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Imitation games: Power-law sensitivity to initial conditions and nonextensivity

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ABSTRACT

We exhibit, at the self-organized critical state, a power-law sensitivity to the initial conditions in the system of competing logistic maps introduced by Suzuki and Kaneko (1994), and which modelizes the battle of birds for defending their territories. From the associated exponent we obtain the value for the entropic index q which defines the recently introduced nonextensive generalization of the Boltzmann-Gibbs thermostatistics. In addition to that, we calculate the dynamical exponent z. We obtained $q \simeq -0.69$ and $z \simeq 1.32$ for a mean-field-type calculation ($d = \infty$), and $q \simeq -0.39$ and $z \simeq 1.12$ for the square lattice ($d = 2$).

Key-words: Nonextensivity; Generalized Thermostatistics; Competing logistic maps; Self-organized criticality.

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Self-organized criticality is an interesting and quite ubiquitous phenomenon. Indeed, it is experimentally observed and theoretically studied in a wide variety of physical systems, which include sandpiles, ricepiles, earthquakes and others [1]. It has also been observed in the model introduced by Suzuki and Kaneko [2] who propose a set of coupled logistic maps to mimic the singing imitation games which constitute the basis of the battle for territory defense in various species of birds. This model has successfully described a variety of biologically relevant features. However, its sensitivity to the initial conditions has never been focused up to now, to the best of our knowledge. This property, which essentially characterizes chaotic behavior, is herein quantitatively studied. Moreover, we show that it is intimately related to thermostatistical nonextensivity in the sense we now describe.

It has been known for many years [3] that standard statistical mechanics fails to describe pathological systems such as those which include long-range interactions (as well as long-range microscopic memory, or fractal boundary conditions in space-time, among others). To discuss this type of anomalous systems, one of us proposed [4] a thermostatistics based on the following generalized entropic form:

$$S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \tag{1}$$

where W is the total number of configurations, $\{p_i\}$ are the associated probabilities, k is some suitable positive constant and $q \in \mathcal{R}$ is the index that allows for the generalization. It is straighforwardly verified that, in the $q \to 1$ limit, Eq. (1) reduces (by using $p_i^{q-1} \sim 1+(q-1)\;\ln p_i$) to the well known expression:

$$S_1 = -k_B\sum_{i=1}^{W} p_i \ln p_i. \tag{2}$$

Let us mention that if we consider a system composed by two *independent* subsystems A and B (in the sense that the composed probabilities *factorize* into those of A and B), we easily verify that $S_q(A+B) = S_q(A)+S_q(B)+(1-q)S_q(A)S_q(B)$, which clearly exhibits *nonextensivity* if $q \neq 1$.

The above generalization has been applied to a wide variety of physical situations including self-gravitational systems [5,6], turbulence in pure-electron plasma [6], dynamic linear response for nonextensive systems [7], Lévy-like [8] and correlated-like [9] anomalous diffusions , solar neutrino problem [10], cosmology [11], long-range fluid and magnetic systems [12], optimization techniques [13].

This formalism has been shown [14,15] to be connected to the sensitivity of nonlinear dynamical systems. More precisely, the numerical analysis of logistic-like maps as well as of the Bak-Sneppen model for biological evolution has shown that, whenever quantities like the Lyapunov exponent vanish (*e.g.*, at the onset of chaos), the standard, exponential type, sensitivity to the initial conditions is often replaced by a weaker, power-law type, one. Furthermore, it has been shown [14] that the associated critical exponent equals $1/(1-q)$ (*i.e.*, if $t \to \infty$, $lim_{\Delta x(0)\to 0}\frac{\Delta x(t)}{\Delta x(0)} \propto t^{1/(1-q)}$ for $d = 1$ nonlinear maps), which provides q. Also, it has been possible to generalize, for arbitrary q, the Pesin connection between the Kolmogorov-Sinai entropy and the Lyapunov exponent.

In the present work, we show how these ideas apply to the Suzuki and Kaneko imitation games [2]. After some transient, this model spontaneously evolves towards a critical state located at the edge between chaos and periodic windows , *i.e.*, in a parameter region where the Lyapunov exponent of individual maps is close to zero. This peculiar steady-state-like behavior is sometimes referred to as *intermittent chaos* or *weakly chaotic.*

We use, for the imitation games, the logistic maps

$$x_{n+1}(i) = 1 - a(i)\, x_n^2(i) \tag{3}$$

that play by pairs *(i,j)*; $(x_n(i) \in [-1,1]; a(i) \in [0,2]; i = 1,2,...,N)$. Initially, each map starts with random values for both $x_0(i)$ and $a(i)$, and repeats its dynamics for a time T_{rel} (*rel* stands for relaxation) long enough to approach its individual attractor. Then, during a time T_{imi} (*imi* stands for imitation), the i-th map (for all values of i) modifies its dynamics (imitates) with a feedback from the j-th map (we describe later on how is chosen the sequence for j)

$$x_{n+1}(i) = 1 - a(i)\left[(1-\epsilon(i))\, x_n(i) + \epsilon(i)\, x_n(j)\right]^2 \tag{4}$$

where the $\{\epsilon(i)\}$ are the parameters (random numbers uniformly distributed between zero and one, and chosen once for ever) that characterize how strong the imitation is. After this time, the i-th map returns to its own dynamics (this is to say, it runs with Eq. (3) starting from its current value for x_n) during a period T_D. During time T_D a quantity $D(i,j)$ is calculated that measures the distance between the imitated values $\{x_n(i)\}$ and the original ones $\{x_n(j)\}$:

$$D(i,j) = \sum_{n=T_{rel}+T_{imi}+1}^{T_{rel}+T_{imi}+T_D} |x_n(i) - x_n(j)|^2 . \qquad (5)$$

For the present work, and following reference [13], we adopt $T_{rel} = T_{imi} = T_D = 30$.

By inverting the role of the two maps, and repeating the same procedure, the quantity $D(j,i)$ is calculated. If $D(i,j) < D(j,i)$ then we will say that the i-th map imitates better the j-th map than the other way around, hence the i-th map "wins". By "wins" we mean that the parameter $a(j)$ is substituted by a value in the vicinity of $a(i)$, more precisely, it becomes $a(i)+\delta$, where δ is a small random number, representing a *mutational* possibility; following [2], δ is extracted from the Lorentzian distribution $P(\delta) = \mu/(\mu^2 + \delta^2)$ with $\mu = 0.001$. If $D(i,j) > D(j,i)$, the "winner" now is the j-th map, and we proceed as before, i.e., we change the parameters of the i-th map. We have not used homogenous random distributions for determining δ because, as noted in [2] and confirmed by us, this can lead to parameters $\{a\}$ which get trapped at intermediate, meaningless, values. Let us add that no further variations were performed on the ϵ-parameter distribution because, as noted by Susuki and Kaneko [2] and confirmed by our own simulations, the distribution of this parameter appears to be essentially "irrelevant".

In order to determine the values of a that often win, and following the lines of [2], we define a score for each value of a. We increase by one a counter associated with the value of a of the winning map, and by zero the counter corresponding to the looser. In Figure 1 we show the score as a function of a. Note the importance of the period-three and period-four windows. Figure 2 shows the score as a function of the logistic-map Lyapunov exponent (λ). Note now the peak on the zero value. Let us recall that, for a single logistic map, $\lambda = 0$ precisely corresponds to $q \neq 1$, i.e., to a power law (instead of exponential) dependence of the sensitivity to the initial conditions.

The Hamming distance between two systems (the *original* and its *replica*) of N maps each, at any time n, is defined as:

$$H_n = \sum_{i=1}^{N} \frac{|a_n(i) - a'_n(i)|}{N} \tag{6}$$

where the prime stands for the replica sample. Note that the Hamming distance is calculated over the parameters $\{a\}$ of the maps, and not over the entire phase space (which includes the x variables) because it is the $\{a\}$ (and not the $\{x\}$) which self-organize. In our simulations, we have let a system of N maps ($N = 125,\ 250,\ 500,\ 1000$) to relax, and then we have constructed a replica by randomly changing (in ± 0.001) the parameters $\{a\}$ of (typically) 10 randomly chosen maps. After this is done, we calculate the "initial" damage or Hamming distance H_0. It is clear that, for increasingly large values of N and since we have fixed in 10 the number of different maps in the replica, the Hamming distance tends to zero. By so doing we numerically approach the definition of the Lyapunov exponent, which demands a vanishingly small initial discrepancy between the replicas. By allowing both the original and the replica systems to follow the previously described dynamics, *with identical sequences of random numbers* (in accordance with the standard spreading-of-damage procedure), we calculated the ratio H_n/H_0. We use as unit time (n is increased by one) each single game between two birds. In Figure 3 it is shown, for the mean-field-like model (every bird plays with each one of the others), the log-log time dependence of the average (over 50 realizations) of H_n/H_0 for various system sizes . It is a general feature that H_n/H_0 grows with time following a power-law (basically the same for all sizes) up to a size-dependent point, after which a stationary plateau is reached. The slope of the first part of the curves is 0.59 ± 0.02. Since this value equals [14] $1/(1-q)$, we determine $q = -0.69 \pm 0.02$. For the d=2 square lattice model (with periodic boundary conditions) we have obtained $q = -0.39 \pm 0.02$. These values can be compared with those associated with other models, namely the logistic map at its threshold to chaos ($q = 0.24$

[14]), the Bak and Sneppen model for biological evolution at the self-organized critical state ($q = -2.1$ [15]), and the ricepile model ($q = -0.12$ [16]).

The dynamical exponent z is usually defined through $\tau \sim L^z$ where τ is the time during which the systems behaves dynamically and L is the linear size of the system. More precisely, the time τ is defined as the time needed for the damage to reach, except for statistical fluctuations, its stationary value. So, τ was estimated through the intersection, in Figure 3, of the two straight lines respectively defined by the plateau-like stationary value and by the power-law increasing regime. It was found (see Figure 4) that the dynamical exponent z equals 1.32 ± 0.01. For the $d = 2$ square lattice model we obtained $z = 1.12 \pm 0.02$. These values can be compared with those obtained for the Bak and Sneppen model ($z = 1.56$ [15]), the ricepile model ($z = 1.3$ [16]), and the square lattice Ising ferromagnet ($z = 2.16$ [17]).

In addition to the above results, we have checked the sensitivity to the initial conditions of the set of values $\{x\}$. After a simple transient, a rather trivial diffusive-like behavior is observed, which reconfirms that, in the present problem, it is in the space of the $\{a\}$ that the nontrivial results are conveniently revealed.

Summarizing, we have studied, at the self-organized critical state, the sensitivity to initial conditions of the Suzuki and Kaneko model for imitation games between birds. We have exhibited nontrivial power-law behaviors, which enable, among others, the connection with nonextensive statistics. In other words, we have herein illustrated how the entropic exponent q can be derived from the knowledge of the microscopic dynamics.

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FIGURE CAPTIONS

Figure 1.- Stationary score *versus* a for 100 logistic maps playing all with all (mean-field-like model). The sum was calculated during $\sim 10^8$ time steps after a long relaxation period. One time step is a single two-birds game. Period three window, period four window and the onset of chaos are pointed by arrows; other remarkable peaks correspond to other windows and bifurcations.

Figure 2.- Stationary score *versus* Lyapunov exponent (λ). Is the result of calculating on Figure 1. Note the peak around $\lambda = 0$. It was used a bin size of 0.0001.

Figure 3.- Log-log plot of the Hamming distance *versus* time for various system sizes. All the curves coincide rather well on the straight line part. The time step is a game between two maps.

Figure 4.-Log-log plot of time τ needed to reach the "knee" (in Fig. 3) *versus* system size. The slope of the straight line is 1.32 with good accuracy. The time units are the same as in Figure 3.

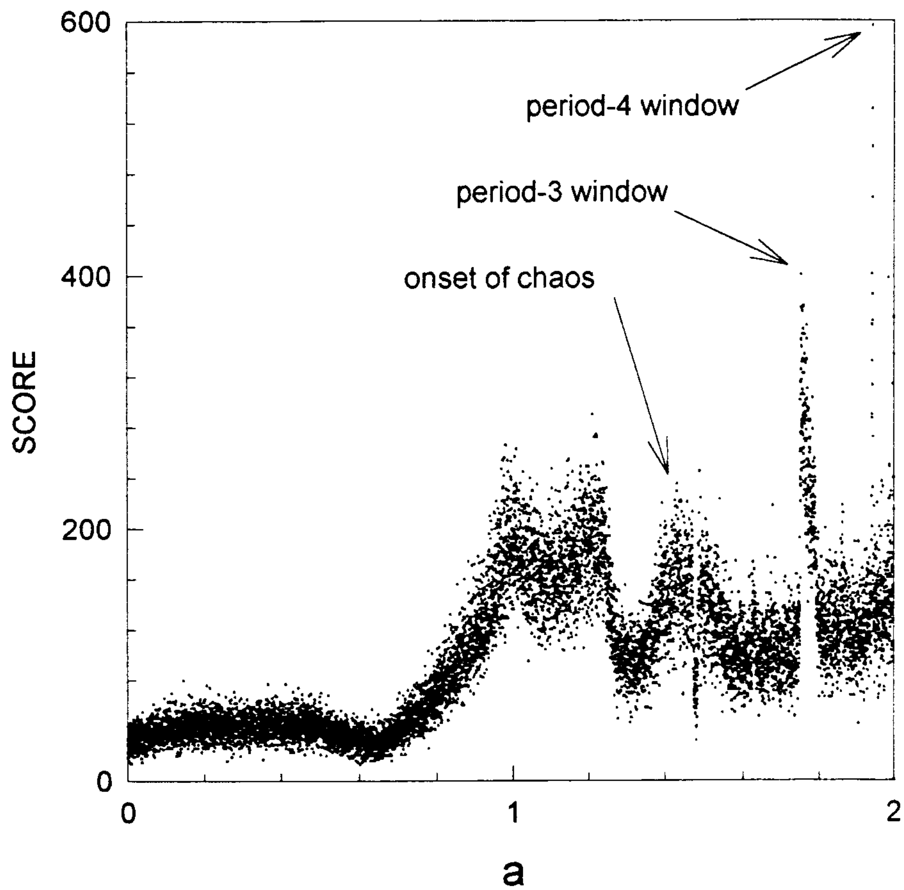
600
400
200
0
SCORE
0
1
2
a
period-4 window
period-3 window
onset of chaos

Figure 1

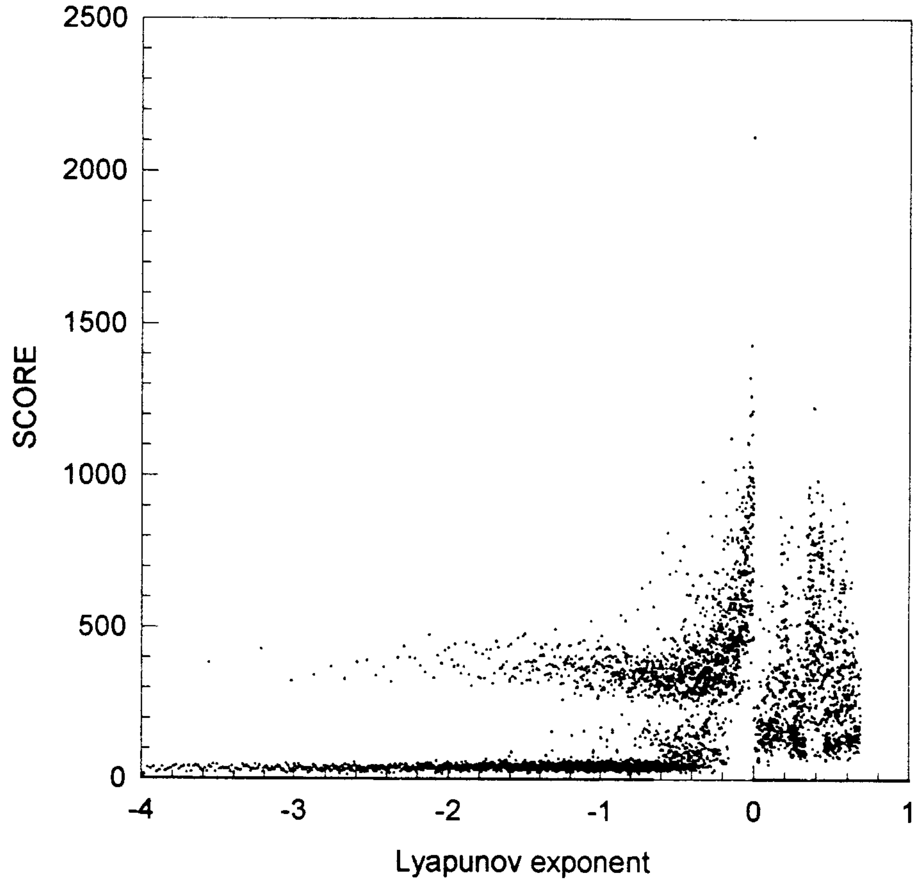

Figure 2

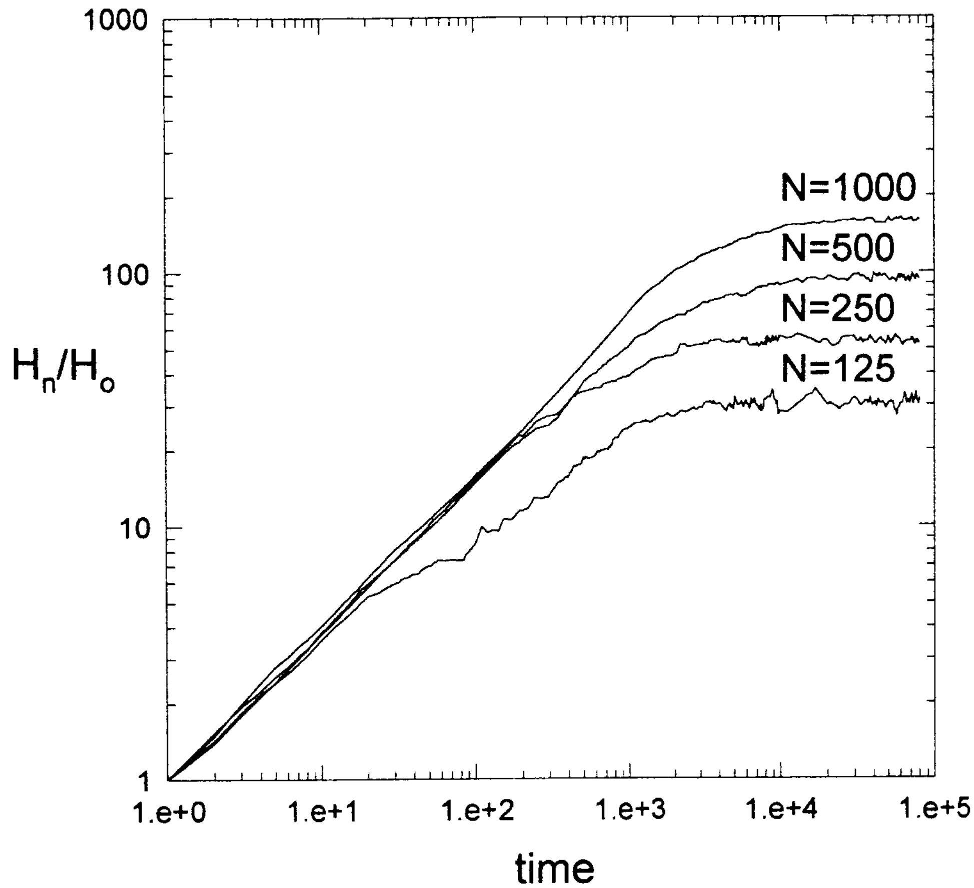
1000
100
10
1
H_n/H_o
1.e+0
1.e+1
1.e+2
1.e+3
1.e+4
1.e+5
time
N=1000
N=500
N=250
N=125

Figure 3

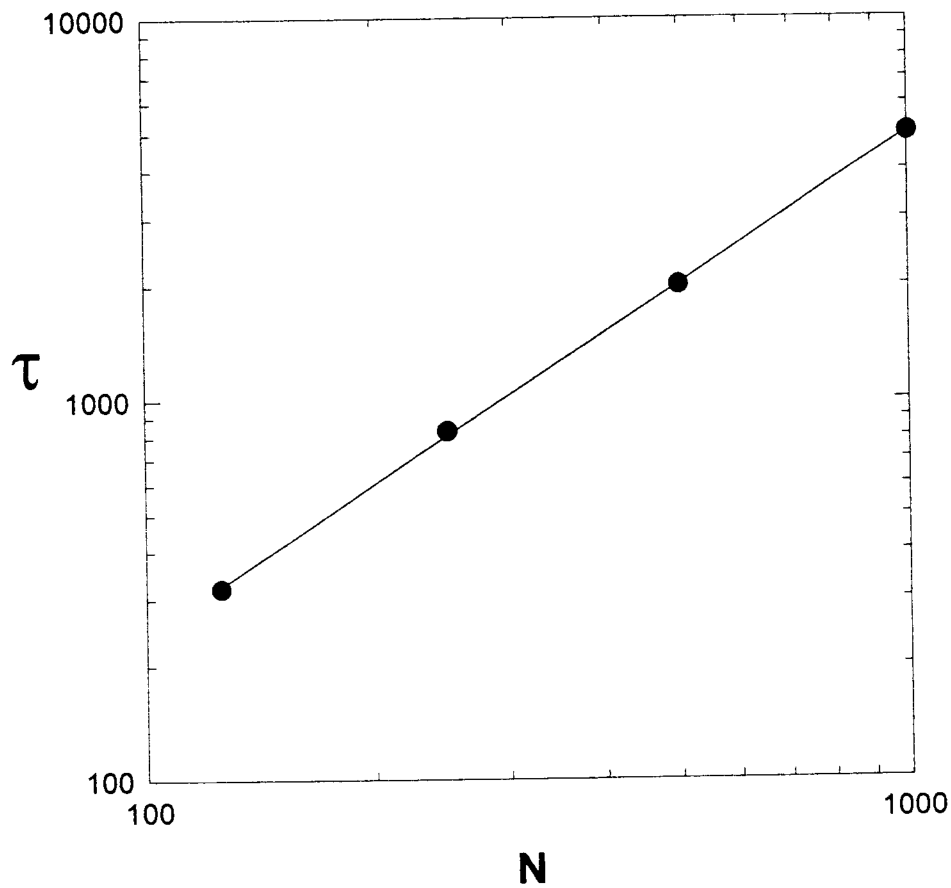

Figure 4