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Distinguishing marks of simply-connected universes

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Abstract

A statistical quantity suitable for distinguishing simply-connected Robertson-Walker (RW) universes is introduced, and its explicit expressions for the three possible classes of simply-connected RW universes with an uniform distribution of matter are determined. Graphs of the distinguishing mark for each class of RW universes are presented and analyzed. There sprout from our results an improvement on the procedure to extract the topological signature of multiply-connected RW universes, and a refined understanding of that topological signature of these universes studied in previous works.

Key-words: Cosmic Topology; Topological Signature in Cosmology; Friedmann-Robertson-Walker Spacetimes; Topology and Cosmology; Cosmic Crystallography; Euclidean Universes; Hyperbolic Universes; Elliptic Universes; Compact Universes; Robertson-Walker Models.

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1 Introduction

Current observational data favor the locally homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models as approximate descriptions of our universe at least since the recombination time. Thus in the framework of the general relativity theory it can be described through a Robertson-Walker (RW) metric

$$ds^2 = dt^2 - R^2(t)\, d\sigma^2 \,, \tag{1.1}$$

where t is a cosmic time, and $d\sigma^2 = d\chi^2 + f^2(\chi)\,[\,d\theta^2 + \sin^2\theta\, d\phi^2\,]$ with $f(\chi) = \chi\,,\ \sin\chi\,,\ \sinh\chi\,,$ depending on the sign of the constant spatial curvature ($k = 0, \pm 1$). These descriptions, however, are only local and do not fix the global shape (topology) of our universe.

Despite the infinitely many possibilities for its global topology, it is often assumed that spacetime is simply-connected leaving aside the hypothesis that the universe may be multiply-connected, and compact (finite) even in the cases $k = 0$ and $k = -1$. In other words, it is often assumed that the $t = const$ spatial sections M of a RW spacetime manifold are one of the following simply-connected spaces: Euclidean E^3 ($k = 0$), elliptic S^3 ($k = 1$), or the hyperbolic H^3 ($k = -1$). However, the connectedness (either simply or multiply) for our three-space has not been settled by cosmological observations. Thus, the space M where we live may also be any one of the possible multiply-connected quotient three-spaces $M = \widetilde{M}/\Gamma$, where $\widetilde{M}$ stands for E^3, S^3 or H^3, and Γ is a discrete group of isometries of the covering space $\widetilde{M}$ acting freely on $\widetilde{M}$ [1].

Whether we live in a simply or multiply-connected, finite (compact) or infinite (non-compact) space, and what is the size and the shape of the universe are open problems modern cosmology seeks to solve [2]. The most immediate consequence of multiply-connectedness of the universe is that the sky may show multiple images of cosmic objects periodically distributed in the space. This periodic distribution of images arises from the correlations in their positions dictated by the discrete isometries of the covering group Γ of the three-manifold used to model its space section.

One way to tackle the problems regarding the topology of the universe is through a suitable statistical analysis applied to catalogs of discrete cosmic sources to find out whether or not there are multiple correlated images of cosmic objects, and eventually determine the topological features of the universe from the pattern of images the sky shows.

The correlations among the images of cosmic objects in multiply-connected universes can be couched in terms of distance correlations between the images. Indeed, one way of looking for distance correlations between cosmic images in multiply-connected universes is by using pair separations histograms (PSH), which are functions $\Phi(s_i)$ that count the number of pair of images separated by a distance that lies in intervals (bins) J_i. The embryonic expectation was that the distance correlations would manifest as topological spikes in PSH's, and that the spike spectrum would be a definite signature of the non-trivial topology [3]. However, this initial expectation turned out to be false [4] – [7]. Nevertheless, the most striking evidence of multiply-connectedness in PSH's is indeed the presence of such topological spikes, which arise from translational isometries $g_t \in \Gamma$. The non-translational isometries $g_{nt} \in \Gamma$, however, manifest as rather tiny deformations of the expected pair separation histogram $\Phi^{sc}_{exp}(s_i)$ corresponding to the underlying simply-connected universe. However, from computer simulations it becomes clear that the expected pair separation histogram (EPSH) corresponding to a multiply-connected universe $\Phi_{exp}(s_i)$, which is nothing but an PSH from which the statistical noise has been withdrawn, is not a suitable quantity for revealing the topology of *multiply-connected* universes [8].

In a recent article, Gomero *et al.* [8] (see also [8]) have proposed a way of extracting the topological signature of any *multiply-connected* universe of constant curvature by using a *new* quantity $\varphi^{mc}(s_i) \equiv (n-1)\,[\,\Phi_{exp}(s_i) - \Phi^{sc}_{exp}(s_i)\,]$, where n is the number of images. Note, however, that this quantity cannot be used as distinguishing marks of *simply-connected* universes since it vanishes identically for such universes. This amounts to saying that the graphs of $\varphi^{mc}(s_i)$ for all three classes of RW simply-connected universes which arise from (simulated or real) catalogs exhibit nothing but statistical noise, and thus $\varphi^{mc}(s_i)$ should not be used as identifying markings in the simply-connected cases. As a matter of fact, the scheme discussed in [8] as well as the approaches that make use of the cosmic microwave background radiation [9] – [20] were fundamentally devised to reveal the possible non-trivial topology of *small* universes. However, neither the multiply nor the simply-connectedness for our universe has been discarded or confirmed by the current astrophysical observations.

In computer-aided simulations the histograms such as the PSH's $\Phi(s_i)$ contain statistical fluctuations, which can give rise to sharp peaks of statistical (non-topological) origin, or can hide (or mask) the tiny deformations due to non-translational isometries. The most immediate approach to cope with fluctuation problems in PSH's is by using the mean pair separation histogram (MPSH) scheme to obtain the mean PSH

$<\Phi(s_i)>$ rather than a single PSH $\Phi(s_i)$. In ref. [8] they have used the MPSH technique to extract the topological signature of RW multiply-connected universes. This technique consists in the use of K (say) computer-generated comparable catalogs to obtain the mean pair separation histograms $<\Phi(s_i)>$ and $<\Phi^{sc}(s_i)>$; and use them as approximations for $\Phi_{exp}(s_i)$ and $\Phi^{sc}_{exp}(s_i)$, to construct the topological signature $\varphi^{mc}(s_i) \simeq (n-1)\,[<\Phi(s_i)> - <\Phi^{sc}(s_i)>]$. Obviously the greater is the number K of catalogs the better are the approximations $<\Phi(s_i)> \simeq \Phi_{exp}(s_i)$ and $<\Phi^{sc}(s_i)> \simeq \Phi^{sc}_{exp}(s_i)$.

In this article we point out that the statistical quantity $\phi^{sc}(s_i) = \Phi^{sc}_{exp}(s_i)$ is indeed a suitable distinguishing mark of the simply-connected RW universes, and rederive its explicit expressions for Euclidean, hyperbolic and elliptic *simply-connected* universes (spherical balls $\mathcal{B}_a$ with radius a) fulfilled with an uniform distribution of cosmic objects. In doing so, on the one hand we obtain the exact (free from statistical fluctuation) expressions for the distinguishing mark $\phi^{sc}(s_i)$ of the three possible classes of simply-connected RW universes; on the other hand one attains a refined statistical meaning of the signature $\varphi^{mc}(s_i)$ and also obtains an improvement on the procedure to extract the topological signature of *multiply-connected* RW universes devised in ref. [8].

In the next section we set our framework, define the basic notation, and derive the expressions for the distinguishing marks for the three possible classes of simply-connected RW universes (Euclidean, hyperbolic, elliptic). There we also present and analyze graphs of the distinguishing mark $\phi^{sc}(s_i)$ of simply-connected RW universes, and discuss the improvement we have obtained in the procedure to extract the topological signature of multiply-connected universes studied in ref. [8]. In the last section we summarize and discuss our main results and present the concluding remarks.

To close this section a word of clarification: although throughout this article we loosely use the terminology topological signature of a universe and/or of a manifold, it should be noted that the topological signature actually corresponds to an observed universe, which in this paper is a spherical ball $\mathcal{B}_a \subset \widetilde{M}$ of radius a, which contains the set of the observed images.

2 Distinguishing marks

In this section we will first set the notation and then recast in a unified and compact way the explicit expressions for the probability densities obtained in [21] so as to show that they can be used as distinguishing mark for the three possible classes ($k = 0, \pm 1$) of simply-connected RW universes fulfilled with an uniform distribution of cosmic objects.

Let us start by recalling that a *catalog* $\mathcal{C}$ is a set of *observed images*, subset of the set $\mathcal{O}$ of *observable* images ($\mathcal{C} \subset \mathcal{O}$), which are clearly contained in the *observable universe*, which in turn is the part of the universal covering manifold $\widetilde{M}$ causally connected to an image of a given observer. The *observed universe* is the part of the observable universe which contains all the sources registered in the catalog. Our observational limitations are formulated through selection rules which dictate how the subset $\mathcal{C}$ arises from $\mathcal{O}$. Catalogs whose images obey the same (well-behaved) distribution law and that follow the same selection rules are said to be *comparable catalogs* [6]. It should be noted that in the process of construction of catalogs it is assumed a RW geometry (needed to convert redshift into distance) and that a particular type of sources (clusters of galaxies, quasars, etc) is chosen from the outset. So, for our purpose in the present work the relevant information registered in a given catalog is the redshift corresponding to each image in the catalog.

Consider a catalog $\mathcal{C}$ with n cosmic images and denote by $\eta(s)$ the number of pairs of images whose separation is s. Consider also that our observed universe is a ball of radius a and divide the interval $(0, 2a]$ in m equal subintervals J_i of length $\delta s = 2a/m$. Each of such subintervals has the form

$$J_i = (s_i - \frac{\delta s}{2} \, , \, s_i + \frac{\delta s}{2}] \qquad ; \qquad i = 1, 2, \, \dots \, , m \; , \tag{2.1}$$

and is centered at

$$s_i = (i - \frac{1}{2}) \; \delta s \; .$$

The PSH is a normalized function which counts the number of pair of images separated by a distance that lies in the subinterval J_i. Thus the function PSH is given by

$$\Phi(s_i) = \frac{2}{n(n-1)} \, \frac{1}{\delta s} \sum_{s \in J_i} \eta(s) \; , \tag{2.2}$$

and is clearly subjected to the normalizing condition

$$\sum_{i=1}^{m} \Phi(s_i) \; \delta s = 1 \; . \tag{2.3}$$

If one considers an ensemble of comparable catalogs[1] with the same number n of images, and corresponding to the same three-manifold M of constant curvature, one can

[1]Note that a typical catalog of the ensemble reflects (corresponds to) a distribution of images in the observed universe $\mathcal{B}_a$.

compute probabilities and expected values of quantities which depend on the images in the catalogs of the ensemble. In particular, we can compute the expected number $\eta_{exp}(s_i)$ of pairs of cosmic images in a catalog $\mathcal{C}$ of the ensemble with separations in J_i. This quantity is quite relevant because from it one has the normalized expected pair separation histogram (EPSH) which clearly is given by

$$\Phi_{exp}(s_i) = \frac{1}{N}\,\frac{1}{\delta s}\;\eta_{exp}(s_i) = \frac{1}{\delta s}\;F(s_i)\;, \tag{2.4}$$

where obviously $N = n(n-1)/2$ is the total number of pairs of cosmic images in $\mathcal{C}$, and $F(s_i) = \eta_{exp}(s_i)/N$ is the probability that a pair of images be separated by a distance that lies in the interval J_i.

In what follows we shall consider that we have an ensemble of comparable catalogs whose underlying observed universe (spherical ball $\mathcal{B}_a$ with radius a) are simply-connected and fulfilled with an uniform distribution of pointlike objects. We will take $\phi^{sc}(s_i) \equiv \Phi^{sc}_{exp}(s_i)$ as distinguishing mark for these three classes of simply-connected universes. Clearly for this uniform distribution of objects all separations $0 < s_i \leq 2a$ are allowed, so the identifying markings $\phi^{sc}(s_i)$ are continuous functions of s given by

$$\phi^{sc}(s) = \Phi^{sc}_{exp}\,(s) = \frac{1}{\delta s}\,F_{sc}(s)\;, \tag{2.5}$$

where $F_{sc}(s)$ is the probability that a pair of images in a catalog $\mathcal{C}$, corresponding to a simply-connected universe, be separated by a distance s. For the sake of simplicity hereafter we will drop the subscript of $F_{sc}(s)$.

To make explicit that the distinguishing mark depends upon the radius of the observed universe we rewrite (2.5) in the form

$$\phi^{sc}(a,s) = \Phi^{sc}_{exp}\,(a,s) = \frac{1}{\delta s}\,F(a,s) = \mathcal{F}\,(a,s)\;, \tag{2.6}$$

where $\mathcal{F}\,(a,s)$ clearly is the probability density, i.e. it is such that $F(a,s) = \mathcal{F}\,(a,s)\,ds$ gives the probability that two arbitrary points in the ball $\mathcal{B}_a$ be separated by a distance between s and $s+ds$. Equation (2.6) makes apparent that the $\phi^{sc}(a,s)$ gives essentially the distribution of probability for all s in the ball $\mathcal{B}_a$. Moreover, since the way one measures the distances varies for each constant curvature universe, it is clearly expected that the expression for distinguishing mark $\phi^{sc}(a,s)$ changes with the three-geometry of these simply-connected universes. In what follows we shall recast in a compact way the expressions of $\phi^{sc}(a,s)$ for Euclidean, hyperbolic and elliptic simply-connected universes [21].

Consider in either of the simply-connected three-spaces a ball $\mathcal{B}_a$ centered at the origin O, and let P and Q be two arbitrary points in the ball. Denote by $r \in [0, a]$ the radial position of P, and by $s \leq 2a$ the distance from P to Q (see figure 1).[2]

Consider now the quantity $\mathcal{F}(a, r, s)\, dr\, ds$, which is the probability that P lies in a position between r and $r + dr$, times the probability that the separation between P and Q lies between s and $s + ds$. Clearly for the simply-connected cases we are concerned the probability density $\mathcal{F}(a, r, s)$ is proportional to the following two areas: (i) $\mathcal{A}_S(r)$ which is the area of the locus of the points P located at a distance r from the origin 0; and (ii) the area of the locus of the points Q that are separated from P by s. Note, however, that when $r + s < a$ the latter locus is a two-sphere S^2 with area $\mathcal{A}_S(s)$, whereas when $r + s > a$ it changes into a spherical calotte (cap) with area $\mathcal{A}_C(a, r, s)$.

To sum up we have that the expression for the probability density $\mathcal{F}(a, r, s)$ for the configuration in which P is between r and $r + dr$, and is separated from Q by a distance between s and $s + ds$, can be written in the general form

$$\mathcal{F}(a, r, s) = \zeta\; \mathcal{A}_S(r)\; [\, \mathcal{A}_S(s)\; \Theta(a - s - r) + \mathcal{A}_C(a, r, s)\; \Theta(s + r - a)\,] \;, \qquad (2.7)$$

where ζ is a normalization constant and Θ is the Heaviside function. Obviously, due to the cut-off effects of the function Θ the first term of the sum in the right-hand side of (2.7) is nonzero only for $r + s < a$, whereas the second term is non-null only when $r + s > a$.

Now since the area $\mathcal{A}_S(r)$ of the two-sphere as well as the area of the spherical calotte $\mathcal{A}_C(a, r, s)$ depend on what is the simply-connected three-space where the ball $\mathcal{B}_a$ is considered, then to obtain $\mathcal{F}(a, r, s)$ for each class of simply-connected universes we are interested one ought to: (i) calculate the areas $\mathcal{A}_S(r)$ and $\mathcal{A}_C(a, r, s)$; (ii) insert in (2.7) and integrate $\mathcal{F}(a, r, s)$ from $r = 0$ to $r = a$; and (iii) impose the normalization condition

$$\int_0^{2a} \mathcal{F}(a, s)\, ds = \int_0^{2a} \phi^{sc}(a, s)\, ds = 1 \;, \qquad (2.8)$$

to obtain the value of the normalization constant ζ. In what follows we shall use this systematic scheme to determine $\mathcal{F}(a, r, s)$ for the three classes of simply-connected universes we are concerned.

[2] In order to encompass the elliptic class in our compact approach we shall initially treat only the elliptic cases in which the radius a of the universe $\mathcal{B}_a \subset S^3$ is such that $2a < \pi R$, where R is the radius of the curvature of the geometry, i.e. the scale factor of RW metric (1.1) for a given time $t = t_0$. Further, for the sake of simplicity and without loss of generality we shall also set $R = 1$ for both the hyperbolic and elliptic cases.

For Euclidean universe ($\mathcal{B}_a \subset E^3$) one obviously has $\mathcal{A}_S(r) = 4\pi r^2$, and straightforward calculations furnish $\mathcal{A}_C(a,r,s) = (\pi s/r)\,[\,a^2 - (s-r)^2\,]$. According to the above-outlined scheme inserting these expressions in (2.7), integrating $\mathcal{F}(a,r,s)$ from $r=0$ to $r=a$, and using the condition (2.8) together with (2.6), one obtains that the expression for the distinguishing mark $\phi_E^{sc}(a,s)$ of a Euclidean universe is given by

$$\phi_E^{sc}(a,s) = \frac{3}{16\,a^6}\; s^2\,(2a-s)^2\,(s+4a)\;, \tag{2.9}$$

which holds for $s \in (0, 2a]$, and where the value of the normalization constant was found to be $\zeta = 9/(16\pi^2 a^6)$.

Before proceeding to the next class it should be noticed that the shape of the signature $\phi_E^{sc}(a,s)$ does not depend on the value of the radius a. Indeed, in terms of a new variable $s' = s/a$ the expression (2.9) can be rewritten in the form

$$\phi_E^{sc}(a,s') = \frac{3}{16\,a}\; s'^2\,(2-s')^2\,(s'+4)\;, \tag{2.10}$$

which makes clear that for distinct radii a one has different constant multiplying factors $3/(16a)$, but without changing the functional dependence of $\phi_E^{sc}(a,s)$ with s. So, the shape of the graph of the distinguishing mark function for Euclidean universes does not depend on the value of the radius a.

For hyperbolic universes ($\mathcal{B}_a \subset H^3$) one obtains $\mathcal{A}_S(r) = 4\pi\sinh^2 r$ and $\mathcal{A}_C(a,r,s) = 2\pi \sinh s\,[\,\sinh s - \cosh s\,\coth\,r + \cosh a\,\operatorname{csch} r\,]$. Again, through the second and third steps of the above-mentioned systematic scheme one finds the following expression for the distinguishing mark of a simply-connected hyperbolic universe fulfilled with an uniform distribution of objects:

$$\phi_H^{sc}(a,s) = \frac{8\sinh^2 s}{(\sinh 2a - 2a)^2}\;\left[\cosh a\;\operatorname{sech}(s/2)\;\sinh(a-s/2)\;-\;(a-s/2)\right]\;, \tag{2.11}$$

which holds for $s \in (0, 2a]$, and where the value of the normalization constant in this case was found to be $\zeta = [\,\pi\,(\sinh 2a - 2a)\,]^{-2}$.

As for the elliptic universes ($\mathcal{B}_a \subset S^3$) when the diameter $2a$ is less than the separation πR between antipodal points of S^3 the above scheme can be similarly used. For this case one obtains $\mathcal{A}_S(r) = 4\pi\sin^2 r$ and $\mathcal{A}_C(a,r,s) = 2\pi\sin s\,[-\sin s - \cos s\;\cot\,r + \cos a\;\csc\,r\,]$. Using these expressions in (2.7) and following the above-outlined general procedure one finds

$$\phi_S^{sc}(a,s) = \frac{8\sin^2 s}{(2a - \sin 2a)^2}\;\left[(a-s/2) - \cos\,a\;\sec(s/2)\;\sin(a-s/2)\right]\;, \tag{2.12}$$

which hold for $2a < \pi$, where we have taken $R = 1$. The value of the normalization constant in this case was found to be $\zeta = [\,\pi\,(2a - \sin 2a)\,]^{-2}$.

The elliptic universes ($\mathcal{B}_a \subset S^3$) for which $2a > \pi R$ cannot be included in the above general scheme. They are trickier to be handled due to the connectivity of the spherical space S^3 and the additional requirement that s must not exceed πR, which is needed to ensure that one is taking the shortest geodesic part between two points of S^3. For the sake of brevity and completeness we shall present here only the final expression for $\phi^{sc}(a,s)$. It turns out that the general expression of the distinguishing mark which holds for all elliptic universes with $a \in (0,\pi]$ and fulfilled with a uniform distribution of objects is given by

$$\begin{aligned} \phi^{sc}_S(a,s) \;\; = \;\; \frac{8\sin^2 s}{(2a-\sin 2a)^2}\; \{2a - \sin 2a - \pi + \Theta(2\pi - 2a - s)\; [\,\sin 2a \,+\, \pi \\ - a - s/2 - \cos a\;\; \sec(s/2)\;\; \sin(a - s/2)\;\,]\;\, \}\;\; , \end{aligned} \tag{2.13}$$

where $s \in (0, \min(2a,\pi)]$.

In what follows we shall present and analyze a few graphs of the distinguishing mark for each class or RW simply-connected universes.

Figure 2 shows the distinguishing mark $\phi^{sc}_E(a,s)$ for a Euclidean universe $\mathcal{B}_a$ with radius $a = 0.5$. This marking also gives the probability distribution of the pair separation distance s for $s \in (0, 2a]$. A close inspection of this figure reveals that the most likely separation between two arbitrary pointlike objects in the Euclidean observed universe $\mathcal{B}_a$ is slightly greater than the radius a of the ball.

Figure 3 shows the distinguishing mark $\phi^{sc}_H(a,s)$ for three values of the radius a. For $a \ll 1$ this function behaves approximately as that we have derived for the Euclidean universe (figure 2), as one would expect from the beginning. For increasing values of the radius a of the observed universe the maximum of the mark $\phi^{sc}_H(a,s)$ moves towards the greater values of s. In other words, the most likely value of s increases (the maximum of $\phi^{sc}_H(a,s)$ takes place later) for increasing values of the radius a.

In figure 4 four graphs of the distinguishing mark $\phi^{sc}_S(a,s)$ for different values of the radius a of the universe are shown. For increasing values of a from 0 to π the maximum of the signature $\phi^{sc}_S(a,s)$ moves continuously towards the smaller values of s. Contrarily to the hyperbolic case, here for increasing values of the radius a the maximum of $\phi^{sc}_S(a,s)$ moves toward the origin (smaller values of s). This is also illustrated in figure 4, where the most likely value of the separation s decreases (the maximum of $\phi^{sc}_S(a,s)$ takes place earlier) for increasing values of a.

It also should be mentioned that for an arbitrary radius of curvature of the geometry R the expressions for $\phi_H^{sc}(a,s)$ and $\phi_S^{sc}(a,s)$ can be obtained by multiplying the right-hand side of (2.11) and (2.13) by $1/R$, and simultaneously by changing $a \to a/R$ and $s \to s/R$.

In the remainder of this section we will discuss the improvement of the method to extract the topological signature

$$\varphi^{mc}(s_i) = (n-1)\,[\,\Phi_{exp}(s_i) - \Phi^{sc}_{exp}(s_i)\,]\;, \tag{2.14}$$

of multiply-connected universes studied in [8] (see also [8]). To this end, the relevant point to be noted is that an EPSH $\Phi_{exp}(s_i)$ is essentially a typical PSH from which the statistical noise has been withdrawn. Hence we have

$$\begin{aligned} \Phi_{exp}(s_i) &= \Phi(s_i) - \rho(s_i)\;, && (2.15)\\ \Phi^{sc}_{exp}(s_i) &= \Phi^{sc}(s_i) - \rho^{sc}(s_i)\;, && (2.16) \end{aligned}$$

where $\rho(s_i)$ and $\rho^{sc}(s_i)$ represent the statistical noises that arise in the corresponding PSH's. Using now the decompositions (2.15) and (2.16) together with (2.14) one readily obtains

$$\varphi^{mc}(s_i) = \; (n-1)\,[\,\Phi(s_i) - \Phi^{sc}(s_i) + \rho^{sc}(s_i) - \rho(s_i)\,]\;, \tag{2.17}$$

which clearly gives the topological signature intermixed with two statistical fluctuations. Now, from equations (2.14) – (2.17) it is clear that one can approach the topological signature of multiply-connected universes $\varphi^{mc}(s_i)$ by reducing the statistical fluctuations, i.e. by making $\rho(s_i) \to 0$ as well as $\rho^{sc}(s_i) \to 0$ through any suitable statistical method to lower the noises. An improvement of the method devised in [8] to extract the topological signature $\varphi^{mc}(s_i)$ of multiply-connected universes with a uniform distribution of matter comes out from the very fact that having the derived expressions (2.9), (2.11) and (2.13) one has from the beginning $\rho^{sc}(s_i) = 0$ for those universes. Thus, for example, if the MPSH is the technique one uses to reduce the statistical fluctuations, the topological signature (2.14) in these cases reduces to the form $\varphi^{mc}(s_i) \simeq (n-1)\,[<\Phi(s_i)> - \Phi^{sc}_{exp}(s_i)\,]$, with the exact expression for $\Phi^{sc}_{exp}(s_i)$ rather than the approximate mean $<\Phi^{sc}(s_i)>$.

3 Concluding remarks

To a certain extent it is well-known that RW geometry (1.1) does not fix the global shape (topology) of the spacetime, and that there is an infinite number of topologically distinct $t = const$ spatial sections M for the RW spacetime manifold. Nevertheless, it is often (implicitly or explicitly) assumed that the $t = const$ spatial sections M of a RW spacetime manifold are one of the following simply-connected spaces: E^3 $(k = 0)$, S^3 $(k = 1)$, or H^3 $(k = -1)$. However, this assumption of simply-connectedness for our three-space has not been settled by cosmological observations. As a matter of fact, neither the simply nor the multiply-connectedness for the three-space where we live has been discarded or confirmed by the available astrophysical data.

The two main approaches to constrain or determine the topology of the our three-space rely on the existence of multiple (topological) images of either cosmic objects or spots of microwave background radiation, and thus they aim at non-trivial topology of *small* universes — a possible simply-connectedness (trivial topology) of the universe has not been suitably considered in these approaches.

A special method to determine possible non-trivial topologies of RW universes, and which relies on the existence of multiple images, was recently discussed in [8]. There it is suggested that the quantity $\varphi^{mc}(s_i) \equiv (n-1)\,[\,\Phi_{exp}(s_i) - \Phi^{sc}_{exp}(s_i)\,]$ is a suitable measure of the topological signature of the multiply-connected RW universes. However, $\varphi^{mc}\,(s_i)$ cannot be used as the topological signature for simply-connected universes, since it vanishes identically. This means that if we live in RW simply-connected universe the graphs of $\varphi^{mc}(s_i)$ which arise from real (or simulated) catalogs will exhibit nothing but statistical noise. Thus $\varphi^{mc}(s_i)$ can be used not only to extract the topological signature of multiply-connected universes but also to decide between multiply or simply-connectedness of the universe, since in this latter case it gives rise simply to statistical noise.

One might think at first sight that the vanishing of $\varphi^{mc}(s_i)$ (which means that it gives rise to nothing but statistical fluctuations) would lead *only* to the simply-connectedness without separating among the three possible classes of simply-connected RW universes. In practice, though, the vanishing of $\varphi^{mc}(s_i)$ takes place for one underlying RW metric used to convert redshift into distance to have the pair separations, and thus it also gives the underlying manifold of the corresponding simply-connected universe, as there is a clear correspondence between geometry and the covering manifold in these cases. Nevertheless, since $\varphi^{mc}(s_i)$ vanishes identically (gives rise to nothing but statistical noise) for all classes of RW universes it is not an univocal (or unequivocal) distinguishing mark of these universes. Thus we have been led to take the quantity $\phi^{sc}(a, s)$ as distinguishing mark of the simply-connected RW universes. Clearly

$\phi^{sc}(a,s)$: (i) does not vanish; (ii) can be used to separate the three possible classes of simply-connected RW universes; and (iii) it is the quantity which really matters (in this statistical context) for the simply-connected cases. Actually $\phi^{sc}(a,s)$ can also be used to distinguish RW universes with different radius a. Further, note that since the way one measures the distances varies for each constant curvature universe, it was really expected from the outset that the expression for the distinguishing mark $\phi^{sc}(a,s)$ would be distinct for different simply-connected universes.

We have also presented the explicit expressions of the distinguishing marks $\phi^{sc}_E(a,s)$, $\phi^{sc}_H(a,s)$ and $\phi^{sc}_S(a,s)$ for, respectively, Euclidean, hyperbolic and elliptic simply-connected RW universes fulfilled with an uniform distribution of cosmic objects. Besides, we have presented and analyzed graphs of this signature for the simply-connected RW universes, and discussed the improvement that these exact expressions bring to the method to extract the topological signature of multiply-connected universes discussed in [8].

The distinguishing marks for the simply-connected RW universes $\phi^{sc}_E(a,s)$, $\phi^{sc}_H(a,s)$ and $\phi^{sc}_S(a,s)$, which we have studied in is this work give, in each case, the probability distributions of the pair separation $s \in (0,2a]$. If one takes these probability distributions as *ground* distributions, then the topological signature of multiply-connected RW universes $\varphi^{mc}(s_i)$ studied in [8] can be understood as a measure of the deviation between the pair separation probability distribution in the multiply-connected cases [given by $\Phi_{exp}(s_i)$] and the corresponding ground pair separation probability distribution. The isometries g of the covering group Γ modify the ground pair separation probability distribution, and the quantity $\varphi^{mc}(s_i)$ measures that deviation of topological origin.

It is worth mentioning that in the cases of multiply-connected universes for which the smaller length of the fundamental polyhedron $\mathcal{P}$ of M ($\mathcal{P} \subset \widetilde{M}$) is greater than the diameter $2R_H$ (R_H is the particle horizon) of the observed universe $\mathcal{B}_{R_H} \subset \widetilde{M}$, no multiple images can be observed. These multiply-connected universes are therefore indistinguishable from the simply-connected universes with the same covering space, equal radius, and identical distribution of cosmic sources. In these multiply-connected cases in which the scale of the multiply-connectedness is greater than the radius R_H universe $\mathcal{B}_{R_H}$ no sign of the multiply-connectedness will arise, and the distinguishing marks we have discussed in this work can play a relevant role, when there is a homogeneous distribution of matter, of course.

To close this article it is worth mentioning that the ultimate goal in the statistical approaches to extract the topological signature (mark) is the comparison of the graphs (signature or mark) obtained either theoretically or from simulated catalogs against similar graphs obtained from real catalogs. To do so, one clearly has to have either the exact explicit expression or the simulated patterns of the topological signatures of the possible universes. The expressions we have found for the distinguishing mark of simply-connected RW universes with a uniform distribution of matter can certainly be used in such comparisons. Note, however, that even if the universe turns out to be simply-connected one still has to face the remaining problem of reducing the noise from just one or even a few real catalogs of cosmic sources.

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Captions for the figures

Figure 1. Two-dimensional schematic figure of observed universe: spherical ball $\mathcal{B}_a$ of radius a, which contains the set of the observed images. The circular arc with center in P represents a spherical calotte (cap) that changes into a sphere S^2 when $r + s \leq a$.

Figure 2. The distinguishing mark $\phi_E^{sc}(a,s)$ for a Euclidean simply-connected universe $\mathcal{B}_a$ for a radius $a = 0.5$. The horizontal axis gives the pair separation s while the vertical axis gives the normalized number of pairs. This curve also gives the probability distribution of the pair separation distance s in this universe $\mathcal{B}_a$. A close inspection reveals that the most likely separation between two arbitrary images in a Euclidean universe $\mathcal{B}_a$ is slightly greater than the radius a of the universe.

Figure 3. Graphs of the distinguishing mark $\phi_H^{sc}(a,s)$ of hyperbolic simply-connected universes $\mathcal{B}_a$ for three different values of the radius a. The normalized number of pairs in the vertical axis is given in unit of $1/(2a)$, while in the horizontal axis the

unit of length is equal to $2a$. Note that for $a \ll 1$ the signature $\phi_H^{sc}(a,s)$ behaves approximately as its Euclidean counterpart $\phi_E^{sc}(a,s)$, whose graph is shown in figure 2. For increasing values of the radius a of the universe $\mathcal{B}_a$ the maximum of the signature $\phi_H^{sc}(a,s)$ shifts to the right. The most likely value of s increases for increasing values of the radius a, and for $a \gg 1$ there is noticeable concentration of large values of s near the extreme value $s = 2a$.

Figure 4. Graphs of the distinguishing mark $\phi_S^{sc}(a,s)$ of elliptic simply-connected universes $\mathcal{B}_a$ for four different values of the radius a. The normalized number of pairs in the vertical axis is given in unit of $1/(2a)$, while in the horizontal axis the unit of length is equal to $2a$. For increasing values of a from 0 to π the maximum of the signature $\phi_S^{sc}(a,s)$ moves towards the smaller values of s. This behavior is the opposite of that corresponding to the hyperbolic case shown in figure 3. The most likely value of the separation s decreases for increasing values of a.

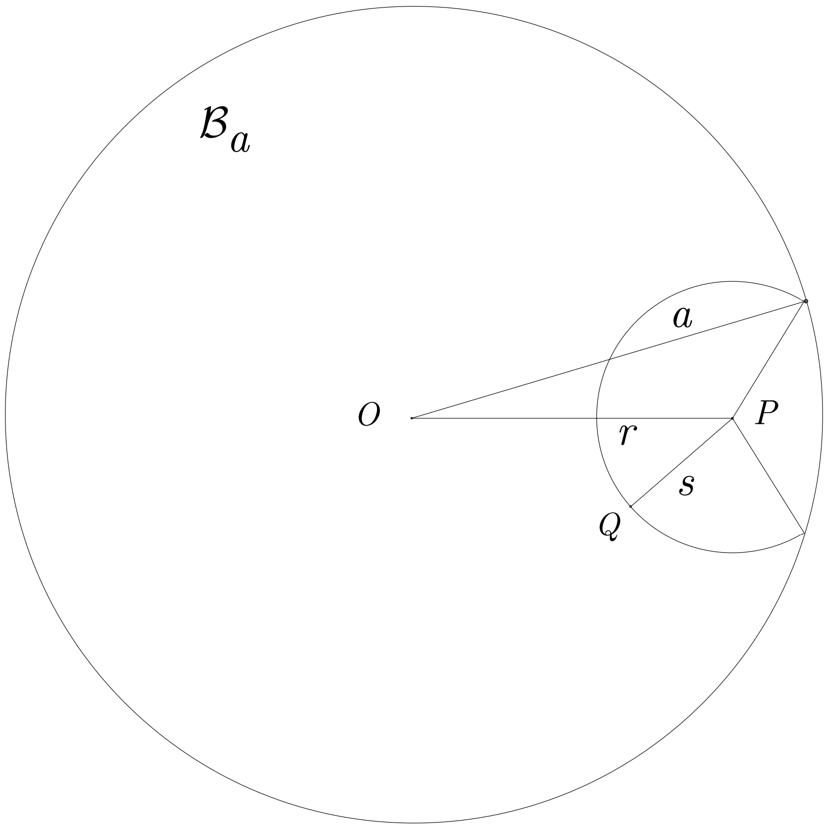

Figure 1

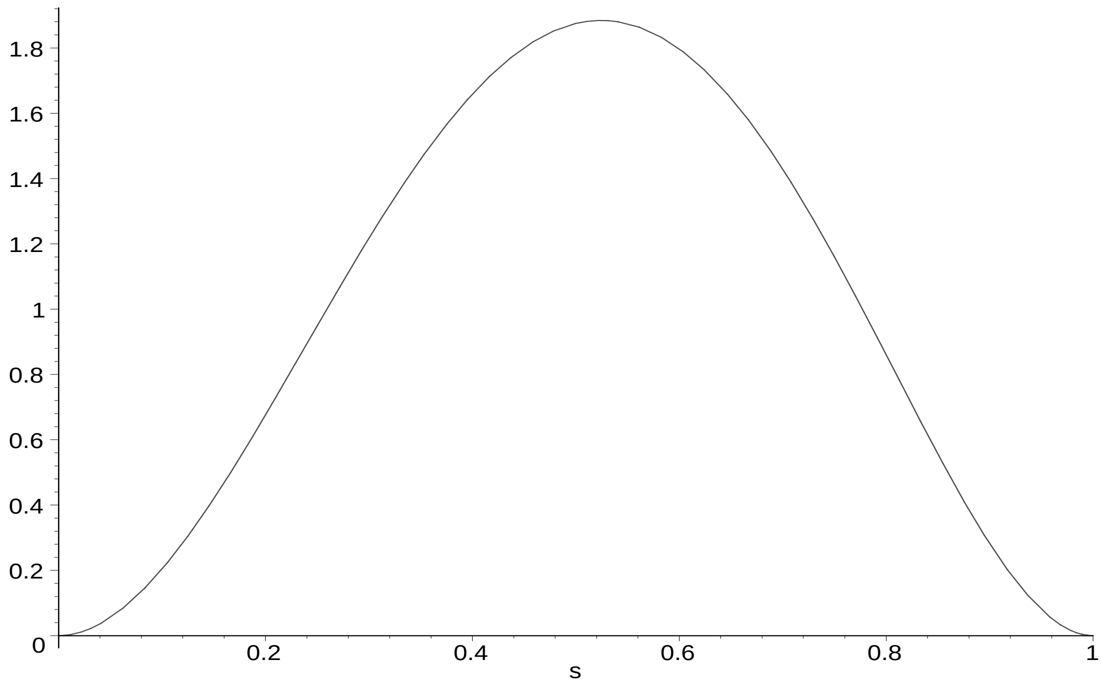

Figure 2

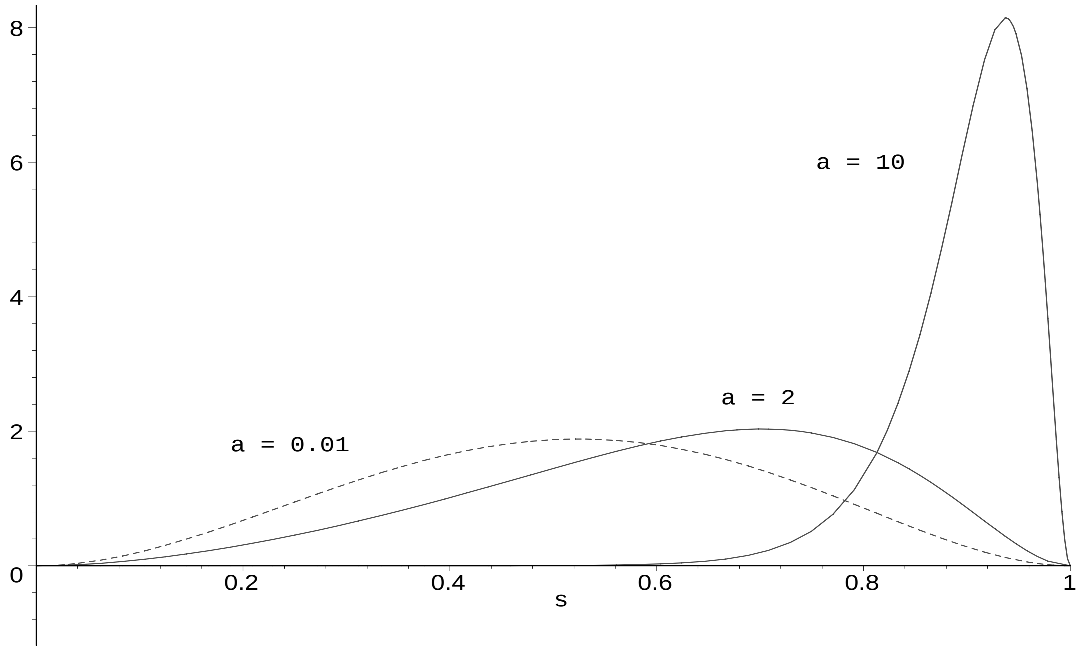

Figure 3

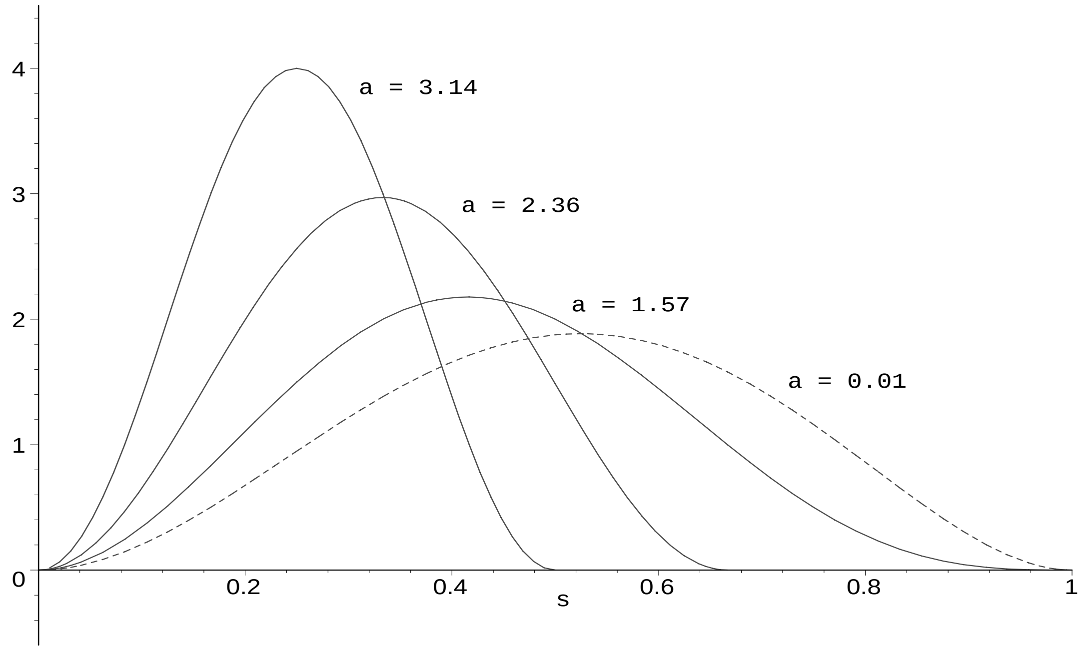

Figure 4

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