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GÖDEL-TYPE METRIC IN EINSTEIN-CARTAN SPACES*

by

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ABSTRACT

An extension of the known class of Gödel-type space-time homogeneous metrics which covers all the range $-\infty < m^2 < \infty$ is obtained in Einstein-Cartan spaces generated by a perfect Weyssenhoff fluid. Also general solutions for non-homogeneous rotating spaces are obtained.

It has been shown by Rebouças and Tiomno[1] that all space-time homogeneous Gödel-type Riemannian metrics which generalize the Gödel metric[2] and which are of the form (cylindrical coordinates)

$$ds^2 = [dt + H(r)\, d\phi]^2 - dr^2 - D^2(r)\, d\phi^2 - dz^2 \tag{1}$$

are given either by

$$D(r) = \frac{1}{m} \sinh (mr), \; H(r) = -\frac{4\Omega}{m^2} \sinh^2 (\frac{mr}{2}), \tag{2}$$

if $m^2 > 0$ (constant), where

$$m^2 = \frac{d^2 D}{dr^2} D^{-1}, \tag{3}$$

or by

$$D\, (r) = \frac{1}{\mu} \sin \mu r \;, \; H(r) = -\frac{4\Omega}{\mu^2} \sin^2 (\mu r/2), \tag{4}$$

if $\mu^2 = -m^2 \geqslant 0$ (constant).

Here the constant Ω is the vorticity or the angular velocity of these rigidly rotating universes relative to the compass of inertia. In the Gödel universe $m^2 = 2\Omega^2$.

The solutions of the Einstein-Maxwell-scalar massless field equations with a perfect fluid and Λ with homogeneous Gödel-type metric[1] correspond, for given Ω, to values of m^2 in the range

$$-\infty < m^2 \leqslant 4\Omega^2. \tag{5}$$

The case $m^2 = 4\Omega^2$ corresponds to a pure scalar field with a convenient value of the cosmological constant[1]:

$$\Lambda = -2\Omega^2 , \rho = p = 0 . \quad (6)$$

We assume in this paper unities

$$8\pi G = c = 1 . \quad (7)$$

The present paper rose from the observation that the upper limitation on m^2 resulted from the positiveness of matter density. Thus, since in the Einstein-Cartan-Sciama-Kibble-Hehl (E.C.S.K.H.) theory of gravitation [3] the spin gives negative contributions to the effective density and pressure,

$$\rho_{eff} = \rho - S^2/4 , p_{eff} = p - S^2/4 , \quad (8)$$

we expected that this would enlarge upwards the range of m^2.

In the (E.C.S.K.H.) theory of gravitation [3] the spin properties of the fluid are related to the anti-symmetric part of the connection. We consider here the formulation where the source is only a Weyssenhoff fluid, with the spin quantities $S_{\rho\sigma}{}^{\mu} = S_{\rho\sigma}V^{\mu}$ proportional to the quadrivelocity (V^{μ}) and to the spin tensor density ($S_{\rho\sigma}$), with $S_{\mu\nu} V^{\nu} = 0$

In this theory, Einstein equations in the tetradic frame given by [1]

$$ds^2 = (e^{(0)}{}_{\mu} dx^{\mu})^2 - \sum_{i=1}^{3} (e^{(i)}{}_{\mu} dx^{\mu})^2 = \eta_{AB} e^{A}{}_{\mu} e^{B}{}_{\nu} dx^{\mu} dx^{\nu} \quad (9)$$

with $(0, 1, 2, 3) \rightarrow (t, r, \phi, z)$ are [3],[4]

$$G^{E}{}_{AB} = T_{AB} = (\rho + p - S^2/2) V_A V_B - (p - S^2/4) \eta_{AB} -$$

$$- e_{A}{}^{\alpha} e_{B}{}^{\beta} (S^{\mu}{}_{(\alpha} V_{\beta)})_{;\mu} + V^{\mu} V^{\nu} e_{A}{}^{\alpha} e_{B}{}^{\beta} (V_{(\alpha} S_{\beta)\nu})_{;\mu} , \quad (10)$$

where $S^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu}$ and; denotes the Christoffel covariant derivative. They lead, with the metric (1), to the equations (tetradic components)

$$G_{00} \equiv 3\,\Omega^2 - m^2 = \rho - S^2/4 + 2\Omega S, \tag{10a}$$

$$G_{02} \equiv \Omega' = S'/2, \tag{10b}$$

$$G_{11} = G_{22} \equiv \Omega^2 = p - S^2/4 + \Omega S, \tag{10c}$$

$$G_{33} \equiv m^2 - \Omega^2 = p - S^2/4, \tag{10d}$$

where the prime indicates derivative relative to r. We have imposed $S_{AB} = (\delta^1{}_A\, \delta^2{}_B - \delta^1{}_B\, \delta^2{}_A)\, S$ or $S = S_{12} = -S_{21}$. The functions Ω (r), m (r) are defined by

$$\Omega\,(r) = -\,H'\,/\,2D, \tag{11}$$

$$m^2\,(r) = D''\,/\,D. \tag{12}$$

A further equation in this theory, in our case

$$2\,V_{[\mu}\,S_{\nu]\lambda;\rho}\,V^\lambda\,V^\rho + (S_{\mu\nu}\,V^\rho)_{;\rho} = 0, \tag{13}$$

is identically satisfied for a Gödel-type metric.

From equs. (10) we obtain (Ω_0 = const.)

$$\Omega\,(r) = \Omega_0 + S(r)/2\,, \tag{14}$$

$$\rho = p = \Omega_0^2\,, \tag{15}$$

$$m^2 = 2\,\Omega_0\,\Omega\,(r) = 2\,\Omega^2 - \Omega\,S\,. \tag{16}$$

We find easily that for $\Omega(r)$ = constant the universe is S - T homogeneous with the metric given by (2) (or 4) with $m^2 > 2\Omega^2$ (or $m^2 < 2\Omega^2$) if $\Omega S < 0$ (or $\Omega S > 0$).

We also see that for $m^2 > \Omega^2$ both ρ and $\rho_{eff} = p_{eff} = m^2 - \Omega^2$

are positive but ρ_{eff} is negative for $m^2 < \Omega^2$. However, contrary to our intuition, not the terms in S^2 of equs.(10) but the terms in S,S' are the ones which allowed us to reach the range $4\Omega^2 < m^2 < \infty$ (with $-\Omega S > 2\Omega^2$ as seen from (16)).

Finally we should mention that equations (11, 12,14, 15, 16) allow us to determine uniquely the metric and sources when Ω_0 (constant) and D(r) are given, D(r) being arbitrary except for the condition $D(r) \simeq r$ near the origin.

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