin\left(\frac{\sqrt{4m^2w_0^2-\nu^2}}{2m}t\right)\right]^{\gamma(\nu,m)} \tag{A.13}$$

with

$$A(\nu,m) = \frac{4m^2w_0^2}{4m^2w_0^2-\nu^2}\left(\frac{\nu}{2m}\right)\left(1-\left(\frac{\nu}{2mw_0}\right)^2\right) \tag{A.14}$$

$$\gamma(\nu,m) = -\left(1-\left(\frac{\nu}{2mw_0}\right)^2\right)\frac{4m^2w_0^2}{4m^2w_0^2-\nu^2} \tag{A.15}$$

and

$$\phi(t) = -ky + \frac{\hbar}{2}\int_0^t d\sigma \left(me^{\left(\frac{\nu}{m}\right)\sigma}\right)[p^2(\sigma) - i\alpha(\sigma)] \tag{A.16}$$

The Green function searched has the following integral representation

$$G(x,y,t) = \int_{-\infty}^{+\infty} dk\ e^{-iky}\left\{ e^{i\frac{x^2}{2}\alpha(t)} e^{ix\ p(t)}\ e^{\frac{i\hbar}{2}\int_0^t d\sigma(p^2(\sigma)m} e^{\left(\frac{\nu}{n}\right)\sigma)} \right.$$
$$\left. e^{+\frac{\hbar}{2}\int_0^t d\sigma(\alpha(\sigma)} me^{\left(\frac{\nu}{m}\right)\sigma)} \right\} \tag{A.17}$$

We have not attempted to evaluate eq. (A-17), bu the integral representation given by this equation have potentialities for precise numerical (appropriate) evaluations of the theory's physical observables.

Finally, let us make the important remark that the one-dimensional Burger equation in the presence of a spatial derivative external forcing ([8]) and given initial-time conditions

$$\frac{\partial U(x,t)}{\partial t} + U(x,t)\ \frac{\partial}{\partial x}U(x,t) = \nu\ \frac{\partial^2 U(x,t)}{\partial^2 x} + \frac{\partial}{\partial x}F(x,t)\ U(x,0) = f(x) \tag{A.18}$$

can be easily mapped on the one-dimensional version of eq. (A-1)-eq. (A-2) by means of the Hopf-Cole transformation (singular for $\nu \to 0^+$)

$$w(x,t) = \frac{1}{\nu}\ exp\ \left(-\frac{1}{2\nu}\int_{-\infty}^x dy\ U(y,t)\right) \tag{A.19}$$

yielding, thus, the following Euclidean-Schrödinger equation

$$\nu\ \frac{\partial w(x,t)}{\partial t} = \nu^2\ \frac{\partial^2 w(x,t)}{\partial^2 x} - \frac{1}{2}\ F(x,t)\cdot w(x,t) \tag{A.20}$$

with the initial condition

$$w(x,0) = \frac{1}{\nu}\ exp\left(-\frac{1}{2\nu}\ \int_{-\infty}^x\ dy\ f(y)\right) \tag{A.21}$$