GENERALIZED LADDER OPERATORS FOR THE DIRAC-COULOMB PROBLEM VIA SUSY QM

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Abstract

The supersymmetry in quantum mechanics and shape invariance condition are applied as an algebraic method to solving the Dirac-Coulomb problem. The ground state and the excited states are investigated via new generalized ladder operators.

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I. INTRODUCTION

The supersymmetry (SUSY) algebra in quantum mechanics (QM) initiated with the work of Nicolai [1] and elegantly formulated by Witten [2], has attracted interest and found many applications in order to construct the spectral resolution of solvable potentials in various fields of physics [3]. SUSY QM was first formulated by Witten [2] and also Gendenshtein who used shape invariance property [4]. The hydrogen atom was studied via SUSY QM in non-relativistic context by Kostelecky and Nieto [5]. They used the SUSY QM for spectral resolution and also for calculating transition probabilities for alkali-metal atoms.

The Dirac-Coulomb problem is an exactly solvable problem in relativistic quantum mechanics and the solution can be found in some books on quantum mechanics, for instance [6]. Dirac-Coulomb problem has also been studied via SUSY QM [8–16]. Our purpose in this paper is to obtain the complete energy spectrum and to point out energy eigenfunctions of the Dirac-Coulomb problem via shape invariance, and new generalized ladder operators in SUSY QM, whose Lie algebraic structure for shape invariant potentials has been presented by Fukui-Aizawa [17] and Balantekin [18], which (has) been applied for exactly soluble potentials in non-relativistic quantum mechanics [19,20].

It is particularly simple to apply SUSY QM for shape-invariant potentials (as) their SUSY partners are similar in shape and differ only in the parameters that appear in them. More specifically, if $V_-(x;a_1)$ is any potential, adjusted to have zero ground state energy $E_-^{(0)}=0$, its SUSY partner $V_+(x;a_0)$ must satisfy the requirement $V_+(x;a_0) = V_-(x;a_1) + R(a_1)$, where a_0 is a set of parameters, a_1 a function of the parameters a_0 and $R(a_1)$ is a remainder independent of x. Indeed, $f^{(s)}(a_1) = f(f\cdots(f(a_1))\cdots)$, i.e. the function f applied s times. In this case, one can determine the energy levels for $V_+(x;a_0)$ to be $E_-^{(n)} = \sum_{i=1}^{n} R(a_i)$. Recently, some relativistic shape invariant potentials have been investigated [21].

The SUSY hierarchical prescription was utilized by Sukumar to solve the energy spectrum of the Dirac-Coulomb problem [9]. In this work the Fukui-Aizawa-Balantekin [17,18] approach to the Dirac-Coulomb problem is investigated via SUSY QM.

This present paper is organized in the following way. In the section II we realize a graded Lie algebra structure in terms of the 4x4 matrix supercharges analogous to Witten's SUSY algebra for the Dirac radial equation associated with the hydrogen atom. In section III, we deduce new generalized ladder operators in relativistic quantum mechanics via supersymmetry, in order to build up the energy eigenvalue and eigenfunctions of supersymmetric partner potentials, using the shape invariance condition. In section IV concluding remarks are given.

II. THE DIRAC-COULOMB PROBLEM AND SUSY

In this section, we adopt the Sukumar approach to construct the 4x4 matrix supercharges. The Dirac radial equation for the hydrogen atom can be written as

$$\begin{pmatrix} \frac{dG}{dr} & 0 \\ 0 & \frac{dG}{dr} \end{pmatrix} + \frac{1}{r}\begin{pmatrix} k & -\gamma \\ \gamma & -k \end{pmatrix}\begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix}\begin{pmatrix} G \\ F \end{pmatrix}, \tag{1}$$

where k is an eigenvalue of the Dirac operator $K = \beta(\vec{\Sigma}\cdot\vec{L} + \mathbf{1})$, $\gamma = \frac{ze^2}{c\hbar}$, $\alpha_1 = m + E$, $\alpha_2 = m - E$, $\mid k \mid = j + \frac{1}{2}$ $(k = \pm 1, \pm 2, \pm 3, \cdots,)$ and $\mathbf{1}$ is the 2x2 unity matrix. The operator D given by

$$D = s + k - \gamma\sigma_1 \tag{2}$$

diagonalizes the matrix that appears in the interaction term,

$$D^{-1}(k\sigma_3 - i\gamma\sigma_2)D = s\sigma_3, \tag{3}$$

so that we obtain

$$\left(\frac{k}{s} + \frac{m}{E}\right)\tilde{F} = \left(\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right)\tilde{G}, \tag{4a}$$

$$\left(\frac{k}{s} - \frac{m}{E}\right)\tilde{G} = \left(-\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right)\tilde{F}, \tag{4b}$$

where

$$\begin{pmatrix} \tilde{G} \\ \tilde{F} \end{pmatrix} = D \begin{pmatrix} G \\ F \end{pmatrix}, \quad \rho = Er. \tag{5}$$

The eigenvalue equations for $k =\mid k \mid$ and $k = - \mid k \mid$, respectively, become

$$\left(\frac{\mid k \mid}{s} + \frac{m}{E} \right) \tilde{F_+} = \left(\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s} \right) \tilde{G_+}, \tag{6a}$$

$$\left(\frac{\mid k \mid}{s} - \frac{m}{E} \right) \tilde{G_+} = \left(-\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s} \right) \tilde{F_+}, \tag{6b}$$

$$\left(\frac{- \mid k \mid}{s} + \frac{m}{E} \right) \tilde{F_-} = \left(\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s} \right) \tilde{G_-}, \tag{6c}$$

$$\left(\frac{- \mid k \mid}{s} - \frac{m}{E} \right) \tilde{G_-} = \left(-\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s} \right) \tilde{F_-}, \tag{6d}$$

where $\tilde{F_+} = \tilde{F}(+ \mid k \mid)$, $\quad \tilde{G_+} = \tilde{G}(+ \mid k \mid)$, $\quad \tilde{F_-} = \tilde{F}(- \mid k \mid)$ and $\tilde{G_-} = \tilde{G}(- \mid k \mid)$.

Defining the intertwining operators in their matrix form

$$A_0^{(+)} = \frac{d}{d\rho} + \left(\frac{s}{\rho} - \frac{\gamma}{s} \right) \sigma_3, \tag{7}$$

$$A_0^{(-)} = -\frac{d}{d\rho} + \left(\frac{s}{\rho} - \frac{\gamma}{s} \right) \sigma_3 \tag{8}$$

and

$$\mathbf{O} = \frac{\mid k \mid}{s} \sigma_1 + i \frac{m}{E} \sigma_2, \tag{9}$$

we get

$$A_0^{(+)} \begin{pmatrix} \tilde{G_+} \\ \tilde{F_+} \end{pmatrix} = \mathbf{O} \begin{pmatrix} \tilde{G_+} \\ \tilde{F_+} \end{pmatrix}, \tag{10}$$

$$A_0^{(-)} \begin{pmatrix} \tilde{F_-} \\ \tilde{G_-} \end{pmatrix} = -\mathbf{O} \begin{pmatrix} \tilde{F_-} \\ \tilde{G_-} \end{pmatrix}. \tag{11}$$

From (9), (10) and (11), we obtain:

$$\begin{pmatrix} \tilde{F}_- \\ \tilde{G}_- \end{pmatrix} \propto \mathbf{O} \begin{pmatrix} \tilde{G}_+ \\ \tilde{F}_+ \end{pmatrix} = A_0^{(+)} \begin{pmatrix} \tilde{G}_+ \\ \tilde{F}_+ \end{pmatrix}. \tag{12}$$

In similar way, we find:

$$\begin{pmatrix} \tilde{G}_+ \\ \tilde{F}_+ \end{pmatrix} \propto -\mathbf{O} \begin{pmatrix} \tilde{F}_- \\ \tilde{G}_- \end{pmatrix} = A_0^{(-)} \begin{pmatrix} \tilde{F}_- \\ \tilde{G}_- \end{pmatrix}. \tag{13}$$

From equations (7), (8), (10) and (11), we see that there exists the following supersymmetric partner eigenvalue equations:

$$A_0^{(-)} A_0^{(+)} \begin{pmatrix} \tilde{G}_+ \\ \tilde{F}_+ \end{pmatrix} = \left(1 + \frac{\gamma^2}{s^2} - \frac{m^2}{E^2}\right) \begin{pmatrix} \tilde{G}_+ \\ \tilde{F}_+ \end{pmatrix} \tag{14}$$

and

$$A_0^{(+)} A_0^{(-)} \begin{pmatrix} \tilde{F}_- \\ \tilde{G}_- \end{pmatrix} = \left(1 + \frac{\gamma^2}{s^2} - \frac{m^2}{E^2}\right) \begin{pmatrix} \tilde{F}_- \\ \tilde{G}_- \end{pmatrix}. \tag{15}$$

The mutually adjoint non-Hermitian supercharge operators for Witten's model are given by

$$Q_+ = A_0^{(+)} \sigma_- = \begin{pmatrix} 0 & A_0^{(+)} \\ 0 & 0 \end{pmatrix}_{4x4}, \quad Q_- = A_0^{(-)} \sigma_+ = \begin{pmatrix} 0 & 0 \\ A_0^{(-)} & 0 \end{pmatrix}_{4x4}, \tag{16}$$

so that the SUSY Hamiltonian H satisfies $[H, Q_\pm]_- = 0$, and takes the form

$$\begin{aligned} H = [Q_+, Q_-]_+ &= \frac{1}{2}\left(p_\rho^2 + W^2(\rho) - \sigma_3 \frac{d}{d\rho} W(\rho)\right) \\ &= \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}, \end{aligned} \tag{17}$$

where the matrix superpotential $W(\rho) = \left(\frac{s}{\rho} - \frac{\lambda}{s}\right)\sigma_3$. At this point we would like to call attention to the fact that the above Hamiltonian operators is a 4x4 matrix and as far as we know this is the first time where a 2x2 matrix superpotential operator appears to the Dirac-Coulomb problem. We also can see that the pair of SUSY Hamiltonians is given by $H_- = A_0^{(+)} A_0^{(-)}, \quad H_+ = A_0^{(-)} A_0^{(+)}$ and σ_3 is the Pauli diagonal matrix.

III. SPECTRAL RESOLUTION VIA LADDER OPERATORS

Let us now build up the energy eigenvalues and eigenfunctions of supersymmetric partner potentials, using the shape invariance condition and the generalized ladder operators. From the last section one obtains the following matrix forms of the pair of SUSY potentials:

$$V_{-}(\rho,\lambda,s)=\begin{pmatrix}\frac{s(s-1)}{\rho^2}-\frac{2\lambda}{\rho}+\frac{\lambda^2}{s^2} & 0\\ 0 & \frac{s(s+1)}{\rho^2}-\frac{2\lambda}{\rho}+\frac{\lambda^2}{s^2}\end{pmatrix}, \tag{18}$$

$$V_{+}(\rho,\lambda,s)=\begin{pmatrix}\frac{s(s+1)}{\rho^2}-\frac{2\lambda}{\rho}+\frac{\lambda^2}{s^2} & 0\\ 0 & \frac{s(s-1)}{\rho^2}-\frac{2\lambda}{\rho}+\frac{\lambda^2}{s^2}\end{pmatrix}. \tag{19}$$

Althought the SUSY partner potentials $V_{(\pm)}$ are not shape invariant, we can see that their respective components are:

$$V_{(+)11}(\rho,\lambda,s)=V_{(-)11}(\rho,\lambda,s+1)-\frac{\lambda^2}{(s+1)^2}+\frac{\lambda^2}{s^2}, \tag{20}$$

$$V_{(-)22}(\rho,\lambda,s)=V_{(+)22}(\rho,\lambda,s+1)-\frac{\lambda^2}{(s+1)^2}+\frac{\lambda^2}{s^2}. \tag{21}$$

From (27), (20) and (21) one can written

$$R_{11}(a_1)=R_{22}(a_1)=-\frac{\lambda^2}{(s+1)^2}+\frac{\lambda^2}{s^2}=\frac{\lambda^2}{a_0^2}-\frac{\lambda^2}{a_1^2}, \tag{22}$$

so that $R_{11}(a_i)=\frac{-\lambda^2}{(s+i)^2}+\frac{\lambda^2}{s^2}$ for $i=1,2\cdots$. Thus we get the following energy eigenvalues of $H_{(-)11}=H_{(+)22}$:

$$E_{-11}^{(n)}=E_{+22}^{(n)}=\sum_{i=1}^{n}R_{11}(a_i)=\frac{-\lambda^2}{(s+n)^2}+\frac{\lambda^2}{s^2}, \tag{23}$$

where $a_i=f^{i-1}(a_0)$. By comparison (23) with the eigenvalue equations we obtain the energy eigenvalues of the hydrogen relativistic atom,

$$1+\frac{\lambda^2}{s^2}-\frac{m^2}{E^{(n)2}}=\frac{\lambda^2}{s^2}-\frac{\lambda^2}{(s+n)^2},\quad s=\sqrt{k^2-\lambda^2}, \tag{24}$$

providing

$$E^{(n)} = \sqrt{\frac{m^2}{1+\frac{\lambda^2}{(\sqrt{k^2-\lambda^2}+n)^2}}}, \quad n = 0, 1, 2, \cdots, \tag{25}$$

which is in agreement with the result obtained by Sukumar using the SUSY Hamiltonian hierarchy method [9].

Note that the shape invariance condition is associated with translation of the parameter a's, so that the Eq. (10) can be written in the following form:

$$\left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx} \Rightarrow p_x^\dagger = p_x \tag{26}$$

$$A_{11}^{(+)}(a_0)A_{11}^{(-)}(a_0) = A_{11}^{(+)}(a_1)A_{11}^{(-)}(a_1) + R(a_1), \tag{27}$$

where

$$A_{11}^{(\pm)} = \pm\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}. \tag{28}$$

Following Fukui-Aizawa-Balantekin [17,18], we obtain the following ladder operators:

$$B_-(s) = T^\dagger(s)A_{11}^-(s), \quad B_+(s) = B_-^\dagger(s), \tag{29}$$

$T^\dagger(s)$ being a translation operator defined by

$$T(s) = e^{\frac{\partial}{\partial s}}, \quad \text{so that} \quad T^\dagger(s) = e^{-\frac{\partial}{\partial s}}, \tag{30}$$

where $a_0 = s$ so that $R(a_n) = T(a_n)R(a_{n-1})T^\dagger(a_0)$ and $R(a_n)B_+(a_0) = B_+(a_0)R(a_{n-1})$. Thus, it is easy to see that the operators $B_\pm(a_0)$ and $R(a_n)$ satisfy the following commutation relation:

$$[H_{(-)11}, B_+^n] = (R(a_1) + R(a_2) + \cdots + R(a_n))B_+^n, \quad n = 1, 2, \cdots, \tag{31}$$

with $H_{(-)11} = B_+B_- = A_{11}^{(+)}A_{11}^{(-)}$. Consequently we see that the $\tilde{F}_-^{(n)}$, component eigenfunction the n-th excited stated, is given by

$$\tilde{F}_-^{(n)} \propto B_+^n(s)\tilde{G}_-^{(0)}(\rho; s), \quad n = 1, 2, 3, \cdots. \tag{32}$$

The ground state eigenfunction must be annihilated by $B_-(s)$, so

$$A^-(s)\tilde{G}_-^{(0)}(\rho;s) = 0, \tag{33}$$

which lead us the following physically acceptable solution:

$$\tilde{G}_-^{(0)}(\rho;s) = N_G \rho^s e^{-\frac{\gamma}{s}\rho}, \tag{34}$$

N_G being the normalization constant.

From (29) and (30) we see that the raising operator may be written as

$$B_+(s) = \left(\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right) e^{\frac{\partial}{\partial s}}. \tag{35}$$

Consequently, for the first excited state one may write

$$\begin{aligned}\tilde{F}_-^{(1)}(\rho;s) &= B_+(s)\tilde{G}_-^{(0)}(\rho;s)\\ &\propto \left(\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right) e^{\frac{\partial}{\partial s}} \rho^s e^{-\frac{\gamma}{s}\rho}\\ &= \left[(2s+1)\left(\frac{1}{\rho} - \frac{\gamma}{s(s+1)}\right)\right] \rho^{s+1} e^{-\frac{\gamma\rho}{s+1}}.\end{aligned} \tag{36}$$

Finally we would like to call attention that the above formalism may be applied to the exactly solvable potentials in relativistic quantum mechanics [21,22].

IV. CONCLUSION

In this paper we investigate the Dirac-Coulomb problem via supersymmetry in quantum mechanics. The shape invariant formalism for the supersymmetric partners is applied to obtain the complete energy spectrum and eigenfunctions of the Dirac-Coulomb problem.

The generalized ladder operators have been used to obtain the complete set of the energy spectrum and eigenfunctions for the Dirac-Coulomb type potential in supersymmetric quantum mechanics formalism. Solving this problem a Schrödinger-like equation for shape invariant potentials is obtained for the upper component. The lower component can, consequently, be obtained the upper component one.

The first review work on SUSY QM with application for the Dirac-Coulomb problem was reported by Haymaker-Rau [10] but they do not consider the Sukumar's method. Recently, the confinement of neutral fermions by a pseudoscalar double-step potential in the Dirac equation in (1+1) dimensions has been investigated [22].

The complete energy spectrum and eigenfunctions of the Dirac-Coulomb problem are deduced via a new approach of the SUSY QM. Indeed, one has used the algebraic structure for shape invariant potential, recently proposed by Fukui-Aizawa-Balantekin [17,18] to construct the energy eigenfunctions for the Dirac-Coulomb problem. This approach is different from the SUSY Hamiltonian hierarchy method applied by Sukumar [9].

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