CBPF-NF-029/99

A Convergence Theorem for Asymptotic Expansions of Feynman Amplitudes

A.P.C. Malbouisson

Centro Brasileiro de Pesquisas Físicas – CBPF

Rua Dr.Xavier Sigaud 150

Rio de Janeiro, RJ 22290-180 Brazil

E-mail: adolfo@lafex.cbpf.br

Abstract

The Mellin representation of Feynman integrals is revisited. From this representation, an asymptotic expansion for generic Feynman amplitudes, for any set of invariants going to zero or to ∞, may be obtained. In the case of all masses going to zero in Euclidean metric, we show that the truncated expansion has a rest compatible with convergence of the series.

Key-words: Infrared divergences; Asymptotic behaviours.

1 Introduction

Infrared divergences may be seen as a special case of a general class of asymptotic behaviours of Feynman amplitudes in a field theory, as some of the involved masses tend to zero. Actually these divergences appear at different levels. For Green functions in Minkowskian metric it has been shown a long time ago that for some theories (e.g. QED) Green's functions exist at the zero-mass limit for some particles, as distributions on the 4-momenta, i.e., Green's functions are well defined quantities in the infrared limit. [1, 2]. For particles on mass shell Green's functions generally does not have a limit for those theories, even if they are well defined off mass shell Green's functions. The oldest and best known example is infrared divergences in scattering amplitudes in QED. This problem has been investigated exhaustivelly (classical papers on the subject are in refs.([3, 4])), since the celebrated work of Bloch and Nordsieck [5]. These investigations have been done using essentially power counting in momentum space. In this way it has been possible to show that suitable quantities, probabilities densities or probabilities amplitudes in which the effect of radiated soft photons are taken into account, are finite at the zero photon mass limit, due to compensations between infrared divergences from soft photons and from radiactive corrections. This result holds order by order in perturbation theory.

Another class of problems arise at the Green's functions level in Euclidean metric, when besides the zero-mass limit we take also vanishingly small values for the external momenta. In this case, we speak of the infrared behaviour of correlation functions. These divergences, which are seen as a "pathological" behaviour in the context of the aplications of field theories to particle physics, are associated with the large distance correlations in statistical systems and play a crucial role in the study of critical phenomena and phase transitions (a complete presentation of the use of field theory in describing critical phenomena is done in [6]).

In this note, we study the asymptotic behaviour of Feynman amplitudes in Euclidean metric. We make use of Mellin transform techniques to represent Feynman integrals, along similar lines as it has been done to study renormalization and asymptotic behaviours of scattering amplitudes in refs. [7, 8, 9] , and to study the heat kernel expansion as in ref.

[10] To fix our framework we consider a theory involving scalar fields $\varphi_i(x)$ having masses m_i, defined on a Euclidean space. For simplicity we may think of a single scalar field $\varphi(x)$ having a mass m. A generic Feynman graph G is a set of I internal lines, L loops, q connected components (a graph is disconnected if $q > 1$) and n vertices linked by some (polynomial) potential. To each vertex are attibuted external momenta $\{p_i\}$ and internal ones $\{k_a\}$. A subgraph $S \subset G$ is a graph such that all the lines vertices and loops belong to G and a quotient graph $\frac{G}{S}$ is a graph obtained from G reducing S to a point.

2 The Mellin Representation of Feynman Integrals and Asymptotic Expansions

The Feynman amplitude $G(\{a_k\})$ corresponding to G is is a function of the set of invariants $\{a_k\}$ built from external momenta $\sum p^2$ and squared masses m_i^2; it is defined in the Schwinger-Bogoliubov representation by, (see for instance refs. [1, 2])

$$G(a_k) = \int_0^\infty \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}}(\alpha) e^{-\frac{V(\alpha)}{U(\alpha)}}, \tag{1}$$

where D is the space dimension with positive metric.

In the above formula, the Symanzik polynomials $U(\alpha)$ and $W(\alpha)$ are constructed from the graph G by the prescription,

$$U(\alpha) = \sum_{1.T} \prod_{i \notin 1.T} \alpha_i \tag{2}$$

and

$$V(\alpha) = \sum_{2.T} (\sum p_j)^2 \prod_{i \notin 2.T} + (\sum_{j \in G} m_j^2 \alpha_j) U(\alpha) \tag{3}$$

where the simbols $\sum_{1.T}$ and $\sum_{2.T}$ means respectively summation over the trees and two-trees (disconnected trees having two connected components) of G passing by all the vertices. The sum $\sum p_j$ is the total external momentum entering one of the two-tree connected components. Notice that $U(\alpha)$ and $W(\alpha)$ are homogeneous polynomials in the α-variables, of degrees L and $L + 1$ respectively.

In the following we have in mind as a physical situation, the infared behaviour, , but we would like to emphasize that our study is quite general, in the sense that it applies to any asymptotic limit in Euclidean metric (any choice of the subset a_l below), for arbitrarily given external momenta, generic or exceptional, and for arbitrary vanishing or finite masses. If we perform a scale transformation on the subset $\{a_l\}$ of invariants, $a_l \to \lambda a_l$, the polynomial V is split into two parts,

$$V(\lambda a_m) = \lambda W(a_l, \alpha) + R(a_q, \alpha) \tag{4}$$

where the polynomials $W(a_l, \alpha)$ and $R(a_q, \alpha)$ are also homogeneous of degree $L+1$ in the α-variables.

To be concrete we consider here, a special situation with the external momenta $\{p\}$ fixed and we investigate the limit $\lambda \to 0$ corresponding to vanishing masses. In this case W is just the second term in Equ. (3). As we have noted above, the method applies along the same lines to any other class of asymptotic behaviour. Incidentally we note that from a dimensional argument,

$$G(\frac{a_l}{\lambda}, a_q) = \lambda^{\omega} G(a_l, \lambda a_q), \tag{5}$$

the study of a given subset going to zero is equivalent to study the $\lambda \to \infty$ limit on the complementary subset of invariants.

Under the λ-scaling performed in Equ.(4) G becomes a function of λ, $G(\lambda)$, and its Mellin transform, $M(x) = \int_0^\infty d\lambda \lambda^{-x-1} F(\lambda)$ may be written in the form,

$$M(z) = \Gamma(-z) \int_0^\infty \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}} e^{-\frac{R}{U}} (\frac{W}{U})^z. \tag{6}$$

The scaled amplitute associated to the Feynman graph G, $G(\lambda)$, may be obtained by the inverse Mellin transform,

$$G(\lambda) = \frac{1}{2i\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} dx \lambda^z M(z) \tag{7}$$

where $\sigma = Re(z) < 0$ belongs to the analyticity domain of $M(z)$.

Since the integrand of Equ.(7) vanishes exponentially at $\sigma \pm i\infty$ due to the behaviour of $\Gamma(z)$ at large values of Imz, the integration contour may be displaced to the right by

Cauchy's theorem, picking up successivelly the poles of the integrand, provided we can desingularize the integral in Equ.(6). Such a problem has been studied by an appropriate choice of local coordinates in [11] and also in [7] using Hepp sectors and a Multiple Mellin representation. In these works it has been possible to show that the meromorphic structure of $M(z)$ has the form,

$$M(z) = \sum_{n,q} \frac{A_{nq} q!}{(z-n)^{q+1}}. \tag{8}$$

It results from the displacement of the integration contour in the inverse Mellin transform, an expansion for small values of λ, of the form,

$$G(\lambda) = \sum_{n=n_0}^{N} \lambda^n \sum_{q=0}^{q_{max}(n)} A_{nq} ln^q(\lambda) + R_N(\lambda) \tag{9}$$

where the coefficients $A_n(\{p\})$ and the powers of logarithms come from the residues at the poles $z = n$.

The rest of the expansion $R_N(\lambda)$ is given by

$$R_N(\lambda) = \int_{-\infty}^{+\infty} \frac{d(Imz)}{2i\pi} \lambda^z \Gamma(-z) F(z), \tag{10}$$

with

$$N < Re(z) < N+1, Re(z) = N + \eta, 0 < \eta < 1 \tag{11}$$

and where,

$$F(z) = \int_0^{\infty} \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}} e^{-\frac{R}{U}} (\frac{W}{U})^z \tag{12}$$

Remark: Hepp sectors, UV divergent graphs, renormalization

To perform the α-integrations in the expressions above we can divide the α-domain of integration into $I!$ Hepp sectors by ordering the α's in all possible ways. To a given permutation $h = (i_1, i_2, ..., i_I)$ corresponds a sector,

$$\alpha_{i_1} \leq ... \leq \alpha_{i_I} \tag{13}$$

with corresponding sector variables,

$$\alpha_{i_I} = \beta_I, \alpha_{i_{I-1}} = \beta_I \beta_{I-1}, ..., \alpha_{i_1} = \beta_I \beta_{I-1} ... \beta_2 \beta_1 \tag{14}$$

Since the integrand in $M(x)$ has homogeneity properties, the integration over β_I can be made explicitly and the remaining integration domain is $0 \leq \beta_i \leq 1, i = 1, ...I-1$. Although it is not allways the case in Minkowskian metric as explained in [8], in Euclidean metric the Symanzik polynomials are factorized in each Hepp sector, which allows the use of simple Mellin transforms as is done in this paper. In more general situations a splitting of the polynomial V into factorizable parts ("FINE" parts, in the language of ref. [7]) is necessary and a multiple Mellin transform must be used. In our case it is not difficult to see that in each sector the α ordering induces for a polynomial $P(\alpha)$ ($W(\alpha)$ and $U(\alpha)$) a factorization of the form,

$$P(\{\alpha\}) \to (\prod_i \beta^{r_i})Q(\{\beta\}) \tag{15}$$

such that $Q(0, ...0) \neq 0$. The behaviour of $P(\{\alpha\})$ around zero is governed by its vanishing when subsets of α-variables go to zero linearly. Convergence or divergence of the integral $\int_0 \prod_i \alpha_i P(\{\alpha_i\})$ can be determined by power counting. For the same reason, eventual divergences in such an integral can be removed by Taylor subtractions [13, 14]. The case in which zeros of $U(\alpha)$ induce ultraviolet divergences can be treated as the convergent case along the following lines: in the α-parametric representation, these divergences are renormalized by Taylor subtractions. But the remainder of the Taylor expansion may be written as in [13, 14],

$$(1-\tau^k)f(x) = \int_0^1 d\xi \frac{(1-\xi)^{k+\nu}}{(k+\nu)!}(\frac{d}{d\xi})^{k+\nu+1}[\xi^{\nu+k}f(\xi x)]. \tag{16}$$

By regrouping the nests of subgraphs that belong to the same equivalence class as explained in refs. [13, 14], we obtain in each sector and for each equivalence class, a finite sum of convergent integrals which are exactly of the same type as in the convergent case, provided the various ξ-variables are simply renamed as suplementary Hepp β-variables. In the following we keep the notations corresponding to convergent graphs.

3 The Rest of the Expansion

Introducing the notation $z = \sigma + i\beta$ we have a first bound to the rest R_N in the truncated expansion above,

$$|R_N(\lambda)| \le \lambda^{N+\eta} Q_N \tag{17}$$

with

$$Q_N = \int_{-\infty}^{+\infty} \frac{d\beta}{2\pi} |\Gamma(-\sigma - i\beta) F(\sigma + i\beta)|. \tag{18}$$

Using recurrence formulas we may relate $\Gamma(-\sigma - i\beta)$ to a gamma function which has positive real part of the argument. We get, remembering Equ. (11) above,

$$\Gamma(-\sigma - i\beta) = \Gamma(2 - \eta - i\beta) \prod_{j=0}^{N+1} \frac{1}{(-N - \eta + j) - i\beta} \tag{19}$$

Now, it may be shown [12] that for $c > 0$ the gamma function $\Gamma(c - i\beta)$ is bounded in absolute value,

$$|\Gamma(c - i\beta)| \le e^{-\epsilon|\beta|} \int_{-\infty}^{\infty} du e^{cu - e^u cos\epsilon} \tag{20}$$

where $\epsilon < \frac{\pi}{2}$ is a positive constant and $c = 2 - \eta$ is also a positive constant. Thus the bound has the form,

$$|\Gamma(c - i\beta)| < c' e^{-\epsilon|\beta|} \tag{21}$$

From Equs. (18), (19), and (21) we have,

$$Q_N < c' \int_{-\infty}^{\infty} \frac{d\beta}{2\pi} |F(N + \eta + i\beta))| \prod_{j=0}^{N+1} \frac{1}{[(-N - \eta + j)^2 + \beta^2]^{\frac{1}{2}}} \tag{22}$$

Also we have the inequalities,

$$\prod_{j=0}^{N+1} \frac{1}{[(-N - \eta + j)^2 + \beta^2]^{\frac{1}{2}}} \le \prod_{j=0}^{N+1} \frac{1}{|(-N - \eta + j)|} < \frac{1}{N!\eta|(1 - \eta)|} \tag{23}$$

The first one is obvious, since $\beta^2 \ge 0$. To see the second one, let us recall the notation $\sigma = N + \eta$, and write,

$$\prod_{j=0}^{N+1} \frac{1}{| - \sigma + j|} = \prod_{j=0}^{N+1} \frac{1}{|\sigma - j|} = \frac{1}{\sigma(\sigma - 1)...(\eta + 1)\eta|\eta - 1|}. \tag{24}$$

From $\sigma > N$, $\sigma - 1 > N - 1$,..., $\eta + 1 > 1$, we find,

$$\prod_{j=0}^{N+1} \frac{1}{|-\sigma + j|} < \frac{1}{N!\eta(1-\eta)} \tag{25}$$

Combining Equs. (23), (22) and (17) we obtain a bound for the rest of the truncated expansion for the λ-scaled amplitude $G(\lambda)$,

$$|R_N(\lambda)| < \frac{\lambda^N . c'}{N!\eta|(\eta - 1)|} \int_{-\infty}^{+\infty} \frac{d\beta}{2\pi} e^{-\epsilon|\beta|} |F(N + \eta + i\beta)| \tag{26}$$

Displacing indefinitely the integration path would generate instead of the truncated expansion, a series, provided the rest R_N have an appropiate behaviour as $N \to \infty$. Let us particularize to the limit of all masses going to zero. In this case the function $F(z)$ in Equ. (12) has the form,

$$F(z) = (\mu^2)^z \int_0^\infty \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}} e^{-\frac{R}{U}} (\frac{W'}{U})^z \tag{27}$$

where μ is a constant mass parameter and $W' = (\sum_{j\in G} \alpha_j) U(\alpha)$. For the absolute value of $F(z)$ we obtain a bound,

$$|F(z)| \leq (\mu^2)^N g(N, \{p\}) \tag{28}$$

with

$$g(N, \{p\}) = \int_0^\infty \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}} e^{-\frac{R}{U}} (W'/U)^{N+\eta} \tag{29}$$

where $\{p\}$ stands for the external momenta.

Taking I-dimensional spherical coordinates, the radial integration may be explicitly performed taking into account the homogeneity properties of the polynomials R, W' and U in the α variables ($R(\alpha)$ and $W'(\alpha)$ are homogeneous of degree $L + 1$ and $U(\alpha)$ is homogeneous of degree L). We obtain an expression in terms of an integral over the I-dimensional angular variables Ω,

$$g(N, \{p\}) = \Gamma(I + N + \eta - \frac{DL}{2} + 1) \int d\Omega f(\Omega) [g(\Omega)]^{N+\eta}, \tag{30}$$

$f(\Omega)$ and $g(\Omega)$ being regular functions, and since the above integral in over angular variables, it has an upper bound $K_N(\{p\})$, K_N being a positive quantity. From Equs. (26), (28) and (30) we see that,

$$|R_N(\lambda)| < \frac{\Gamma(N + \eta + I - \frac{DL}{2} + 1)}{\Gamma(N + 1)} \frac{c'}{\eta(1-\eta)} (\mu^2)^{N+\eta} K_N(\{p\}) \lambda^{N+\eta} \tag{31}$$

For $I - \frac{DL}{2} > 0$ (which is just the condition for UV convergence for the graph), the ratio $\kappa = \frac{\Gamma(N+\eta+I-(DL/2)+1)}{\Gamma(N+1)}$ is clearly a finite positive quantity. Thus renaming the various constants appearing in the expressions above, the rest of the asymptotic expansion may be written in the form,

$$|R_N(\lambda)| < K_1 K_N(\{p\})(\mu^2)^N \lambda^N \tag{32}$$

The scaling parameter λ is arbitrarily small in the limit of the masses going to zero. Therefore the factor $(\lambda\mu^2)^N$ in the bound above makes the sequence of the remainders $R_N(\lambda)$ converge to zero as $N \to \infty$ which is a condition for convergence of the asymptotic expansion.

4 Concluding Remarks

The study of the asymptotic behaviour of Feynman amplitudes has a long history. Weinberg [15] proved such a theorem on asymptotic behaviours for the specific case of scaling by λ all external momenta of an Euclidean convergent amplitude. Later the theorem was extended to divergent amplitudes [16, 14]. Perhaps it could be objected that the present study, since it concerns a detailed analysis of the behaviour of a single Feynman amplitude, although generic, should not be a real progress, in view of some limitations to the use of perturbation methods in field theory. Nevertheless, in applications of field theory to critical phenomena, the examples of models of field theory that have been found to give relevant information, are controled by the free field fixed point, or by fixed points that approach the free field fixed point in some limit. This means that Feynman diagram approach to field theory plays an important role in understanding physical situations in critical phenomena. This is one of the reasons why we hope the analysis presented in this note could be interesting.

5 Acknowlegements

We would like to thank M.B. da Silva Neto and N.F. Svaiter by fruitful discussions and a critical reading of the mauscript. This paper was supported by Conselho Nacional de

Desenvolvimento Cientifico e Tecnologico (CNPq) - Brazil.

References

[1] J.H. Lowenstein, W. Zimmermann, Nucl. Phys. **B86**, 77 (1975).

[2] M.C. Bergere, Y.M.P. Lam, Commun. Math. Phys. **39**,1 (1974).

[3] D.R. Yennie, S.C. Frautshi, H. Suura, Ann. Phys. **13** 379 (1961).

[4] T.W.B. Kibble, Phys. Rev. **133**, 1527 (1968), Phys. Rev. **174**, 1882 (1968), Phys. Rev. **175**, 1624 (1968).

[5] F. Bloch, A. Nordsieck, Phys. Rev. **52**, 54 (1937).

[6] Jean Zinn-Justin, Quantum Field Theory and Critical Phenomena, Clarendon Press - Oxford (1996).

[7] M.C. Bergere, C. de Calan, A.P.C. Malbouisson, Comm. Math. Phys. **62**, 137 (1978).

[8] C. de Calan, A.P.C. Malbouisson, Ann. Inst. Henri Poincaré, **32**, 91, (1980).

[9] C. de Calan, A.P.C. Malbouisson, Comm. Math. Phys., **90**, 413 (1983).

[10] A.P.C. Malbouisson, J. Math. Phys. **35**, 479, (1994)

[11] A.N. Vartchenko, Functional Analysis and its Applications, **10**, 13 (1976)

[12] J. Duistermaat, V.W. Guillemin, Inventiones Math. **29**, 39 (1975).

[13] M.C. Bergere, J.B. Zuber, Commun. Math. Phys. **35**, 113 (1974).

[14] M.C. Bergere, Y.M.P. Lam, J. Math. Phys. **17**, 1546 (1976).

[15] S. Weinberg, Phys. Rev. **118**, 838 (1960).

[16] J.P.Fink, J. Math. Phys. **9**, 1389 (1968)