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PROJECTOR IN CONSTRAINED QUANTUM DYNAMICS

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Abstract

We develop a method which allows us to deal with a constrained many particle system. We use the projector technique together with the path integral formulation to obtain the quantum dynamics of holonomic and non-holonomic systems.

In the non-holonomic case the Feynman's integrals are defined only locally and the wave function acquires a path dependent phase factor.

Key-words: Projector; Constrained system; Non-holonomic systems.

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1.- Introduction

We propose a method of quantization of constrained systems, based on the use of a projector technique together with the path integral approach.

The projector technique was developed by Amaral[1] and Pitanga[2,3] in order to deal with classical Lagrangian and Hamiltonian systems. In this technique neither Lagrange multipliers nor the elimination of coordinates is required to obtain the classical equation of a motion. As is well known, the Lagrange multipliers are disadvantageous for quantization, since the canonical conjugated momenta are zero. If we work with all coordinates, without elimination, we may choose a cartesian system of coordinates in the configuration space. In this system the Feynman integrals are very easy to handle.

All we need in the projector technique is to have a local vector space generated by the constraints. Following Whittaker[4] the constraints must be imposed on the virtual displacement, and not on the trajectories. This remark enables us to consider a generalized variational principle extended to non-holonomic systems. We can use the Path integrals formulation[5,6] to construct the Hamiltonian quantum operator $\hat{H}$, for classical Lagrangian submitted to a linear suplementary conditions. In this case we can write a constrained holonomic or non holonomic geodesic.Observing that the main contribuitions to Feynman integral are just theses geodesics, we arrive to a quantum Hamiltonian by considering paths around the constrained geodesics. For a many particle system the quantum Hamiltonian is equivalent to that established by Amiot et al[7], Takahashi[8] and Eden[9], with the aditional curvature term dued the constraints.

For a general non-holonomic systems quantization may be introduced by means of a geometric procedure or the path-space formalism, developed by Zaccaria et al[10] and Balachandran[11].

In sec.2 we obtain the geodesics using the projector method.

In sec.3 we apply the path integral approach in order to determine the quantum Hamiltonian. Section 4 we dedicate to final remarks.

2.- Constrained System in Configuration Space

Let us consider a system described by a Lagrangian $L = L(r_\mu, \dot{r}_\mu, t)$ together with a set of K independent constraints;

$$\phi^{\alpha}(r_\mu, \dot{r}_\mu) = 0 \quad , \quad \alpha = 1, ..., K \quad \text{and} \quad \mu = 1, ..., N \quad . \tag{1}$$

The Lagrangian of the system to be considered in this paper is of the form,

$$L(r_\mu, \dot{r}_\mu, t) = \frac{1}{2} m_\mu \dot{r}_\mu^2 - V(r_\mu) \tag{2}$$

Here we have adopted the summation convention over repeated indices.

The classical action is,

(3) $$S = \int_{t_1}^{t_2} L(r_\mu,\dot{r}_\mu,t)dt \quad .$$

The constraints define a hypersurface in the configuration-velocities space. The K normal vectors[12] to the hypersurface are described by a vectorial basis $\{ |e_\nu\rangle \}$:

(4) $$|e^a\rangle = \partial_\nu \phi^a \; |e_\nu\rangle \qquad , \qquad a = 1 ,..., K \qquad .$$

Here and in what follows $\partial_\nu = (\partial/\partial\dot{x}_\nu, \partial/\partial\dot{y}_\nu, \partial/\partial\dot{z}_\nu)$.

The vectors $|e^a\rangle$ span a local geometry whose metric is non singular;

(5) $$\langle e^a|e^b\rangle = g^{ab} \quad .$$

The displacements dr^* which are compatible with the constraints must be orthogonal to this vector,

(6) $$\langle dr^*|e^a\rangle = \partial_\nu \phi^a dr^*_\nu = 0 \qquad .$$

A generalized variational principle may be extended to a non-holonomic case by imposing constraints on the virtual displacement.

The infinitesimal displacement must be projected on the hyper-surface orthogonal to the vectors $\{ |e^a\rangle \}$. We can do this by means of a projector;

(7) $$\Lambda = I - \varepsilon_{ab}|e^b\rangle\langle e^a| \quad ,$$

whose components are;

(8) $$\langle e_\mu|\Lambda|e_\nu\rangle = \Lambda_{\mu\nu} = \delta_{\mu\nu} - \varepsilon_{ab}\partial_\mu\phi^b\partial_\nu\phi^a \equiv \delta_{\mu\nu} - Q_{\mu\nu} \quad .$$

The components of the virtual displacements on the hypersurface are;

(9) $$\delta r^*_\mu = \Lambda_{\mu\nu}\delta r_\nu \quad .$$

We see from (6) that (9) is automatically satisfied, because $\Lambda|e^b\rangle = 0$. We assume that the infinitesimal virtual displacement is identical to the infinitesimal possible displacement,then we write,

(10) $$\delta r^*_\mu = \Lambda_{\mu\nu}[\, r_\nu(t+\epsilon) - r_\nu(t)] \equiv \Lambda_{\mu\nu}\eta_\nu \quad .$$

Making a variation of the action along δr^*_μ (we may chose a local system of coordinates in which $\delta d - d\delta = 0$) the generalized variational principle yelds:

(11) $$\Lambda_{\mu\nu}E_\nu = \Lambda_{\mu\nu}\left(\frac{d}{dt}(\partial_{\dot\nu}L) - \partial_\nu L \right) = 0 \quad .$$

This system of equations together with the constraints given by eq.(1) determines the equations of motion of the system([18]).

The line element of the path along the hypersurface allowed by the constraints is,

(12) $$ds^{2\|} = (E - V)\Lambda_{\mu\nu}dr_{\mu}dr_{\nu} \quad .$$

A constrained geodesic is a curve whose lenght is stationary and satisfy the equation of constraint. From the generalized Jacobi's principle we have;

(13)

$$\delta\int\sqrt{\tilde{\Lambda}_{\mu\nu}d_{\lambda}r^{\mu}d_{\lambda}r^{\nu}}\,d\lambda = \int\left[d_{\lambda}(\tilde{\Lambda}_{\mu\rho}d_{\lambda}r^{\rho}) - \partial_{\rho}\tilde{\Lambda}_{\mu\nu}d_{\lambda}r^{\rho}d_{\lambda}r^{\nu}\right]\Lambda_{\mu\alpha}\delta r^{\alpha} = 0$$

where $\tilde{\Lambda}_{\mu\rho} = (E-V)\Lambda_{\mu\rho}$.

Making use of the constraints $\Lambda_{\mu\nu}\dot{r}^{\nu} = \dot{r}_{\mu}$ we have:

(14) $$\ddot{r}^{\mu} - \Gamma^{\mu}_{\rho\nu}\dot{r}^{\nu}\dot{r}^{\rho} = 0$$

where $\Gamma^{\mu}_{\rho\nu} = \partial_{\rho}\tilde{\Lambda}_{\mu\nu} + \partial_{\nu}\tilde{\Lambda}_{\mu\rho} - \partial_{\mu}\tilde{\Lambda}_{\rho\nu}$.

These are the same equations obtained by Synge([14]) valid to holonomic or non-holonomic geometry and are the curves of motion of a conservative constraind system.

3.- Path Integral

From the path integral formulation we have that the wave functions $\psi(t)$ and $\psi(t+\epsilon)$ at the instants t and t+ε can be related by using the exponential of the classical action; thus, we have,

(15) $$\psi(r(t+\epsilon),t+\epsilon) = \int \exp(\tfrac{i}{\hbar}S)\psi(r(t),t)D[r(t)] \quad ,$$

the symbol $D[r(t)]$ means integrations over all trajectories.

In our case we take the time evolution of the system in the limit $\epsilon \to 0$. We have then that the classical action can be approximated by,

(16) $$S = \lim_{\epsilon \to 0} \int_t^{t+\epsilon} L(r_\mu,\dot{r}_\mu,t)dt$$

$$\simeq \epsilon\, L \left(\frac{r(t+\epsilon)+r(t)}{2},\frac{r(t+\epsilon)-r(t)}{\epsilon}\right) \quad .$$

In order to take into account the constraints the above equation must be written as :

(17) $$S^* \simeq \epsilon\, L \left(\frac{r(t+\epsilon)+r(t)}{2},\left(\frac{r(t+\epsilon)-r(t)}{\epsilon}\right)\right) \quad .$$

$$\simeq \frac{1}{2\epsilon}\, g_{\mu\nu}\Big[\eta^\mu\eta^\nu - \Gamma^\mu_{\alpha\rho}\eta^\nu\eta^\alpha\eta^\rho + \tfrac{1}{4}\Gamma^\mu_{\rho\tau}\Gamma^\nu_{\alpha\omega}\eta^\rho\eta^\tau\eta^\alpha\eta^\omega +$$

$$+ \frac{1}{3}\left(\partial_\tau\Gamma^\mu_{\alpha\omega} + \tfrac{1}{4}\Gamma^\mu_{\rho\tau}\Gamma^\rho_{\alpha\omega}\right)\eta^\nu\eta^\tau\eta^\alpha\eta^\omega + \ldots\Big] - \epsilon\Big(V_o - \eta^\mu V_\mu +$$

$$+ \tfrac{1}{2}\eta^\mu\eta^\nu V_{\mu\nu} - \tfrac{1}{2}\eta^\mu\eta^\nu\eta^\rho V_{\mu\nu\rho} - \tfrac{1}{4!}\eta^\mu\eta^\nu\eta^\alpha\eta^\rho V_{\mu\nu\alpha\rho} + \ldots\Big)$$

where we have used the equation (14) since $\Gamma^\mu_{\rho\nu} = \partial_\rho\tilde{\Lambda}_{\mu\nu}$ describes the conections of the allowed submanifold in the cartesian frame and:

(18) $$\eta^\mu = r^\mu(t+\epsilon) - r^\mu(t) \quad ,$$

(19) $$\mathcal{V}_\alpha = \partial_\alpha V \quad , \quad \mathcal{V}_{\alpha\rho} = \frac{1}{2!}\partial_\alpha\partial_\rho V \quad ,\ldots \quad .$$

The method to obtain the Schrödinger equation from the path integrals consists of the three following steps([15,16]);

a) first, apply a full expansion in power series to both sides of equation (15) using (16). We expand the left hand side is expanded with respect to time and the right hand side with respect to coordinates;

(20) $$\psi(r(t+\epsilon),t) + \epsilon\partial_t\psi(r) + \frac{\epsilon^2}{2!}\partial_t^2\,\psi(r) + \ldots =$$

$$= \int\Big\{\exp\big(\frac{i}{2\hbar\epsilon}\eta^*_\mu\eta^*_\nu\big)\big[\; + \frac{i}{\hbar}\epsilon L + \big(\frac{i}{\hbar}\big)^2\frac{1}{2!}\,\epsilon^2 L^2 + \ldots\big] \times$$

$$\times\;\big[\psi(r(t+\epsilon),t) - \eta^*_\nu\partial_\nu\psi + \frac{1}{2!}\eta^*_\mu\eta^*_\nu\partial_\mu\partial_\nu\psi + \ldots\big]\Big\}D(\eta^*(t))$$

where $\eta^*_\mu = \Lambda_{\mu\nu}\eta^\nu$ is the projection of the infinetesimal displacement.

b) second, use the properties of the gaussian integrals;

c) and finally, compare the coefficients of the ϵ-monomials in both sides of equation (20) after the expansion and integration. Hence we obtain the following Schrödinger equation,

(21) $$i\hbar\partial_t\psi = -\frac{\hbar^2}{2}\,\partial_\mu\Lambda_{\mu\nu}\partial_\nu + V(r) + \frac{\hbar^2}{6}\,\Lambda_{\mu\nu}R_{\nu\mu} = (\hat{H} + \frac{\hbar^2}{6}R)\psi$$

where R is the scalar curvature. When the constraints are linear in the spatial coordenates R = 0 and the equation above is the same obtained by Takahashi[8].

We can see that this formulation enables us to obtain directly the symmetrized operator $\hat{H}$.

In the non-holonomic case the wave function acquires a non-integrable phase factor, dued the non-integrability of the constraints. From (8) and (15) we have :

$$\Psi = \int \exp(\tfrac{i}{\hbar}S)\left[\exp(-\tfrac{i}{\hbar}\int \varepsilon_{ab}\partial_{\mu}\phi^{b}\partial_{\nu}\phi^{a}\dot{r}_{\nu}dr_{\mu})\psi\right]D[r(t)] \tag{22}$$

$$= \int \exp(\tfrac{i}{\hbar}S)\ \psi' D[r(t)] \quad .$$

It is easy to show that the non integrability of the constraints leads , by using Stokes' theorem, to the expression

$$I = \oint \varepsilon_{ab}\partial_{\mu}\phi^{b}\partial_{\nu}\phi^{a}\dot{r}_{\nu}dr^{\mu} = \iint \tfrac{1}{2}\ \dot{r}_{\nu}(\Gamma^{\nu}_{\rho\mu}-\Gamma^{\nu}_{\mu\rho})dr^{\mu}\wedge dr^{\rho} \tag{23}$$

where

$$\Gamma^{\nu}_{\rho\mu}-\Gamma^{\nu}_{\mu\rho} = e^{\nu}_{a}(\partial_{\rho}e^{a}_{\mu} - \partial_{\mu}e^{a}_{\rho}) \quad . \tag{24}$$

For holonomic system the above is equal to zero, by Frobenius theorem. We must have for non-holonomic systems :

(25) $$I = \oint \varepsilon_{ab}\partial_\mu\phi^b\partial_\nu\phi^a t_\nu dr_\mu = 2\pi\beta$$

In order to avoid ambiguities in the wave function or in the propagator we must have :

(26) $$1 = \exp(I)$$

that is $\beta 2\pi = 2\pi n\hbar$, where β is a constant.

4.- Final Remarks

We have shown in this paper that is possible to develop a quantum theory for a constrained classical system, without destroying symmetries or making use of Lagrange's multipliers.

The projector methods enable us to extend the variational principle to a non-holonomic system. In this formulation we do not need Lagrange multipliers.However, the quantum dynamics of non-holonomic systems belong to a class having a local Lagrangian and a global Hamiltonian description. The action of a non holonomic system is path dependent.We can see this from equation (17).The wave function ψ' being path dependent must be valid in patchs of a non trivial U(1) fibre bundle. The same is true for the Feynman's propagator. In order to avoid the ambiguity on the wave function the precondition of quantization leads to $\beta = n\hbar$ where β is a constant dependent on the topology induced by the constraint.

Finally we remark that the linear non-holonomic constraints define a Finsler geometry called S-Riemannian[17] because we do not have torsion and the connection are antisymmetric. In this particular case $\Lambda_{\mu\nu}$ is only function of the position. The general case, where $\Lambda_{\mu\nu} = \Lambda_{\mu\nu}(x,\dot{x})$, the Feynman integral must be adapted to a tangent fibre bundle. We will do this in futures works.

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