On Anomalies in Classical Dynamical Systems

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Abstract

The definition of "classical anomaly" is introduced. It describes the situation in which a purely classical dynamical system which presents both a lagrangian and a hamiltonian formulation admits symmetries of the action for which the Nöther conserved charges, endowed with the Poisson brackets structure, close an algebra which is just the centrally extended version of the original symmetry algebra. The consistency conditions for this to occur are derived. Explicit examples are given based on simple two-dimensional models. Applications of the above scheme and lines of further investigations are suggested.

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1   Introduction.

Here I define as *"classically anomalous"* any classical dynamical system, described both
in the lagrangian and in the hamiltonian formalism, whose symmetries of the action
produce conserved Nöther charges which, under the Poisson-bracket algebra, satisfy a
centrally extended version of the original symmetry algebra.

In many and perhaps even most of the texts discussing quantum mechanics and quan-
tum field theories it is commonly stated that anomalies are a purely quantum-mechanical
effect. This statement reflects a widespread (but erroneous) belief in the scientific com-
munity, mostly shared by researchers who do not have a direct working experience with
anomalies. While the specialists in the field are aware that some specific features, which
can be reasonably named "anomalies", can be present even in purely classical dynamical
systems, it seems, however, that this correct interpretation passes largely unnoticed. One
of the reasons is due to the fact that most of the results concerning anomalous effects
in classical dynamical systems are scattered in the literature. Moreover, they appear in
rather technical contexts and it seems that very little effort (if any) has been made in
order to place them in a more general framework.

The aim of this paper is to furnish some clarification, emphasizing one single aspect
of the appearance of "classical anomalies". In the interpretation here proposed "classical
anomalies", as previously defined, lay and can be detected in the interplay between the
lagrangian-versus-hamiltonian description of a dynamical system presenting a symmetry
of the action. Very basic examples are explicitly constructed and analyzed. No new result
will be here discussed, rather a re-interpretation of known results and techniques will be
given. Due to the mainly pedagogical character of the present note, a minimum level of
mathematical sophistication has been purposely kept throughout the following discussion.

The present work is so organized. In the next section a drastically sketchy and far
from complete resume of the history and importance of anomalies in quantum field theory
will be made. The result obtained by Gervais and Neveu [1] in analyzing the Liouville
theory will be mentioned. To my knowledge, they have been the first authors who have
noticed an anomalous effect in a classical system. To the phenomenon they observed it is
applicable the definition of "classical anomaly" as previously stated.

In the following section a general argument is given, suggesting the mechanism which
gives rise to what has been here named of *classical anomalies*. The remaining sections
are devoted to work out some specific examples with concrete models, by analyzing in
each specific case whether the symmetries of the actions are preserved or not by the Pois-
son brackets structure. All the given examples are worked out for two-dimensional field
theories. There is nothing special, however, about $2D$. The models below have been
chosen both for their relevance to applications and for their relative simplicity, useful for
illustrative purposes. In order of presentation, the following systems will be analyzed:
i) the free chiral fermion,
ii) the massless free boson,
iii) the Floreanini-Jackiw [2] chiral boson (FJ) model (introduced to complete the discus-
sion of free and chiral models),
iv) finally, the Liouville theory will be revisited in the light of the present interpretation.

In the conclusions some further remarks and comments will be made and possible

future lines of development will be suggested.

2   A bit of history

In the last thirty years the investigation of anomalies in quantum field theories has been one of the major areas of research attracting the attention of theoretical physicists. The reason is clear. An objective relevance for physical applications, coupled with a beautiful mathematical structure.

Indeed, the first discovered anomaly (by Adler and Bardeen-Jackiw [3, 4], named ABJ after them) has been the $U(1)$ chiral anomaly of gauge theories which puts a consistency constraint on the existence of a quantized gauge theory. Just to mention an application, the gauge group and the representation multiplets of the standard model for the electroweak interactions are carefully selected in order to guarantee an anomaly-free theory. Much in the same way, the critical dimensionality of a quantized string theory can be determined by requiring the cancellation of an anomaly [5] which can be associated to the Weyl invariance [6].

On the other hand, anomalies in physics are not always unwanted features to be wiped out. E.g. the trace anomalies associated to dilatation invariance enter the Callan-Symanzik equations [7]. For a reference work concerning anomalies in the context of quantum current algebras and their related physical applications one can consult [8].

Since their original discovery, anomalies have been regarded as a feature of quantization. Some folklore was put on this aspect. In the light of the Feynman path-integral approach, Fujikawa [9] developed a celebrated method which relies upon the fact that the functional measure is not always invariant under a symmetry of the classical action.

On the mathematical side, anomalies have been shown to satisfy consistency conditions induced by the (anomalous) Ward identities satisfied by the corresponding quantum field theories [10]. The (covariant) anomalies computed with the Fujikawa method do not,however, satisfy such conditions; nevertheless the relation between the two anomalies (consistent versus covariant) was made explicit in [11]. The reason for this discrepancy is that in the Fujikawa approach one regularizes the jacobian arising from a field transformation instead of the full partition function. A slightly modified version of the Fujikawa technique can be introduced, allowing the regularization of the full partition function. It turns out to be equivalent to the heat kernel technique.

The [10] consistency conditions for the anomalies allowed determining their expressions via purely algebraic methods which make use of the so-called "transgression formula" [12]. An elegant reformulation of such consistency conditions is given in terms of the BRST-cohomology [13]. To summarize, the *form* of the possible anomalies is determined by the possible existence of non-trivial cocycles for a BRST-cohomology associated to the symmetry under consideration. On the other hand, the *coefficients* of such anomalies can be related to the index theorems for elliptic operators via heat kernel computations [14] (it is required for the purpose a one-loop Euclidean version of the quantum field theory under investigation, regularized through zeta-functions [15]). A detailed account of the latter construction is given in [16].

Anomalous symmetries of quantized theories received, therefore, a nice mathematical

interpretation. As is already recalled, they have been regarded as genuine new features of quantization, not present in the underlining classical theories. This point of view is, by the way, still nowadays commonly shared by most of the researchers in the area and popularized in standard textbooks.

As far as I know, Gervais and Neveu were the first authors (in [1], pages 131/2) who observed in a purely classical theory a new phenomenon that, with some good reasons, deserves to be named "classical anomalies" and for which the definition here introduced is applicable. The authors of [1] analyzed the Liouville theory appearing in the partition function of non-critical strings according to [6]. They showed that, even for the *classical* Liouville theory, the generators of (one chiral sector of) the conformal transformations satisfy, under classical Poisson brackets, the Virasoro algebra, i.e., the central extension of the Witt algebra. This remark contained in [1] was not later developed (e.g, the Nöther charges were not explicitly mentioned), since the main focus of their authors was on the quantum version of the Liouville theory (see also their related works [17]). Needless to say, most of the papers written by theoretical physicists on the Liouville field theory deal with the quantum version of the model, as a simple key-word inspection of the electronic bulletin boards can show.

Subsequent works such as [18] and [19] on classical Liouville and Toda-field systems, were mostly concerned with the integrability properties of such models, like the presence of classical Sklyanin r-matrices in their Drinfeld-Sokolov exchange-algebras. No connection of such classical Poisson-brackets structures with the symmetries of the action, even if implicit, was explicitly stated.

3 General considerations on classical anomalies

The class of systems under consideration here consists of the classical dynamical systems which admit both a lagrangian and a hamiltonian description. It will be further assumed that the action $\mathcal{S}$ admits an invariance under a group of symmetries which can be continuous (Lie), infinite-dimensional and/or super. According to the Nöther theorem, in the lagrangian framework, the conserved charges are associated to each generator of the corresponding Lie algebra symmetry. As far as the lagrangian description alone is concerned, no further property can be stated concerning the conserved charges. This situation drastically changes when the hamiltonian formalism is taken into account. The phase space of the theory possesses an algebraic structure given by the Poisson brackets. The existence of such a structure makes, therefore, possible to compute the Poisson bracket between any two given conserved charges. In the standard situation, the Poisson brackets among conserved charges realize a closed algebraic structure which is isomorphic to the original Lie algebra of the symmetries of the action. It turns out, however, as it will be illustrated in the examples which follow, that this is not always the case. Indeed it is logically possible (and concretely realized in some specific cases) that the Poisson brackets among the Nöther charges would correspond to a centrally extended version of the original symmetry algebra. In this particular case we can describe such a situation by stating that the classical system possesses an anomalously realized symmetry or, in short, a "classical anomaly".

The possibility for this phenomenon to occur is based on very simple and nice mathe-

matical consistency conditions, implemented by the Jacobi-identity property of the given symmetry algebra. Let us illustrate this point by considering some generic (but not the most general) scheme. Let us suppose that the (bosonic) generators δ_a's of a symmetry invariance of the action satisfy a linear algebra whose structure constants satisfy the Jacobi identity, i.e.

$$[\delta_a, \delta_b] \quad = \quad f_{ab}{}^c \delta_c, \tag{1}$$

while

$$[\delta_a, [\delta_b, \delta_c]] + [\delta_b, [\delta_c, \delta_a]] + [\delta_c, [\delta_a, \delta_b]] = 0 \quad . \tag{2}$$

The associated Nöther charges Q_a's are further assumed to be the generators of the algebra, i.e., applied on a given field ϕ they produce

$$\delta_a \phi \quad = \quad \{Q_a, \phi\}, \tag{3}$$

where the brackets obviously denote the Poisson-brackets.

The condition

$$[\delta_a, \delta_b]\phi \quad = \quad f_{ab}{}^c \delta_c \phi, \tag{4}$$

puts restriction on the possible Poisson brackets algebra satisfied by the Nöther charges. It is certainly true that

$$\{Q_a, Q_b\} \quad = \quad f_{ab}{}^c Q_c, \tag{5}$$

(which corresponds to the standard case) is consistent with both (3) and (4). However, in a generic case, it is not at all a necessary condition since more general solutions can be found. Indeed, the presence of a central extension, expressed through the relation

$$\{Q_a, Q_b\} \quad = \quad f_{ab}{}^c Q_c + k * \Delta_{ab}, \tag{6}$$

(where k is a central charge and the function Δ_{ab} is antisymmetric in the exchange of a and b), is allowed.

Indeed, since the relation

$$\{Q_a, \{Q_b, \phi\}\} - \{Q_b, \{Q_a, \phi\}\} \quad = \quad \{\{Q_a, Q_b\}, \phi\} \tag{7}$$

holds due to the Jacobi property of the Poisson bracket structure (which is assumed to be satisfied), no contradiction can be found with (4); the right hand side of (7) in fact is given by

$$\{f_{ab}{}^c Q_c + k * \Delta_{ab}, \phi\} = \{f_{ab}{}^c Q_c, \phi\} = f_{ab}{}^c \delta_c \phi \quad , \tag{8}$$

due to the fact that k is a central term and has vanishing Poisson brackets with any field.

This observation on one hand puts restrictions on the possible symmetries for which a classical anomaly can be detected; the symmetries in question, on a purely algebraic ground, must admit a central extension. This is not the case, e.g., for the Lie groups

of symmetry based on finite simple Lie algebras. On the other hand, it is a warning that, whenever a symmetry *does* admit an algebraically consistent central extension, it should be carefully checked, for any specific dynamical model which concretely realizes it, whether it is satisfied exactly or anomalously. This remark already holds at the classical level, not just for purely quantum theories.

Some further points deserve to be mentioned. The first one concerns the fact that the quantization procedure (which, for the cases we are here considering, can be understood as an explicit realization of an abstract Poisson brackets algebra as an algebra of commutators between operators acting on a given Hilbert space) can induce anomalous terms for theories which, in their classical version, are not anomalous in the sense previously specified. It therefore turns out that the occurrence of classical anomalies is a phenomenon which is "more difficult to observe" than the corresponding appearance of quantum anomalies since it occurs more seldom.

A second point concerns the fact that the Poisson brackets algebra, as an abstract algebra, is assumed to satisfy the Leibniz property. This is no longer the case for its concrete realization given by the algebra of commutators. The Nöther charges are in general non-linearly constructed with the original fields ϕ_i (which collectively denote the basic fields and their conjugate momenta) of a given theory. For such a reason it is only true in the classical case that, whenever an anomalous central charge in an infinitesimal linear algebra of symmetries is detected, it can be normalized at will by a simultaneous rescaling of all the fields ϕ_i involved ($\phi_i \mapsto \alpha \cdot \phi_i$) and of the Poisson brackets normalization ($\{.,.\} \mapsto \frac{1}{\alpha}\{.,.\}$), for an arbitrary real constant α. In the classical case any central charge different from zero can therefore be consistently set equal to 1. On the contrary, in the quantum case a specific value of the central charge is fixed by the type of representation of the symmetry algebra associated with the given model and is a genuine physical parameter (the role of the Virasoro central charge in labeling the conformal minimal models is an example). The above argument is not, however, (at least directly) applicable to non-linear symmetries, such as those leading to the classical counterparts of the Fateev-Zamolodchikov W-algebras. Classical non-linear symmetries are out of the scope of the present paper and deserve to be analyzed separately.

It is worth mentioning that in a different context, the appearance of centrally extended algebras has been studied in [20] (and references therein). This analysis is referred to purely lagrangian systems; the object of the investigation is whether the symmetries of a given action are realized exactly on the lagrangian or only up to boundary terms. It is therefore quite a different approach and kind of investigation from the one proposed here.

Furthermore, let me remark that the presence of a centrally extended algebra of classical symmetries is not always a sign of the presence of an anomaly (at least not in the sense here specified). In [21] it was shown that a classical two-dimensional complex bosonic field, coupled to an external constant electromagnetic field, admits a symmetry corresponding to the central extension of the two-dimensional Poincaré algebra. This model is not anomalous, within the definition here proposed, because, due to the presence of the constant external field, the symmetry algebra of the classical action itself is centrally extended and not given by the ordinary $2D$ Poincaré algebra.

Finally, in the present work no effort is made to derive the hamiltonian dynamics associated to a given lagrangian. It is simply assumed to exist, based on the results

furnished in the literature. This is especially true for the chiral boson model in section 6, whose hamiltonian analysis is somewhat delicate, but has nevertheless been performed in [22], see also [23].

4   The free chiral fermion

The first example that will be discussed here concerns the theory of the free chiral fermion. It is described by the Grassmann field $\psi(x,t)$, where x is a one-dimensional space coordinate and t the time. The dynamics is specified by the action

$$\mathcal{S} \;=\; \int dx dt \cdot \psi \partial_- \psi, \tag{9}$$

where

$$z_\pm = x \pm t$$

and

$$\partial_\pm = \frac{1}{2}(\partial_x \pm \partial_t).$$

For our purposes we will assume the space-coordinate x to be compactified on a circle S^1 of radius R and ψ to satisfy periodic boundary conditions.

The equation of motion is given by

$$\partial_- \psi \;=\; 0. \tag{10}$$

The action $\mathcal{S}$, besides being off-shell invariant, as an easy computation shows, under the infinitesimal transformation

$$\delta_\epsilon(\psi) \;=\; \epsilon(z_+)\partial_x \psi + \frac{1}{2}(\partial_x \epsilon(z_+))\psi, \tag{11}$$

admits a global fermionic symmetry given by

$$\delta_\kappa(\psi) \;=\; \kappa, \tag{12}$$

where κ is a global fermionic parameter.

It is convenient to Laurent-expand $\epsilon(z_+)$ according to

$$\epsilon(z_+) \;=\; -\sum_n \epsilon_n (z_+)^{n+1}, \tag{13}$$

The two above symmetries can be expressed through

$$\delta_\epsilon(\psi) \;=\; \sum_n \epsilon_n \cdot l_n \psi,$$
$$\delta_\kappa(\psi) \;=\; \kappa \cdot g \psi, \tag{14}$$

where the operators l_n and g are respectively given by

$$l_n \;=\; -(z_+)^{n+1}\partial_x - \frac{1}{2}(n+1)(z_+)^n,$$

$$g \;=\; \oint dx \cdot \frac{\delta}{\delta\psi(x)}. \tag{15}$$

While the commutators among the l_n's operators realize the Witt algebra

$$[l_n, l_m] \;=\; (n-m)l_{n+m}, \tag{16}$$

the anticommutator of g with itself satisfies

$$[g, g]_+ \;=\; 0, \tag{17}$$

so that g is nilpotent.

The conserved Nöther charges associated to the above symmetries are given by

$$L_n \;=\; -\oint dx (z_+)^{n+1}\psi\partial_x\psi,$$

$$G \;=\; 2\oint dx\,\psi(x,t). \tag{18}$$

In the hamiltonian description the equation of motion is expressed through

$$\dot{\psi} \;=\; \{H, \psi\}_t, \tag{19}$$

where H is the hamiltonian

$$H \;=\; -\oint dx \cdot (\psi\partial_x\psi), \tag{20}$$

while the equal-time Poisson brackets $\{.,.\}_t$ are introduced through

$$\{\psi(x), \psi(y)\}_t \;=\; \frac{1}{2}\delta(x-y). \tag{21}$$

We are now in the position to compute the Poisson brackets among the Nöther conserved charges, which are the generators of the symmetries, according to

$$l_n\psi \;=\; \{L_n, \psi\}_t,$$

$$g\psi \;=\; \{G, \psi\}_t. \tag{22}$$

While the Nöther generators L_n associated with the Witt generators reproduce, under the Poisson brackets structure, the Witt algebra, i.e.

$$\{L_n, L_m\} \;=\; (n-m)L_{n+m}, \tag{23}$$

it is no longer true that the fermionic conserved charge G satisfies the same nilpotency condition as g. Indeed we have that the Poisson bracket of G with itself produces a central element, given by

$$\{G, G\} = 4\pi R. \tag{24}$$

It follows that the generator of the fermionic symmetry is now no longer nilpotent. Even in this trivial free model the presence of a symmetry which presents a classical anomaly can be detected.

In the quantum case, due to the double contractions in the Wick expansion, the quantum analogs of the L_n generators satisfy the centrally extended version of the Witt algebra, i.e. the Virasoro algebra, with central charge $c = \frac{1}{2}$. This is in accordance with the statement that the "quantization" is a more effective way to produce anomalies than the plain introduction of a classical Poisson bracket structure. Still, as the fermionic symmetry shows, in many cases it is sufficient the introduction of a classical Poisson bracket structure to induce anomalies at the level of the Nöther charges.

5   The free massless boson in $2D$

The next example that we are discussing concerns the 2-dimensional free massless boson model, described by the following action

$$\mathcal{S} = -2 \int dx dt \cdot \partial_- \phi \partial_+ \phi. \tag{25}$$

The field $\phi(x,t)$ satisfies the free equation of motion

$$\Box \phi \equiv 4 \partial_- \partial_+ \phi = 0 \tag{26}$$

This system admits an (anomalous-free) two-dimensional conformal invariance which corresponds to the direct sum of two copies of the Witt algebra ($Witt \oplus Witt$). Actually the symmetry algebra of the system is richer. Indeed, the following transformations are symmetries of the action

$$
\begin{aligned}
\delta_+ \phi &= \epsilon(z_+) \partial_+ \phi + \mu(z_+), \\
\delta_- \phi &= \overline{\epsilon}(z_-) \partial_- \phi + \overline{\mu}(z_-),
\end{aligned}
\tag{27}
$$

for arbitrary infinitesimal functions $\epsilon(z_+), \overline{\epsilon}(z_-), \mu(z_+), \overline{\mu}(z_-)$. Such set of transformations is anomalous in the sense here discussed. This point can be easily understood when we specialize $\mu(z_+)$ ($\overline{\mu}(z_-)$) to be given by

$$
\begin{aligned}
\mu(z_+) &= \lambda_+ \partial_+ \epsilon(z_+), \\
\overline{\mu}(z_-) &= \lambda_- \partial_- \overline{\epsilon}(z_-).
\end{aligned}
\tag{28}
$$

for arbitrary fixed values of the parameters $\lambda_\pm$.

In full analogy with the previous case, after Laurent series expansion for $\epsilon(z_+), \overline{\epsilon}(z_-)$,

$$
\begin{aligned}
\epsilon(z_+) &= -\sum_n \epsilon_n (z_+)^{n+1}, \\
\overline{\epsilon}(z_-) &= -\sum_n \overline{\epsilon}_n (z_-)^{n+1},
\end{aligned}
\tag{29}
$$

we obtain two mutually commuting set of λ_+ and λ_--dependent symmetry generators, each set generating a copy of the Witt algebra. They are given by

$$l_n(\lambda_+) = -(z_+)^{n+1} \partial_+ - \lambda_+ (n+1)(z_+)^n \cdot \int \frac{\delta}{\delta \phi(x,t)}$$

$$\bar{l}_n(\lambda_-) = -(z_-)^{n+1}\partial_- - \lambda_-(n+1)(z_-)^n \cdot \int \frac{\delta}{\delta\phi(x,t)} \tag{30}$$

For any given couple of values $\lambda_\pm$, we obtain the closure of the $Witt \oplus Witt$ algebra

$$
\begin{aligned}
[l_n(\lambda_+), l_m(\lambda_+)] &= (n-m)l_{n+m}(\lambda_+), \\
[\bar{l}_n(\lambda_-), \bar{l}_m(\lambda_-)] &= (n-m)\bar{l}_{n+m}(\lambda_-), \\
[l_n(\lambda_+), \bar{l}_m(\lambda_-)] &= 0.
\end{aligned}
\tag{31}
$$

The free massless boson model admits a hamiltonian formulation, with the hamiltonian given by

$$H = \frac{1}{2}\int dx \left(\pi^2 + (\partial_x\phi)^2\right). \tag{32}$$

The equations of motions, expressed through

$$\frac{d}{dt}f = \{H,f\} + \frac{\partial}{\partial t}f \tag{33}$$

imply

$$
\begin{aligned}
\dot{\phi} &= \pi, \\
\dot{\pi} &= (\partial_x)^2\phi.
\end{aligned}
\tag{34}
$$

The equal-time Poisson brackets are obviously given by

$$\{\pi(x), \phi(y)\} = \delta(x-y) \tag{35}$$

and vanishing otherwise.

As a straightforward computation shows, the Nöther conserved charges $L_n(\lambda_+), \overline{L}_n(\lambda_-)$, associated to the symmetry generators $l_n(\lambda_+), \bar{l}_n(\lambda_-)$ respectively, are recovered from the Laurent expansions

$$
\begin{aligned}
L_n(\lambda_+) &= \int dx (z_+)^{n+1} \cdot T, \\
\overline{L}_n(\lambda_-) &= \int dx (z_-)^{n+1} \cdot \overline{T},
\end{aligned}
\tag{36}
$$

where $T, \overline{T}$ are given by

$$
\begin{aligned}
T &= \frac{1}{4}(\pi^2 + (\partial_x\phi)^2 + 2\pi\partial_x\phi - 4\lambda_+\partial_x^2\phi - 4\lambda_+\partial_x\pi), \\
\overline{T} &= -\frac{1}{4}(\pi^2 + (\partial_x\phi)^2 - 2\pi\partial_x\phi - 4\lambda_-\partial_x^2\phi + 4\lambda_-\partial_x\pi).
\end{aligned}
\tag{37}
$$

The conservation law for $L_n, \overline{L}_n$ is a consequence of the (anti-)chiral equations satisfied by T ($\overline{T}$) respectively, i.e.

$$
\begin{aligned}
\partial_- T &= 0, \\
\partial_+ \overline{T} &= 0.
\end{aligned}
\tag{38}
$$

$L_n, \overline{L}_n$ are the generators of the $l_n, \overline{l}_n$ transformations since

$$
\begin{aligned}
l_n \phi &= \{L_n, \phi\}, \\
\overline{l}_n \phi &= \{\overline{L}_n, \phi\}.
\end{aligned}
\tag{39}
$$

$L_n, \overline{L}_n$ generate the direct sum of two copies of the Virasoro algebra, $Vir \oplus Vir$, as it can be directly read from the equal-time Poisson brackets between $T(x), \overline{T}(x)$, namely

$$
\begin{aligned}
\{T(x), T(y)\} &= -2\lambda_+{}^2 \partial_y{}^3 \delta(x-y) + 2T(y)\partial_y \delta(x-y) + \partial_y T(y) \cdot \delta(x-y), \\
\{T(x), \overline{T}(y)\} &= 0, \\
\{\overline{T}(x), \overline{T}(y)\} &= 2\lambda_-{}^2 \partial_y{}^3 \delta(x-y) + 2\overline{T}(y)\partial_y \delta(x-y) + \partial_y \overline{T}(y) \cdot \delta(x-y).
\end{aligned}
\tag{40}
$$

For given values of $\lambda_\pm \neq 0$, central terms are produced which are proportional to $\lambda_\pm{}^2$. The corresponding transformations can therefore be regarded as anomalous.

The two-dimensional conformal symmetry itself however is not anomalous in this free case, since for the choice $\lambda_\pm = 0$, the symmetry is preserved at the Poisson bracket level.

It should be stressed the fact that the freedom in choosing inhomogeneous transformations acting on ϕ, for $\lambda_\pm \neq 0$, can be held as responsible for the preservation (i.e. not anomalous realization) of the two-dimensional conformal invariance even in the quantum case. The choice $\lambda_\pm \neq 0$ corresponds to the introduction of the Feigin-Fuchs term in the Coulomb gas formalism.

6 The Floreanini-Jackiw chiral boson model

For completeness, let us discuss the last chiral and free model, namely the Floreanini-Jackiw chiral boson model [2] introduced through the lagrangian

$$
\mathcal{L} = \partial_t \phi \partial_x \phi - (\partial_x \phi)^2,
\tag{41}
$$

which leads to the equations of motion

$$
\partial_x \partial_- \phi = 0.
\tag{42}
$$

Despite the fact that it is not manifestly Lorentz-invariant, nevertheless it can be proven to be Poincaré invariant in 2 dimensions. This model defines the dynamics of a chiral boson. The treatment is much in the same lines as the free boson model with a notable exception. Since we are in presence of a chiral dynamics the invariance of the model is given by a single (chiral) copy of the Witt algebra and its central extension. A class of λ-dependent infinitesimal symmetries of the above action is given by

$$
\delta_\lambda \phi = \epsilon(z_+) \partial_x \phi + \lambda \partial_x \epsilon(z_+).
\tag{43}
$$

The corresponding Nöther conserved charges are given by the following expressions

$$
L_n(\lambda) = \frac{1}{2} \int dx (z_+)^{n+1} \left((\partial_x \phi)^2 + \lambda \partial_x{}^2 \phi \right).
\tag{44}
$$

The hamiltonian of the system is

$$
H = \frac{1}{2} \int dx (\partial_x \phi)^2,
\tag{45}
$$

while the Poisson-brackets structure system is non-local

$$\{\phi(x), \phi(y)\} \;\; = \;\; \partial_y^{-1} \delta(x - y). \tag{46}$$

Despite its non-locality however, since in (44) only the derivatives of the field ϕ enter, the algebra satisfied by the $L_n(\lambda)$ Nöther charges is a local algebra which, as in the previous example, corresponds to the Virasoro algebra with central extension proportional to λ^2. For $\lambda \neq 0$ we are in presence of an anomaly induced by the Poisson bracket structure. The Nöther charges L_n are, as in the previous example, the generators of the (43) transformations.

7   The Liouville theory revisited

The last model that we are going to discuss is the Liouville theory, which will be here revisited in view of the considerations which motivated the present paper.

The action of the Liouville model can be written as

$$\mathcal{S} \;\; = \;\; -\int dx dt \cdot (2\partial_-\phi\partial_+\phi + e^{2\phi}). \tag{47}$$

The equations of motion are

$$2\partial_-\partial_+\phi \;\; = \;\; e^{2\phi}. \tag{48}$$

In the hamiltonian description the hamiltonian is given by

$$H \;\; = \;\; \oint dx \cdot (\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\phi)^2 + e^{2\phi}), \tag{49}$$

while the Poisson-bracket structure between π, ϕ is the same as in the free case

$$\{\pi(x), \phi(y)\} \;\; = \;\; \delta(x - y). \tag{50}$$

We obtain

$$\dot{\phi} \;\; = \;\; \pi,$$
$$\dot{\pi} \;\; = \;\; \partial_x{}^2\phi - 2e^{2\phi}. \tag{51}$$

The theory is conformally invariant, with transformations given, just like the free-case, by the infinitesimal transformations

$$\delta_+\phi \;\; = \;\; \epsilon(z_+)\partial_x\phi + \lambda_+(\partial_x\epsilon(z_+))\phi,$$
$$\delta_-\phi \;\; = \;\; \bar{\epsilon}(z_-)\partial_x\phi + \lambda_-(\partial_x\bar{\epsilon}(z_-))\phi. \tag{52}$$

However, due to the presence of the potential term, the action is no longer off-shell invariant for arbitrary values of $\lambda_\pm$. The invariance is indeed satisfied only for

$$\lambda_+ = \lambda_- = \tfrac{1}{2} \quad . \tag{53}$$

There is no longer a whole class of $\lambda_\pm$-dependent symmetry transformations, but just a given, point-like in the $\lambda_\pm$ parametric space, set of symmetry transformations. In this

particular case, the analysis plainly follows the one conducted for the free massless boson. The conserved Nöther charges $L_n, \overline{L}_n$ can be introduced through the Laurent-expansion of

$$\int dx \epsilon(z_+) T(x) \quad , \quad \int dx \overline{\epsilon}(z_-) \overline{T}(x). \tag{54}$$

They are conserved provided that

$$\partial_- T = \partial_+ \overline{T} = 0 \quad . \tag{55}$$

$T, \overline{T}$ can be unambiguously fixed to be given by

$$\begin{aligned} T &= \frac{1}{4}(\pi^2 + (\partial_x \phi)^2 + 2\pi \partial_x \phi + e^{2\phi} - 2{\partial_x}^2 \phi - 2\partial_x \pi), \\ \overline{T} &= -\frac{1}{4}(\pi^2 + (\partial_x \phi)^2 - 2\pi \partial_x \phi + e^{2\phi} - 2{\partial_x}^2 \phi + 2\partial_x \pi). \end{aligned} \tag{56}$$

The two sets of Nöther charges, $L_n, \overline{L}_n$, generate the symmetry transformation of the field ϕ according to (39).

   Their Poisson-bracket algebra however is anomalous and coincides with $Vir \oplus Vir$, with fixed values of the two central charges $c_\pm$ given by $c_\pm = \mp 6$, for the given normalization of the field ϕ and of the action,

$$\begin{aligned} \{T(x), T(y)\} &= -\frac{1}{2}{\partial_y}^3 \delta(x-y) + 2T(y)\partial_y \delta(x-y) + \partial_y T(y) \cdot \delta(x-y), \\ \{T(x), \overline{T}(y)\} &= 0, \\ \{\overline{T}(x), \overline{T}(y)\} &= \frac{1}{2}{\lambda_-}^2 {\partial_y}^3 \delta(x-y) + 2\overline{T}(y)\partial_y \delta(x-y) + \partial_y \overline{T}(y) \cdot \delta(x-y). \end{aligned} \tag{57}$$

(I recall that, by definition, the central charge is normalized to be the coefficient in front of the inhomogeneous term δ''' normalized by a factor 12). The conformal invariance of the $2D$ Liouville theory is classically anomalous, satisfying the definition proposed here. It is in contrast with the free massless boson model, where the symmetry can be restored both at the classical and quantum level, as well as the free chiral fermion theory. In that case the chiral (i.e. Witt) invariance is classically preserved, while it is quantum violated for a fixed value of the central charge ($c = 1/2$). The Liouville theory, on the other hand, admits a non-vanishing classical central charge. Its normalization is meaningless in the classical case, since it can always be reabsorbed through a simultaneous rescaling of the fields and of the Poisson brackets, as already recalled. Nevertheless, in the quantum theory, the effect of such "freedom of rescaling" of the underlining classical theory, can be seen in the arbitrariness of the Liouville quantum central charge, which is not fixed by the theory, apart the restriction coming from unitarity requirement. This is in sharp contrast to the free chiral theory, where such a freedom is not allowed.

8   Conclusions

   In the present work I have stressed the fact that features which correspond to an anomalous realization of a symmetry can be present even in purely classical dynamical systems.

I introduced the definition of "classical anomaly" to describe the situation of a classical system whose conserved Nöther charges, which are associated to a symmetry of the action, admit a Poisson brackets algebra, induced by the hamiltonian dynamics, which is only isomorphic to a centrally extended version of the original symmetry algebra.

The underlining mathematical reason which makes possible the realization of such a case has been illustrated in section 3. It has been shown there that eventual central extensions of the original symmetry realized by the Poisson brackets algebra of the Nöther charges satisfy the compatibility constraint required by the simultaneous validity of (3) and (4). Later, a list of simple models which show the concrete application of this mechanism and the appearance of anomalies have been given.

It is certainly true that the quantization is still a "preferred" mechanism to produce anomalies, confront the discussion in the text and the specific example, e.g., of the chiral fermion theory of section 4, which is quantum anomalous under 1-dimensional diffeomorphisms, but satisfies the ordinary Witt algebra for what concerns classical Poisson brackets. It turns out, however, that in general and in many cases of interest, it is not necessary to perform the quantization of a dynamical system in order to induce anomalies. In some cases the introduction of a classical Poisson brackets structure is sufficient for the purpose. The anomalous nilpotent fermionic symmetry of the free chiral fermion of section 4 is perhaps the simplest example, as well as the anomalous conformal symmetry of the classical Liouville equation analyzed in section 7.

More than that, any symmetry which algebraically admits a central extension, is potentially anomalous. The investigation of the Poisson brackets algebra of its Nöther charges realized on specific models can lead to non-trivial results even for classical dynamics.

Specific differences with respect to the "quantum anomalies" have been pointed out throughout the text. At least for the case of symmetries associated to linear algebras (in the present analysis no effort was put in including infinitesimal symmetries of non-linear W-type), the Leibniz rule observed by classical Poisson brackets allow, through the simultaneous rescaling of the fields and the Poisson brackets, to freely normalize the value of the central charge, which can be conveniently chosen.

The examples chosen and the techniques employed in the present work are elementary. The main motivation of this paper is to illustrate, in the simplest possible contexts, the mathematical framework, deep and simple at the same time, behind the appearance of anomalies in classical dynamical systems.

The techniques which are usually encountered in the literature and which appear in disguised form in the analysis of the examples here illustrated (one for all, the introduction of the Feigin-Fuchs term in the Coulomb gas approach to "shift" the value of the quantum central charge), often appear to a layman reader just a set of ad hoc prescriptions to perform technical computations. While it is certainly true that they are technically very helpful, the deep symmetry principles which make them possible are somehow hidden. To place them in the proper context of lagrangian and hamiltonian dynamics is the main issue of the present paper.

The analysis here conducted suggests many possible lines of development. On a purely mathematical ground one can ask which kind of centrally extended algebras can find a dynamical interpretation as (anomalous) symmetry for some given dynamical system.

D. Leites [24] has recently proposed several new ways to superize the Liouville equation

which can be investigated in the light of the present considerations.

On the other hand, the interplay between lagrangian and hamiltonian methods seems quite fruitful. It seems likely and currently under investigation that, by employing superspace techniques, the embedding of certain classes of hamiltonian solitonic equations in some superized system, which admits also a (super-)lagrangian description, could be given. Nice and neat results concerning the symmetry algebra of these systems should be derivable. Needless to say, the presence of central charges in the Virasoro subalgebra is mandatory for any integrable system which contains KdV as its consistent reduction.

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