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SOME RESULTS FROM A MELLIN TRANSFORM EXPANSION FOR THE HEAT KERNEL

by

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ABSTRACT

We calculate, in the case of a differential operator containing a gauge field, coefficients of a new Heat Kernel expansion obtained in a preceeding paper. That expansion allows to show that the meromorphic structure of the generalized zeta-function is much richer than it was previously known. Also, an application to anomalies is done, resulting in a general formula for arbitrary dimension D. The special cases D = 2 and D = 3 are investigated.

Key-words: Heat Kernel; Zeta-function; Anomalies.

I. INTRODUCTION

In a previous note[1] it has been obtained an asymptotic expansion to the diagonal part of the Heat Kernel associated to a given elliptic operator H of order $\underline{m}$, based on the connection, trough a Mellin transform, between the Heat Kernel and the Seeley's Kernel $K(s;x,y)$[2] of the complex s-th power H^s of the operator H, and the meromorphic properties of $K(s;x,x)$. We recall that "Heat Kernel" means the solution of the "Heat equation",

$$\frac{\partial}{\partial t} F(t;x,y) = HF(t;x,y) \quad , \tag{1.1}$$

where t is a "time" or "temperature" parameter, and x and y are, in the case we are interested in, points of a D-dimensional compact manifold, M. The Seeley's Kernel is defined for $Re(s) < -D/\bar{m}$, such that

$$H^s f(x) = \int_M dy K(s;x,y) f(y) \quad .$$

The expansion mentioned above is obtained by analytic continuation of K in the variable s, and reads,

$$F(t;x,x) = -\sum_{\ell=0}^{\infty} t^{\ell}\left(\frac{d\phi}{ds}\right)_{s=\ell} - \sum_{j} t^{\frac{j-D}{\underline{m}}} \Gamma\left(\frac{D-j}{\underline{m}}\right) R_j(x) \quad . \tag{1.2}$$

The sum over j is such that we take $j = 0,1,2,\ldots$ *excluding the terms such that* $(j-D)/\bar{m} = 0,1,2,\ldots$, and $R_j(x)$ is the residue of $K(s;x,x)$ at the pole $s = (j-D)/\bar{m}$,

$$R_j(x) = \frac{1}{i\underline{m}(2\pi)^{D+1}} \int_{|\xi|=1} d\xi \int_\Gamma d\lambda\ \lambda^{\frac{j-D}{\underline{m}}}\ b_{-\underline{m}-j}(x,\xi,\lambda), \qquad (1.3)$$

where Γ is a curve coming from ∞ along a ray of minimal growth, clockwise on a small circle around the origin, then going back to ∞. The quantities $b_{-\underline{m}-j}$ are obtained from the coefficients of the symbol of H (see section 3), and $|\xi| = 1$ means that the set of variables $\{\xi\}$ is constrained to be at the surface of the unit sphere in a D-dimensional space. The function $\phi(s)$ is introduced to account for the coincidence of the poles of the gamma-function $\Gamma(-s)$ and those of $K(s;x,x)$ at the positive intergers ℓ, and is defined by,

$$\Gamma(-s)K(s;x,x) \approx \phi(s) \,|\, (s-\ell)^2 \quad , \qquad (1.4)$$

for $s \approx \ell$.

As was remarked in Ref. 1, the expansion (1.2) is rather different from de Witt's *ansatz* currently used[4]. In particular, it contains fractionary powers at even dimension and even operator order, coming from the second term in the expansion.

In the rest of the paper we explore some consequences of that new expansion. In Section 2 we show that the generalized zeta-function $\xi(s)$ has an infinity of poles at real values of s. In Section 3 we calculate the coefficients of the leading and of the next-lo-leading terms in (1.2). In Section 4 we obtain a general formula for the anomaly in arbitrary dimension D, and particularize to the special cases D = 2 and D = 3.

II. MEROMORPHY OF THE GENERALIZED ZETA-FUNCTION

One of the implications of the series (1.2) is of a mathematical character and concerns the meromorphic structure of the Hawking's generalized zeta-function[3], which is much richer than the structure previously known. This may be easily seen as follows:

The generalized zeta-function is written as,

$$\xi(s) = \frac{1}{\Gamma(s)} \int_0^1 dt\, t^{s-1} F(t;x,x) + Q(s) \quad , \tag{2.1}$$

where Q(s) converges for all s.

Let us take D = 4 and consider an operator of order $\underline{m} = 2$. Replacing in (2.1) F(t;x,x) by the series (1.2) we see that the first term of the expansion gives no poles due to the factor $1/\Gamma(s)$ in front of the integral in (2.1). From the second term of the expansion we have the sum,

$$-\frac{1}{\Gamma(s)} \sum_j \Gamma\left(\frac{4-j}{2}\right) R_j(x) \int_0^1 dt\, t^{s+\frac{j}{2}-3} \quad , \tag{2.2}$$

which gives poles at $s = 2-j/2$, for integer values of j and $(j-4)/2 \neq 0,1,2,\ldots$.

Thus the poles of the generalized zeta-function are not situated just only at $s=1$ $(j=2)$ and $s=2$ $(j=0)$. We also have poles at $s=3/2$ $(j=1)$ and $s=1/2$ $(j=3)$, and for $j=5,7,\ldots$ we have an infinity of poles in s at the negative half-integers. There are no poles at negative integers due to the vanishing of the residues of $K(s;x,x)$ at those values[2]. The residues at the poles are given by the corresponding coefficients $-\left[1/\Gamma(2-j/2)\right]\Gamma\left(\frac{4-j}{2}\right)R_j(x)$ in (2.2).

III. APPLICATION TO A DIFFERENTIAL OPERATOR

Let us consider a differential operator H of order <u>m</u> = 2,

$$H = -\left[g^{\mu\nu}(x)(\partial_\mu + B_\mu(x))(\partial_\nu + B_\nu(x)) + P(x)\right] , \qquad (3.1)$$

acting on a D-dimensional compact manifold M, endowed with a metric $g_{\mu\nu}(x)$ $(\mu,\nu=1,2,\ldots D)$. In (3.1) $P(x)$ is a non-differential operator and,

$$B_\mu(x) = gA_\mu(x) + \eta_\mu(x) , \qquad (3.2)$$

$A_\mu(x)$ and g being respectively the gauge field and a coupling constant (to not confuse with the metric tensor nor with its determinant). The quantity $\eta_\mu(x)$ contains information about curvature and torsion. The usual convention of summation over repeated indices will be adopted.

In Seeley's notation the operator H must be written in the form,

$$H = \sum_{\{\alpha\}|\alpha|\leq 2} (-i)^{|\alpha|} H^{|\alpha|}_{\alpha_1\ldots\alpha_D}(x) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_D^{\alpha_D}} \quad , \tag{3.3}$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_D$.

Expanding (3.1) and comparing with (3.3) we obtain the set of coefficients $H^{|\alpha|}_{\alpha_1\ldots\alpha_D}(x)$,

$$H^{(2)}_{\alpha_1\ldots\alpha_D}(x) = H^{(2)}_{0\ldots01(\mu)0\ldots01(\nu)0\ldots0}(x) = g_{\mu\nu}(x) \tag{3.4a}$$

$$H^{(1)}_{\alpha_1\ldots\alpha_D}(x) = H^{(1)}_{0\ldots01(\nu)\ldots0} = -2ig_{\mu\nu}(x)B^{\mu}(x) \tag{3.4b}$$

$$H^{(0)}_{\alpha_1\ldots\alpha_D}(x) = H^{(0)}_{0\ldots0\ldots0}(x) = -g_{\mu\nu}(x)(\partial^{\mu}B^{\nu}-B^{\mu}B^{\nu}) - P(x). \tag{3.4c}$$

Now, to calculate the coefficients of the second term of the expansion (1.2) we need the quantities b_{-2-j} (see eq. (1.3)), which are expressed in terms of the coefficients $a_{2-k}(x,\xi)$ of the symbol of H^2,

$$a_{2-k}(x,\xi) = \sum_{|\alpha|=2-k} H^{|2-k|}_{\alpha_1\ldots\alpha_D}\xi_1^{\alpha_1}\ldots\xi_D^{\alpha_D} \quad , \tag{3.5}$$

by the following set of equations,

$$\ell = 0: \qquad b_{-2}\left[a_2(x,\xi) - \lambda\right] = 1 \tag{3.6a}$$

$\ell > 0$:

$$b_{-2-\ell}(a_2-\lambda) + \sum_{j,k} (-i)^{|\alpha|} \sum_{\{\alpha\}} \frac{\partial^{|\alpha|} b_{-2-j}}{\partial\xi_1^{\alpha_1}\ldots\partial\xi_D^{\alpha_D}} \frac{\partial^{|\alpha|} a_{2-k}}{\alpha_1!\ldots\alpha_D!\,\partial_{x_1}^{\alpha_1}\ldots\partial_{x_D}^{\alpha_D}} = 0 ,$$

with $j < \ell$, $j + k + |\alpha| = \ell$. (3.6b)

The coefficients a_{2-k} are easily obtained from equs. (3.4),

$$a_2(x,\xi) = g_{\mu\nu}(x)\xi^\mu\xi^\nu \equiv |\xi|^2 \quad (3.7a)$$

$$a_1(x,\xi) = -2ig_{\mu\nu}(x)B^\mu(x)\xi^\nu \quad (3.7b)$$

$$a_o(x,\xi) = -g_{\mu\nu}(x)(\partial^\mu B^\nu - B^\mu B^\nu) - P(x) \quad (3.7c)$$

Then the first two quantities b_{-2-j} that we need for calculating the leading and the next-to-leading contributions in the second term of the expansion (1.2) are given by,

$$b_{-2}(x,\xi,\lambda) = (|\xi|^2 - \lambda)^{-1} \quad (3.8)$$

$$b_{-3}(x,\xi,\lambda) = \frac{2iB.\xi}{(|\xi|^2-\lambda)^2} - \frac{2i\xi.\partial|\xi|^2}{(|\xi|^2-\lambda)^3} , \quad (3.9)$$

where the scalar product is defined with the metric $g_{\mu\nu}(x)$.

From (1.2), (1.3), (3.8) and (3.9), those contributions, the coefficients of the powers $t^{-D/2}$ and $t^{(1-D)/2}$, are given respectively by,

$$-\Gamma(\tfrac{D}{2})R_o(x) = -\Gamma(\tfrac{D}{2})\frac{1}{2i(2\pi)^{D+1}} \int_{|\xi|=1} d\xi \int_\Gamma d\lambda\lambda^{-D/2}(|\xi|^2-\lambda)^{-1}$$

(3.10)

$$-\Gamma\left(\frac{D-1}{2}\right)R_1(x) = -\Gamma\left(\frac{D-1}{2}\right)\frac{1}{(2\pi)^{D+1}}\left[\int_{|\xi|=1} d\xi B.\xi \int_\Gamma \frac{\lambda^{(1-D)/2}d\lambda}{(|\xi|^2-\lambda)^2} - \right.$$

$$\left. - \int_{|\xi|=1} d\xi\xi.\partial|\xi|^2 \int_\Gamma \frac{\lambda^{(1-D)/2}\,d\lambda}{(|\xi|^2-\lambda)^3}\right] \qquad (3.11)$$

where we take the integration path Γ as the curve coming from $-\infty$ along the negative real axis then clockwise along the unit circle around the origin then backwards to $-\infty$ along the negative real axis. Since we must restrict the ξ's to the surface of the unit D-dimensional sphere with the metric $g_{\mu\nu}(x)$, we have $|\xi|^2 = 1$, and to avoid the singularity at $\lambda = 1$, we introduce a regulator $p > 1$.[2] Then (3.10) and (3.11) became,

$$-\Gamma(\tfrac{D}{2})R_o(x) = \Gamma(\tfrac{D}{2})\frac{1}{2.(2\pi)^{D+1}}\int d\xi\left[2\sin\left(\frac{\pi D}{2}\right)\int_{-1}^{-\infty}\frac{d\lambda\,\lambda^{-D/2}}{p-\lambda} - i\int_{\pi}^{-\pi}\frac{d\theta\,e^{i(1-D/2)\theta}}{p-e^{i\theta}}\right]$$

$$(3.12)$$

$$-\Gamma\left(\frac{D-1}{2}\right)R_1(x) = -\Gamma\left(\frac{D-1}{2}\right)\frac{1}{(2\pi)^{D+1}}\int d\xi\left\{B.\xi\left[-2i\sin\left(\frac{\pi(1-D)}{2}\right)\int_{-1}^{-\infty}\frac{d\lambda\,\lambda^{(1-D)/2}}{(p-\lambda)^2} + \right.\right.$$

$$+i\int_{\pi}^{-\pi}\frac{d\theta\,e^{(i/2)(3-D)\theta}}{(p-e^{i\theta})^2}\right] - \xi.\partial||\xi||^2\left[-2i\sin\left(\frac{\pi(1-D)}{2}\right)\int_{-1}^{-\infty}\frac{d\lambda\,\lambda^{(1-D)/2}}{(p-\lambda)^3} + \right.$$

$$\left.\left. +i\int_{\pi}^{-\pi}\frac{d\theta\,e^{(i/2)(3-D)\theta}}{(p-e^{i\theta})^3}\right]\right\}. \qquad (3.13)$$

In (3.12) and (3.13) and in the subsequent formulae, the integrations over the ξ's are constrained to the unit sphere $|\xi| \equiv \sqrt{g_{\mu\nu}(x)\xi^\mu\xi^\nu} = 1$.

In dimension $D = 4$, making the change of variables $p^{-1/2}e^{i\theta/2} = e^{i\phi}$, the integrations over λ and θ may be performed. The results, after suppression of the regularization are,

$$-\Gamma(2)R_o(x) = \frac{i}{(2\pi)^4}\int d\xi \qquad (3.14)$$

and

$$-\Gamma(\tfrac{3}{2})R_1(x) = \frac{1}{(2\pi)^4}\Gamma(\tfrac{3}{2})\left\{(5-3\pi/2-5i)\int d\xi\, B.\xi - \left[\frac{13}{16}+\frac{15\pi}{4}+\frac{21i}{4}\right]\int d\xi\, \xi.\partial(g_{\mu\nu}(x)\xi^\mu\xi^\nu)\right\}$$

(3.15)

Analogously, in dimension $D = 2$, the coefficients of the two first powers of the second term in (1.2) (powers t^{-1} and $t^{-1/2}$ respectively) are obtained from (3.12) and (3.13),

$$-\Gamma(1)R_o(x) = \frac{i}{2(2\pi)^2}\int d\xi \qquad (3.16)$$

$$-\Gamma(\tfrac{1}{2})R_1(x) = \Gamma(\tfrac{1}{2})\frac{1}{2(\pi)^2}(1-\pi/2-i)\int d\xi\, B.\xi -$$

$$-\left[(4+3\pi)/8-i\right]\int d\xi\, \xi.\partial(g_{\mu\nu}(x)\xi^\mu\xi^\nu) \qquad (3.17)$$

As an example we calculate the coefficients (3.16) and (3.17) in the Penrose compactified 2-dimensional Minkowskii space[5] which has the metric $\bar{g}_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In this case the unit sphere $|\xi| = 1$ is the piece of hyperbole depicted in (Fig. 1). Using polar coordinates (r,θ) and the well known formula for the induced metric on a (D-1)-dimensional surface embedded a in D-dimensional metric space, it is easy to see that the integration on the "surface" $|\xi| = 1$ reduces simply to integration over θ between the limits θ_1, θ_2 and $\theta_1 + \pi$, $\theta_2 + \pi$,

$$\int_{|\xi|=1} d\xi\,(\text{Penrose}) = \int_{\theta_1}^{\theta_2} d\theta + \int_{\theta_1+\pi}^{\theta_2+\pi} d\theta$$

with

$$\theta_1 = \text{arctg}(1/\pi^2) \qquad (3.18a)$$

$$\theta_2 = \text{arctg}\ \pi^2 \qquad (3.18b)$$

We obtain,

$$-\Gamma(1)R_0(x)\,(\text{Penrose}) = \frac{i}{(2\pi)^2}(\theta_2 - \theta_1) \qquad (3.19)$$

$$-\Gamma(1/2)R_1(x)\,(\text{Penrose}) = \frac{\Gamma(1/2)}{(2\pi)^3}(1 - \pi/2 - i)\sqrt{2}\left\{\left(\frac{1+i}{2}\right)\cdot\right.$$

$$\left.\Pi\left(\alpha,\ \frac{1+2}{2},\frac{1}{\sqrt{2}}\right) + F\left(\alpha,\ \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha,\ \frac{1}{\sqrt{2}}\right)\right\}\Bigg|_{\theta_1}^{\theta_2} + (\theta_1 \to \theta_1 + \pi, \theta_2 \to \theta_2 + \pi)$$

$$+\ \phi(\theta_1, \theta_2) \qquad (3.20)$$

where Π and E are the elliptic integrals of the third and of the second kind respectivelly, F is the generalized hypergeometric series and ϕ is the function,

$$\phi(\theta_1,\theta_2) = \int_{\theta_1}^{\theta_2} \frac{\cos\theta\, d\theta}{\sqrt{\sin 2\theta}} + \int_{\theta_1+\pi}^{\theta_2+\pi} \frac{\cos\theta\, d\theta}{\sqrt{\sin 2\theta}}$$

IV. ANOMALIES

In this Section we apply our expansion (1.2) to study anomalies using the Heat Kernel method[6]. We borrow some of the notations and methods employed in a recent work by Cognola and Zerbini[7], since they are suitable for our purposes. Using the generalized zeta-function regularization, the anomaly may be written in the form,

$$A = -q \lim_{s\to o} \mathrm{Tr}\left\{(X+Y)\frac{1}{\Gamma(s)}\int_0^1 dt\, t^{s-1}\left[F(t;x,x) - P_o(x,x)\right]\right\} , \tag{4.1}$$

where $q=-1$, 1/2 or 1, for fermions neutral or charged bosons respectively, $X = X_1 + X_2$, $Y = Y_1 + Y_2$ are operators satisfying the relation $\delta K(J) = (\delta J X_1 + Y_1\delta J)K + K(Y_2\delta J + \delta J X_2)$; $K(J)$ is such that $H(J) \propto K(J)$ for bosons and $H(J) = K^2(J)$ for fermions, J being a classical source. P_o is the projector onto the zero modes. For the axial anomaly, $X = Y = i\gamma_5$.

We replace in (4.1), $F(t;x,x)$ by (1,2) and after some simple manipulations we see that the sole contribution to the anomaly comes from the coefficient of the power t^0, giving, for arbitrary dimension D,

$$A = -q\ \mathrm{Tr}\left\{(X+Y)\left[-(d\phi/ds)\Big|_{s=0} - P_0(x,x)\right]\right\} . \qquad (4.2)$$

Now, from (1.4) and the formula,

$$\Gamma(z) = \frac{\Gamma(z+\ell+1)}{z+\ell} \prod_{n=1}^{\ell} \frac{1}{z+\ell-n} \quad ,$$

we have for integer $\ell \geq 0$,

$$\frac{d\phi}{ds}\Big|_{s=\ell} = -2\ \frac{(-1)^\ell}{\ell!}\ K(\ell;x,x) \quad , \qquad (4.3)$$

where the Seeley's Kernel for integer ℓ is[2]

$$K(\ell;x,x) = \frac{1}{(-1)^\ell 2(2\pi)^D} \int d\xi \int_0^\infty dt\, t^\ell b_{-2-2\ell-D}(x,\xi,te^{i\theta}) \qquad (4.4)$$

Thus, taking $\arg\lambda = \theta = \pi$ in (4.4) the anomaly may be obtained *for arbitrary* dimension D, from (4.2), with

$$\frac{d\phi}{ds}\Big|_{s=0} = \frac{-1}{(2\pi)^D} \int d\xi \int_0^\infty dt\, b_{-2-D}(x,\xi,-t) \quad . \qquad (4.5)$$

Next we apply (4.2) to the cases D = 2 and D = 3. The case D = 3 is particularly interesting, since, in spite of the well known difficulties in defining the matrix γ_5 in odd dimension[8], cer tain aspects of even dimensional axial anomaly could appear in odd-dimensional field theories - see Niemi and Semenoff[9] and other references therein. This is due to the fact that the connection between zero modes of Dirac operators and non-triviality of the background field topology is valid for any value of D as showed by Callias[10].

Moreover, there is a technical difficulty to (formally) calculating anomalies in odd-dimension using the de Witt *ansatz* in the Heat Kernel method which is not present with our expansion: when one uses the de Witt *ansatz* for expanding F(t), the anomaly depends on the coefficient of the power $t^{D/2}$, which does not exist for odd values of D, while with our expansion the anomaly depends directly on the coefficient of the zero-th power of t, given by (4.5), for any dimension, even or odd.

Calculations for a general coordinate dependent metric are extremely involved. Here, we restrict ourselves to the simpler situation of a symetric, coordinate independent metric tensor $g_{\mu\nu}$. In this case we obtain,

for D = 2:

$$A_2 = \frac{g}{(2\pi)^2} \mathrm{Tr} \left\{ (X+Y) \int d\xi \left[4\xi^\mu (\partial_\mu B_\nu) \xi^\nu + 1 g_{\mu\nu} (\partial^\mu B^\nu - B^\mu B^\nu) + P(x) + + 2(B.\xi)^2 \right] \right\} \quad (4.6)$$

for D = 3:

$$A_3 = \frac{q}{(2\pi)^3} \operatorname{Tr}\Big\{(X+Y)\int d\xi \Big[-g^{\mu\nu}(\partial_\mu\partial_\nu B^\sigma)\xi_\sigma + 2i(B.\xi)(g_{\mu\nu}(\partial^\mu B^\nu - B^\mu B^\nu) +$$

$$+ P(x)) - 2iB^\mu(\partial_\mu B^\nu)\xi_\nu + i\xi^\mu g_{\rho\sigma}(\partial_\mu \partial^\rho B^\sigma + B^\rho \partial_\mu B^\sigma + (\partial_\mu B^\rho)B^\sigma +$$

$$+ i\xi^\mu \partial_\mu P(x) - \frac{8}{3}(B.\xi)^3 - \frac{4}{3}\,\xi^\mu \xi_\sigma (B.\xi)(\partial_\mu B^\sigma) -$$

$$- \frac{16}{3}\, i\xi^\mu \xi^\nu \xi_\sigma (\partial_\mu \partial_\nu B^\sigma)\Big]\Big\} \tag{4.7}$$

In the Penrose compactified 2-dimensional Minkowskii space[5], (4.6) gives the result,

$$A_2(\text{Penrose}) = \frac{q}{(2\pi)^2} \operatorname{Tr}\Big\{(X+Y)\Big[(\theta_2-\theta_1)\Big(\Big(\frac{1}{2}+i\Big)\Big(\partial B_1/\partial x^0 +$$

$$- \partial B_o/\partial x^1\Big) + (2-i)B_oB_1 + P(x)\Big) + \frac{1}{2}\,\ell n\,\frac{\sin\theta_2}{\sin\theta_1}\Big(\frac{B_1^2}{2} + \frac{\partial B_1}{\partial x^1}\Big) -$$

$$- \frac{1}{2}\,\ell n\,\frac{\cos\theta_2}{\cos\theta_1}\Big(\frac{B_o^2}{2} + \frac{\partial B_o}{\partial x^0}\Big)\Big]\Big\}$$

where the angles θ_1, θ_2 are given by (3.18a,b) and B_o, B_1 are the components of $B_\mu(x)$ given by (3.2).

FIGURE CAPTION

Fig. 1 - ξ-variables submitted to the constraint $g_{\mu\nu}\xi^{\mu}\xi^{\nu} = 1$ in the Penrose compactified 2-dimensional Minkowskii space.

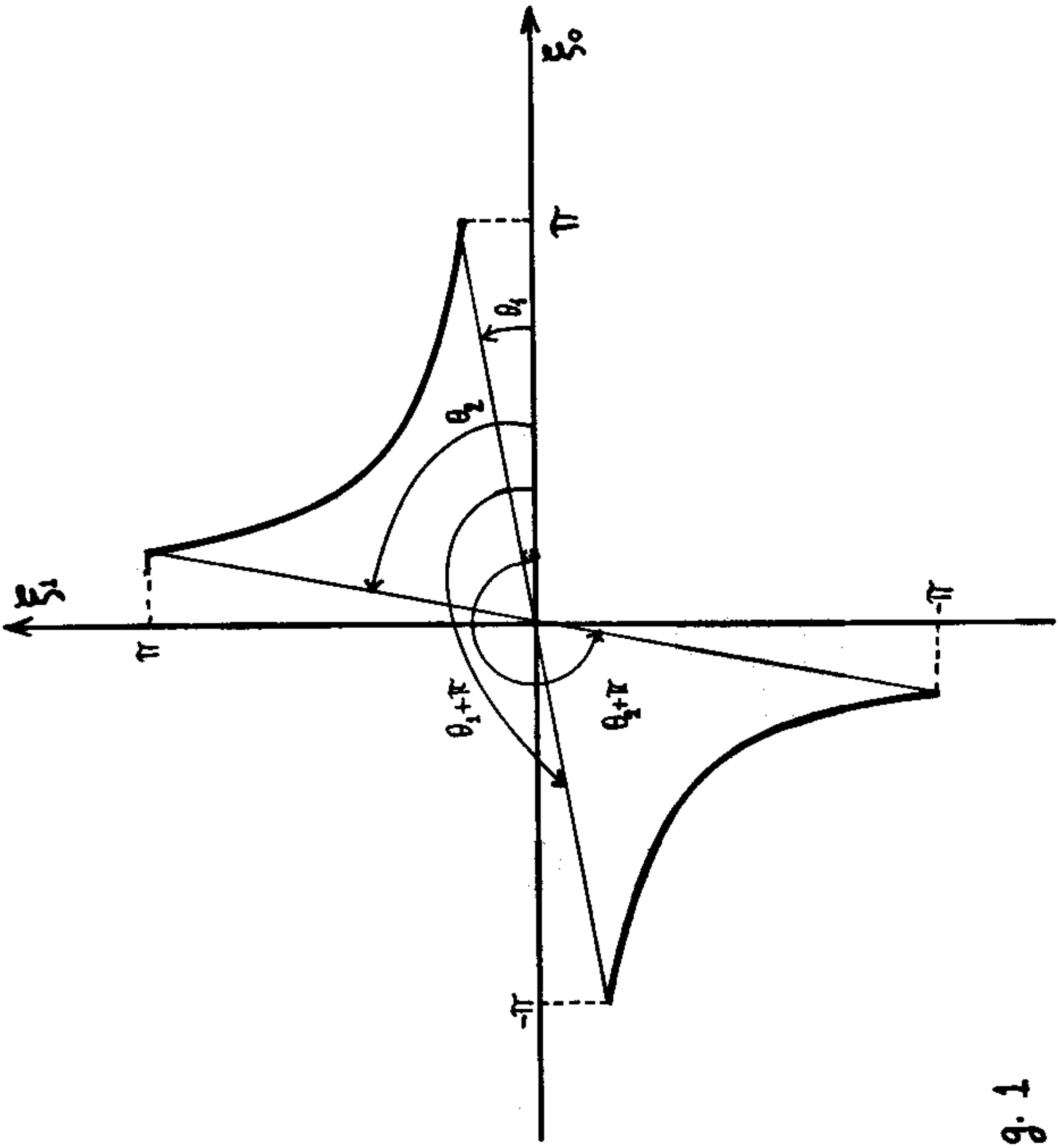

Fig. 1

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