CBPF-NF-029/02
e-print hep-th/0205195

STABILITY EQUATION AND TWO-COMPONENT EIGENMODE FOR DOMAIN WALLS IN A SCALAR POTENTIAL MODEL

G. S. Dias, E. L. Graça and R. de Lima Rodrigues*

Centro Brasileiro de Pesquisas Físicas
Rua Dr. Xavier Sigaud, 150
CEP 22290-180, Rio de Janeiro-RJ, Brazil

† Departamento de Física, Universidade Federal Rural do Rio de Janeiro
Antiga Rodovia Rio-São Paulo Km 47, BR 465, CEP 23.890-000, Seropédica-RJ

August 2002

Abstract

Supersymmetric Quantum Mechanics involving a two-component representation and two-component eigenfunctions is applied to obtain the stability equation associated to a potential model formulated in terms of two coupled real scalar fields. We investigate the question of stability by introducing an operator technique for the Bogomol'nyi-Prasad-Sommerfield (BPS) and non-BPS states on two domain walls in a scalar potential model with minimal $N = 1-$supersymmetry.

PACS numbers: 11.30.Pb, 03.65.Fd, 11.10.Ef

Key-works: Domain wall; Supersymmetry in quantum mechanics; Coupled scalar fields.

Typeset using REVTEX

*Permanent address: Departamento de Ciências Exatas e da Natureza, Universidade Federal de Campina Grande, Cajazeiras – PB, 58.900-000, Brazil. E-mail: rafaelr@cbpf.br or rafael@fisica.ufpb.br. E-mail for GSD is dias@cbpf.br.

I. INTRODUCTION

The algebraic framework of Supersymmetry in Quantum Mechanics (SUSY QM), as formulated by Witten [1] and Sukumar [2], may be elaborated from a 2-dimensional model with minimal $N = 1$ Supersymmetry. The SUSY QM generalization [3,4] of the harmonic oscillator raising and lowering operators for shape-invariant potentials [5], and connections established with supercoherent states for the super-Wigner oscillator systems [6] associated to the isotonic oscillator, have been considered [7]. The super-realization of the Wigner oscillator has also been applied to pure parabosonic systems in the interesting paper of Ref. [8].

The SUSY algebra has also been applied to build up a variety of new one-parameter families of isospectral supersymmetric partner potentials in quantum field theory [9–14]. Recently, one has investigated [15,16] the 2x2-matrix superpotential associated with the linear classical stability from the static solutions for a system of two coupled real scalar fields in (1+1) dimensions that present powers up to sixth-order. In this case, the static field configurations were determined via Rajaraman's method [17]. Also, results on sn-type elliptic functions given in Ref. [18], for which boundary conditions on bound energy levels of a classical system defined by one single scalar field are imposed, have deserved an extension to a relativistic system of two coupled real scalar fields in a finite domain in (1+1) dimensions [19].

A 2x2-matrix superpotential for a neutron in interaction with a static magnetic field generated by a current-carrying straight wire, which is also described by two-component wave functions, has also been worked out [20]. Indeed, Witten's formalism for Supersymmetry was applied to this planar physical system in both the momentum [21] and coordinate [20,22] representations. According to our development, we can readily realize the SUSY QM algebra, in coordinate representation, for a 2-dimensional potential model with minimal $N = 1$-SUSY containing up to fourth-order powers in the fields.

In this work, we show that the two-component eigenmodes of the fluctuation operator may be of two types; of course, for a zero mode we have a corresponding $\Psi_0(z) = \begin{pmatrix} \eta_0 \\ \xi_0 \end{pmatrix}$. Nevertheless, this is not necessarily true for an arbitrary non-trivial real eigenvalue ω_n^2. Here, we shall determine the eigenmodes and indicate the number of bound states. We also build the matrix superpotential for SUSY QM in the case of the stability equation associated to 2-dimensional potential model considered by Shifman et al. [23–25]. First, we consider the classical configurations with domain wall solutions, which are bidimensional structures in 3+1 dimensions. They are static, non-singular, classically stable Bogomol'nyi [53] and Prasad-Sommerfield [27] (BPS) soliton (defect) solutions to field equations with finite localized energy associated with two coupled real scalar fields and non-BPS states. Recently, marginal stability and the metamorphosis of BPS states have been investigated [28], via SUSY QM, and one presents a detailed analysis for a 2-dimensional $N = 2-$Wess-Zumino model with two chiral superfields, and composite dyons in $N = 2$-supersymmetric gauge theories.

Domain walls have been recently exploited in a context that stresses their connection with BPS-bound states [29]. Let us point out that some investigations are interesting in connection with Condensed Matter [30], Cosmology [31], coupled field theories with soliton solutions [32–40] and one-loop quantum corrections to soliton energies and central charges in the supersymmetric ϕ^4 and sine-Gordon models in (1+1)-dimensions [41,42]. Recently, the reconstruction of 2-dimensional scalar field potential models has been considered, and quantum corrections to the solitonic sectors of both potentials has been investigated [43].

The work of Ref. [41] reproduces the results for the quantum mass of the SUSY solitons previously obtained in Ref. [42]. The quantization of two-dimensional supersymmetric solitons is in fact a surprisingly intricate issue in many aspects [44–49]. Indeed, using dimensional regularization and reduction from 2+1 dimensions, which preserve SUSY, Rebhan-Nieuwenhuizen-Wimmer have shown that the existence of an anomalous contribution for a BPS satured domain wall at the quantum level [50], which is in agreement with Shifman *et al.* [25].

Also, recently, the generalized zeta function regularization method has been applied to compute the one-loop quantum corrections of the kink in the sine-Gordon and other scalar potentials [51].

Domain walls have also been recently exploited in the context of dynamical domain ribbons [33], deffects that live inside topological deffects. [34], triple junctions via $N = 1$ supersymmetry theories [35] and non-supersymmetric tiling domains and networks of domain wall [36,37], more complicated field configurations with axial geometry, and domain wall junctions in a class of generalized Wess-Zumino models with $\mathbf{Z}$ symmetry [38]. The set of potential BPS junctions that have been identified in [38] contain the junctions that appear in [36,37]. Recently, the BPS saturated objects with axial geometry (wall junctions, vortices), in generalized Wess-Zumino models, have also been investigated [39].

In the present work, a connection between SUSY QM developments and the description of such a physical system with stability equation is expressed in terms of two-component wave functions. This leads to 4x4 supercharges and supersymmetric Hamiltonians whose bosonic sectors possesses a fluctuation operator (O_F) associated with two-component eigenstates in terms of BPS and non-BPS states.

This work is organized as follows: In Section II, we investigate domain walls configurations for two coupled scalar fields; an extension to supersymmetric non-relativistic quantum mechanics with two-component wave functions is also implemented. In Section III, we investigate the stability of BPS and non-BPS states in the context of SUSY QM. Our Conclusions are presented in Section IV.

II. DOMAIN WALLS FROM TWO COUPLED SCALAR FIELDS

In this section, we investigate a potential model in terms of two coupled real scalar fields in (1+1) dimensions that present classical soliton solutions known as domain walls.

The Lagrangian density for such a non-linear system, in natural units ($c = \hbar = 1$), is written as

$$\mathcal{L}\left(\phi, \chi, \partial_\mu \phi, \partial_\mu \chi\right) = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2}\left(\partial_\mu \chi\right)^2 - V(\phi, \chi), \tag{1}$$

where $\eta^{\mu\nu} = dig(+,-)$ is the metric tensor. Here, the potential $V = V(\phi, \chi)$ is any positive semidefinite function of ϕ and χ, which must have at least two different zeroes in order to present domain walls as possible solutions. The general classical configurations obey the equations:

$$\Box\phi + \frac{\partial}{\partial\phi}V = 0, \qquad \Box\chi + \frac{\partial}{\partial\chi}V = 0, \tag{2}$$

where $\Box = \partial^\mu\partial_\mu$. For static soliton solutions, the equations of motion become the following system of non-linear differential equations:

$$\phi'' = \frac{\partial}{\partial\phi}V, \quad \chi'' = \frac{\partial}{\partial\chi}V, \tag{3}$$

where primes stand for differentiations with respect to the space variable. There appears in the literature a trial orbit method for the attainment of static solutions for certain positive potentials. This method yields, at best, some solutions to Eq. (3) and by no means to all classes of potentials [17]. Recently, the trial orbit method has been applied to systems of two coupled scalar fields containing up to sixth-order powers in the fields [40].

Let us consider a positive potential, $V(\phi,\chi)$, with the following explicit form:

$$V(\phi,\chi) = \frac{1}{2}\lambda^2\left(\phi^2 - \frac{m^2}{\lambda^2}\right)^2 + \frac{1}{2}\alpha^2\chi^2(\chi^2 + 4\phi^2) + \alpha\lambda\chi^2(\phi^2 - \frac{m^2}{\lambda^2}), \tag{4}$$

where $\alpha, \lambda > 0$. This potencial is of interest for it has solutions like BPS and non-BPS; moreover, it presents four supersymmetric minima. Note that this potential has the discrete symmetry: $\phi \to -\phi$ and $\chi \to -\chi$, so that we have a necessary (but non-sufficient) condition that it must have at least two zeroes in order that domain walls can exist.

In this case, the Bogomol'nyi form of the energy, consisting of a sum of squares and surface terms, becomes

$$E \geq \left| \int dz \frac{\partial}{\partial z} W[\phi(z),\chi(z)] \right|, \tag{5}$$

where the superpotential $W[\phi(z),\chi(z)]$ shall be discussed below. It is required that ϕ and χ satisfy the BPS state conditions [53]:

$$\begin{aligned} \phi' &= -\lambda\phi^2 - \alpha\chi^2 + \frac{m^2}{\lambda}, \\ \chi' &= -2\alpha\phi\chi, \end{aligned} \tag{6}$$

with

$$\begin{aligned} \frac{\partial W}{\partial\phi} &= \phi', \\ \frac{\partial W}{\partial\chi} &= \chi', \end{aligned} \tag{7}$$

The superfield superpotential $W(\Phi,\boldsymbol{\chi})$, as proposed in Ref. [24], yields the component-field potential $V(\phi,\chi)$ of Eq. (4):

$$W(\Phi,\boldsymbol{\chi}) = \frac{m^2}{\lambda}\Phi - \frac{\lambda}{3}\Phi^3 - \alpha\Phi\boldsymbol{\chi}^2, \tag{8}$$

where Φ and $\boldsymbol{\chi}$ are chiral superfields which, in terms of bosonic (ϕ, χ), fermionic (ψ, ξ) and auxiliary fields (F, G), are $\theta-$expanded as shown below:

$$\Phi = \phi + \bar{\theta}\psi + \frac{\theta\bar{\theta}}{2}F,$$
$$\boldsymbol{\chi} = \chi + \theta\xi + \frac{\theta\bar{\theta}}{2}G, \tag{9}$$

where θ and $\bar{\theta} = \theta^*$ are Grassmann variables. The superpotential above, with two interacting chiral superfields, allows for solutions describing string like "domain ribbon" defects embedded within the domain wall. It is energetically favorable for the fermions within the wall to populate the domain ribbons [33]. The supersymmetric vacua are determined by the extrema of the superpotential, so that

$$\frac{\partial W}{\partial \phi} = 0 \tag{10}$$

and

$$\frac{\partial W}{\partial \chi} = 0 \tag{11}$$

provide four vacuum states (ϕ, χ) whose values are listed below:

$$M_1 = \left(-\frac{m}{\lambda}, 0\right)$$
$$M_2 = \left(\frac{m}{\lambda}, 0\right)$$
$$M_3 = \left(0, -\frac{m}{\sqrt{\lambda\alpha}}\right)$$
$$M_4 = \left(0, \frac{m}{\sqrt{\lambda\alpha}}\right). \tag{12}$$

When the wall M_{13} is stable, the two vacuum states, M_1 and M_3, may be adjacent, energy-degenerated with the energies of the three walls M_{23}, M_{14}, M_{24}. Of course, from (12), we see that we have two more possible domain walls, viz., M_{12}, and M_{34}, with nondegeneracy in energy. Indeed, in this work, the potential presents a Z_2xZ_2-symmetry, so that one can build some intersections between the walls.

This generalized system can be solved by the trial orbit development considered in [17]. However, a possible soliton solution occurs ever when we choose $\chi = 0$, so that it implies a domain wall, M_{12}, which is associated to the soliton of the ϕ^4 model:

$$\phi(z) = \frac{m}{\lambda} \tanh(mz). \tag{13}$$

The superpotential, in terms of its components, leads to the correct value for a Bogomol'nyi minimum energy, corresponding to the BPS-satured state. Then, we see that, at classical level, according to Eq. (5), one may put [25]

$$E_B^{min} = \mid W[\phi(z), \chi(z)]_{z=+\infty} - W[\phi(z), \chi(z)]_{z=-\infty} \mid = \frac{4m^3}{3\lambda^2}. \tag{14}$$

Thus, the tension of the wall M_{12} is $T_{12} = \frac{4m^2}{3\lambda^2}$.

III. SUSY QM AND LINEAR STABILITY

Now, let us analyze the classical stability of the domain walls in this non-linear system [12–14,32], which is ensured by considering small perturbations around $\phi(z)$ and $\chi(z)$:

$$\phi(z,t) = \phi(z) + \eta(z,t) \tag{15}$$

and

$$\chi(z,t) = \chi(z) + \xi(z,t). \tag{16}$$

Next, let us expand the fluctuations $\eta(z,t)$ and $\xi(z,t)$ in terms of normal modes:

$$\eta(z,t) = \sum_n \epsilon_n \eta_n(z) e^{i\omega_n t} \tag{17}$$

and

$$\xi(z,t) = \sum_n c_n \xi_n(z) e^{i\tilde{\omega}_n t}, \tag{18}$$

where ϵ_n and c_n are chosen so that $\eta_n(z)$ and $\chi_n(z)$ are real. If $\tilde{\omega}_n = \omega_n$, then the field equations yield a Schrödinger-like equation for two-component wave functions, Ψ_n. However, in general, we obtain

$$O_F \Psi_n = \tilde{\Psi}_n, \quad n = 0, 1, 2, \cdots, \tag{19}$$

where

$$O_F = \left(\begin{array}{cc} -\frac{d^2}{dz^2} + \frac{\partial^2}{\partial\phi^2}V & \frac{\partial^2}{\partial\chi\partial\phi}V \\ \frac{\partial^2}{\partial\phi\partial\chi}V & -\frac{d^2}{dz^2} + \frac{\partial^2}{\partial\chi^2}V \end{array} \right)_{|\phi=\phi(z),\chi=\chi(z)} \tag{20}$$

and the eigenmodes are cast under the form:

$$\tilde{\Psi}_n = \left(\begin{array}{c} \omega_n^2 \eta_n(z) \\ \tilde{\omega}_n^2 \xi_n(z) \end{array} \right). \tag{21}$$

Note that, for the potential model considered in this work, according to Eq. (4), we can readily arrive at:

$$\begin{aligned} \frac{\partial^2}{\partial\chi\partial\phi}V &= \frac{\partial^2}{\partial\phi\partial\chi}V = 4\alpha(2\alpha+\lambda)\phi\chi, \\ \frac{\partial^2}{\partial\phi^2}V &= 6\lambda^2\phi^2 - 2m^2 + 2\alpha(2\alpha+\lambda)\chi^2, \\ \frac{\partial^2}{\partial\chi^2}V &= 6\alpha^2\chi^2 + 2\alpha(2\alpha+\lambda)\phi^2 - \frac{2\alpha m^2}{\lambda}. \end{aligned} \tag{22}$$

We can get the masses of the bosonic particles, using the results above, from the second derivatives of the potential:

$$m_\phi^2 \equiv \frac{\partial^2 V}{\partial \phi^2}|_{z\to\pm\infty}$$
$$m_\chi^2 \equiv \frac{\partial^2 V}{\partial \chi^2}|_{z\to\pm\infty} \,. \tag{23}$$

For the sector $\chi = 0$, and $z \to \pm\infty$, $m_\phi^2 = 4m^2$. In this sector of BPS states, the fluctuation potential term becomes

$$V_{BPS}(z) = m^2 \begin{pmatrix} 6tanh^2(mz) - 2 & 0 \\ 0 & \frac{2\alpha}{\lambda^2}(2\alpha+\lambda)\tanh^2(mz) - \frac{2\alpha}{\lambda} \end{pmatrix}. \tag{24}$$

We can see that V_{BPS} is a diagonal Hermitian matrix, then O_F is also Hermitian. Hence, the eigenvalues ω_n^2 of $O_{F_{11}}$ and $\tilde{\omega}_n^2$ of $O_{F_{22}}$ are all real. We shall now show that ${\omega_n}^2$ are non-negative, the proof of which takes us to a solution of the Pöschl-Teller potential [54].

The mode equations are decoupled and may be of two kinds, given by

$$O_{F_{11}}\eta_n \equiv -\frac{d^2}{dz^2}\eta_n - m^2(6sech^2(mz) - 4)\eta_n = \omega_n^2\eta_n \tag{25}$$

and

$$O_{F_{22}}\xi_n \equiv -\frac{d^2}{dz^2}\xi_n - \frac{2m^2\alpha}{\lambda^2}(2\alpha+\lambda)sech^2(mz)\xi_n + m^2(\frac{2\alpha}{\lambda^2}(2\alpha+\lambda) - \frac{2\alpha}{\lambda})\xi_n = \tilde{\omega}_n^2\xi_n. \tag{26}$$

Note that, according to Eqs. (4) and (22), if $\alpha = 0$, the potencial becomes $V(\phi) = \frac{\lambda^2}{2}(\phi^2 - \frac{m^2}{\lambda^2})^2$, so that the stability equation is given by Eq. (25) and, therefore, there exists only the wall M_{12}.

We can now see that both types of solutions exist only for certain discrete values of ω_n^2. Let us perform the transformation, $mz = y$, so that, upon comparison with the equation (12.3.22) in [54], we obtain the following eigenvalues:

$$\omega_n^2 = m^2\left\{4 - \left[\frac{5}{2} - (n + \frac{1}{2})\right]^2\right\}. \tag{27}$$

In this case, we find only two bound states associated with the eigenvalues $\omega_0^2 = 0$ and $\omega_1^2 = 3m^2$. Thus, the BPS states are stable.

Similarly, for the second type of solutions, we find, from Eqs. (26) and (12.3.22) of Ref. [54], the following eigenvalues:

$$\tilde{\omega}_n^2 = m^2\left\{\frac{2\alpha}{\lambda^2}(2\alpha+\lambda) - \frac{2\alpha}{\lambda} - \left[\sqrt{\frac{2\alpha}{\lambda^2}(2\alpha+\lambda) + \frac{1}{4}} - (n+\frac{1}{2})\right]^2\right\}. \tag{28}$$

In this case, we find the number of bound states as given by

$$n = 0, 1, \cdots < \sqrt{\frac{2\alpha}{\lambda^2}(2\alpha+\lambda) + \frac{1}{4}} - \frac{1}{2},$$

so that, as an example, if we take $\alpha = 2\lambda$, we get

$$\tilde{\omega}_n^2 = m^2 \left\{16 - (4-n)^2\right\}, n = 0, 1, 2, 3. \tag{29}$$

We see that, in this particular case, we have four bound state associated with the eigenvalue $\tilde{\omega}_0^2 = 0$, $\tilde{\omega}_1^2 = 7m^2$, $\tilde{\omega}_2^2 = 12m^2$ and $\tilde{\omega}_3^2 = 15m^2$. Therefore, the BPS state is stable.

Extending to the case of only one single real scalar field [10,12–14], we can realize, *a priori*, the 2x2-matrix superpotential that satisfies the following Riccati equation associated to the flutuaction potential $V_{BPS}(z)$:

$$\mathbf{W}^2 + \mathbf{W}' = V_{BPS}(z), \tag{30}$$

whose solution, in the general case, becomes

$$\mathbf{W} = -2 \left(\begin{array}{cc} \lambda\phi & \alpha\chi \\ \alpha\chi & \alpha\phi \end{array} \right)_{|\phi=\phi(x),\chi=\chi(x)}. \tag{31}$$

Restricted to the sector $\chi = 0$, W becomes diagonal:

$$\mathbf{W} = -2 \left(\begin{array}{cc} m\tanh(mz) & 0 \\ 0 & m\frac{\alpha}{\lambda}\tanh(mz) \end{array} \right)_{|\phi=\phi(z),\chi=0}. \tag{32}$$

Therefore, it is easy to show that the linear stability is satisfied, i.e., $\omega_n^2 = \langle O_F \rangle = \langle \mathcal{A}^+\mathcal{A}^- \rangle = (\mathcal{A}^-\Psi_n)^\dagger(\mathcal{A}^-\Psi_n) =\mid \mathcal{A}^-\Psi_n \mid^2 \geq 0$, as it has been anticipated. Note that we have set $O_F \equiv \mathcal{A}^+\mathcal{A}^-$, where the intertwining operators of SUSY QM must be given in terms of the matrix superpotential, $\mathbf{W}$. According to the Witten' SUSY model [1,5], we have

$$\mathcal{A}^\pm = \pm\mathbf{I}\frac{d}{dz} + \mathbf{W}(z), \quad \Psi^{(n)}_{\mathrm{SUSY}}(z) = \left(\begin{array}{c} \psi_+^{(n)}(z) \\ \psi_-^{(n)}(z) \end{array} \right), \tag{33}$$

where $\mathbf{I}$ is the 2x2 identity matrix.

Indeed, the bosonic sector Hamiltonian of H_{SUSY} is exactly given by O_F, which, as obtained from the stability Eq. (19), has the following zero-eigenmode:

$$\mathcal{A}^-\Psi_+^{(0)}(z) = 0, \qquad \Psi_+^{(0)}(z) = \Psi_0 = \left(\begin{array}{c} \eta_0(z) \\ \xi_0(z) \end{array} \right), \tag{34}$$

as said before, this is not necessarily true for the excited eigenmodes.

Therefore, the two-component normal modes in (21) satisfy ${\omega_n}^2 \geq 0$, so that the stability of the domain wall is ensured.

The non-BPS wall, for $\phi = 0$, is described by the following equation of motion:

$$\frac{d^2\chi}{dz^2} = -2\alpha\chi\left(\frac{m^2}{\lambda} - \alpha\chi^2\right), \tag{35}$$

whose solution connecting the vacua M_3 and M_4 is given by

$$\chi(z) = \frac{m}{\sqrt{\lambda\alpha}}\tanh(Mz), \quad M = \sqrt{\frac{\alpha}{\lambda}}m, \tag{36}$$

so that, in this case, the fluctuation potential term on the wall M_{34} is given by

$$V_{NBPS}(z) = 2m^2 \begin{pmatrix} \frac{2\alpha}{\lambda} + (1 + \frac{2\alpha}{\lambda})sech^2(Mz) & 0 \\ 0 & \frac{\alpha}{2\lambda} + \frac{3\alpha}{\lambda}sech^2(Mz) \end{pmatrix}. \tag{37}$$

As $\alpha \neq \lambda$, the tension of M_{34} is different from the wall tension M_{12}.

The graded Lie algebra of the supersymmetry in quantum mechanics for both the BPS and non-BPS states may be readily realized as

$$H_{SUSY} = [Q_-, Q_+]_+ = \begin{pmatrix} \mathcal{A}^+\mathcal{A}^- & 0 \\ 0 & \mathcal{A}^-\mathcal{A}^+ \end{pmatrix}_{4X4} = \begin{pmatrix} \mathcal{H}_+ & 0 \\ 0 & \mathcal{H}_- \end{pmatrix}, \tag{38}$$

$$[H_{SUSY}, Q_\pm]_- = 0 = (Q_-)^2 = (Q_+)^2, \tag{39}$$

where $Q_\pm$ are the 4 by 4 supercharges of Witten SUSY $N = 2$ model, viz.

$$Q_- = \sigma_- \otimes \mathcal{A}^-, \quad Q_+ = Q_-^\dagger = \begin{pmatrix} 0 & \mathcal{A}^+ \\ 0 & 0 \end{pmatrix} = \sigma_+ \otimes \mathcal{A}^+, \tag{40}$$

with the intertwining operators, $\mathcal{A}^\pm$, in terms of 2x2-matrix superpotential, are given by Eq. (33) and $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, where σ_1 and σ_2 are Pauli matrices. Of course, the bosonic sector Hamiltonian of H_{SUSY} is exactly the fluctuation operator given by $\mathcal{H}_+ = O_F = -\frac{\mathbf{I}d^2}{dz^2} + \mathbf{V}$, where $\mathbf{I}$ is the 2x2 identity matrix, $\mathbf{V} = V_{BPS}$, for the BPS states and $\mathbf{V} = V_{NBPS}$, for non BPS states.

In the case above, where the fluctuating operator is diagonal, we may construct two representations of SUSY in quantum mechanics. Indeed, from (32), we define,

$$W_{ij} = \delta_{ij} W_{(i)}, \tag{41}$$

where W_{ij} stand for the components of $\mathbf{W}(z)$. We are now using the summation convention for repeated indices. A parenthesis suppresses this convention.

The components of $\mathcal{A}^\pm$ follow from (33),

$$\mathcal{A}_{ij}{}^{\pm} = \delta_{ij}(\pm\frac{d}{dz} + W_{(i)}(z)); \tag{42}$$

also, the components of the fluctuaton operator read:

$$O^+_{F_{ij}} = \mathcal{A}^+_{ij}\mathcal{A}^-_{jk} = \delta_{ik}(\frac{d}{dz} + W_{(k)}(z))(-\frac{d}{dz} + W_{(k)}(z)), \tag{43}$$

differently from the previous one. Now, all modes at a general nth-order can vibrate with different frequencies. For this reason, we add another index (k) to account for frequency, so that it distinguishes a k-mode,

$$O^+_{F_{ik}}\psi_k^{(n)} = \mathcal{A}^+_{ij}\mathcal{A}^-_{jk}\psi_k^{(n)} = \delta_{ik}\omega^2_{(k)(n)}\psi_k^{(n)}. \tag{44}$$

This takes us to two representations of SUSY Quantum Mechanics [3], viz.,

$$O^+_{F_{11}}\psi^{(n)}_1 = \mathcal{A}^+_{11}\mathcal{A}^-_{11}\psi^{(n)}_1 = \omega^2_{(1)(n)}\psi^{(n)}_1,$$
$$O^+_{F_{22}}\psi^{(n)}_2 = \mathcal{A}^+_{22}\mathcal{A}^-_{22}\psi^{(n)}_2 = \omega^2_{(2)(n)}\psi^{(n)}_2. \tag{45}$$

It is now possible to build up the respective supersymmetric partners for each SUSY representation, similarly to (38). We have that W_1 and W_2 follow from (31) and (41),

$$W_1 = -2m\tanh(mz),$$
$$W_2 = -2m\frac{\alpha}{\lambda}\tanh(mz), \tag{46}$$

and with the help of [2] (by shape invariance), we have a procedure for constructing a hierarchy of these fluctuation operators, obtaining therefore the eigenfunctions and eigenvalues in these two representations. Here, we present only the eingenfunctions ($\psi^{(0)}_i = c_{(i)}e^{\int dzW_i(z)}$) of the ground state of the operator $O^+_{F_{11}}$ and $O^+_{F_{22}}$, which are respectively given by:

$$\psi^{(0)}_1 = c_{(1)}sech^2(mz),$$
$$\psi^{(0)}_2 = c_{(2)}sech^{\frac{2\alpha}{\lambda}}(mz), \tag{47}$$

where $c_{(i)}$ is the normalization constant of the correponding ground state. The eigenvalue of the ground state becomes $\omega^2_{(i)(0)} = 0$, due to the annihilation condition, viz. $\mathcal{A}^-_{ii}\psi^{(0)}_{(i)} = 0$. The index i identifies which SUSY representation we are referring to. Note that, if $\alpha = \lambda$, we see that the two SUSY representations become equivalent. One should also remark that α and λ have been chosen both positive, so that $\frac{2\alpha}{\lambda} > 0$, so that as $\psi^{(0)}_2(z)$ is a normalisable configuration; also there are 2 normalisable independent zero-mode eigenfuctions such that, $\eta_0 = \psi^{(0)}_1(z)$ and $\xi_0 = \psi^{(0)}_2(z)$.

IV. CONCLUSIONS

In the present work, we investigate, in terms of fluctuation operators, BPS and non-BPS states in 2-dimensional model with minimal $N = 1$ supersymmetry. The corresponding stability equations, for different eingenvalues, have been analyzed without and with supersymmetry in Quantum Mechanics (SUSY QM), so that frequency eigenvalues are identical. A connection between BPS and non-BPS states has been implemented via supersymmetric Quantum Mechanics with two-component wave functions, and the stability equations associated with the soliton solutions of the simple model of two coupled real scalar fields have been investigated, by calculating the tension on two domain walls. In both cases, the domain walls belonging to the BPS and non-BPS states, the zero-mode ground states become two-component eigenfunctions, for the particular case in which we have found the explicit form of different eigenvalues, via two SUSY representations.

A realization of Witten's $N = 2$-SUSY model for this system must necessarily be modified [4]. The essential reason for the necessity of modification is that the Riccati equation given by (30) is reduced to a set of first-order coupled differential equations. In this case, the superpotential is not necessarily and directly defined as $W(z) = -\frac{1}{\psi_+^{(0)}}\frac{d}{dz}\psi_+^{(0)}(z)$, according to the system described by one-component wave functions with $N = 2$ SUSY in the context of non-relativistic quantum mechanics [1,5]. Moreover, as the zero-mode is associated with a two-component eigenfunction, $\Psi_+^{(0)}(z)$, one may write the matrix superpotential only in the form $\frac{d}{dz}\Psi_+^{(0)}(z) = \mathbf{W}\Psi_+^{(0)}(z)$ [4]. However, we can find the eigenmodes of the supersymmetric partner $\mathcal{H}_-$ from those of $\mathcal{H}_+ \equiv O_F$, and the spectral resolution of the hierarchy of matrix Hamiltonians may be achieved in an elegant way, according to the Sukumar' method for SUSY QM [2]. In this case, the intertwining operators $\mathcal{A}^+(\mathcal{A}^-)$ convert an eigenfunction of $\mathcal{H}_-(\mathcal{H}_+)$ in to an eigenfunction of $\mathcal{H}_+(\mathcal{H}_-)$ with the same energy and simultaneously destroys (creates) a node of $\Psi_+^{(n+1)}(z)\left(\Psi_-^n(z)\right)$.

A detailed analysis of this application may be implemented in terms of an elliptical path of the BPS domain walls.

ACKNOWLEDGMENTS

We wish to thank CBPF-MCT and CFP-UFPB of Cajazeiras-PB, Brazil. We would also like to acknowledge J. A. Helaÿel-Neto for many estimulating discussions and criticisms on an earlier manuscript. GSD is grateful to GFT-UCP for the kind hospitality. RLR and GSD would also like to thank the CNPq - Brazil for their post-doctoral and doctoral Graduate fellowships, respectively. The authors would also like to thank Plyushchay, Binosi, Rebhan, Wimmer and Espichán Carrillo for the kind interest in pointing out relevant references on the subject of this paper.

REFERENCES

[1] E. Witten, *Nucl. Phys.* **B185**, 513 (1981); D. L. Pursey, *Phys. Rev.* **D33**, 2267 (1986).

[2] C. Sukumar, *J. Phys. A: Math. Gen.* **18**, L57 (1985); C. Sukumar, *J. Phys. A: Math. Gen.* **18**, 2917 (1985).

[3] G. S. Dias and J. A. Helayel-Neto, "N=2-Supersymmetric Dynamics of a Spin-$\frac{1}{2}$ Particle in an Extended External Field," hep-th/0204220.

[4] R. de Lima Rodrigues, "The Quantum Mechanics SUSY Algebra: an Introductory Review," hep-th/0205017.

[5] L. E. Gendenshtein, *JETP Lett.* **38**, 356 (1983); L. E. Gendenshtein and I. V. Krive, *Sov. Phys. Usp.* **28**, 645 (1985); A. Lahiri, P. K. Roy, and B. Bagchi, *Int. J. Mod. Phys.* **A5**, 1383 (1990); F. Cooper, A. Khare, and U. Sukhatme, *Phys. Rep.* **251**, 267 (1995).

[6] J. Jayaraman and R. de Lima Rodrigues, *J. Phys. A: Math. Gen.* **23**, 3123 (1990); J. Jayaraman and R. de Lima Rodrigues, *Mod. Phys. Lett.* **A9**, 1047 (1994).

[7] J. Jayaraman and R. de Lima Rodrigues, *J. Phys. A: Math. Gen.* **32**, 6643 (1999).

[8] M. S. Plyuschay, *Int. J. of Mod. Phys.* **A15**, 3679 (2000).

[9] Q. Wang, U. P. Sukhatme, and W.-Y. Keung, *Mod. Phys. Lett.* **A5**, 525 (1990); D. S. Kulshreshtha, J.-Q. Liang, and H. J. W. Müller-Kirsten, *Ann. Phys. (N. Y.)* **225**, 191 (1993).

[10] J. Casahorran and S. Nam, *Int. J. Mod. Phys.* **A6**, 5467 (1991); C. V. Sukumar, *J. Phys. A: Math. Gen.* **19**, 2297 (1986); J. Hruby, *ibid.* **22**, 1807 (1989); L. J. Boya. and J. Casahorran, *Ann. Phys. (N. Y.)* **196**, 361 (1989).

[11] C. N. Kumar, *J. Phys. A: Math. Gen.* **20**, 5397 (1989); W.-Y. Keung, U. P. Sukhatme, Q. Wang and T. D. Imbo, *J. Phys. A: Math. Gen.* **22**, L987 (1989).

[12] R. de Lima Rodrigues, *Mod. Phys. Lett* **A10**, 1309 (1995); G. Junker and P. Roy, *Ann. Phys. (N. Y.)* **256**, 302 (1997).

[13] P. Barbosa da Silva Filho, R. de Lima Rodrigues, and A. N. Vaidya, *J. Phys. A: Math. Gen.* **32**, 2395-2402 (1999).

[14] V. Gomes Lima, V. Silva Santos and R. de Lima Rodrigues, *Phys. Lett.* **A298**, 91-97 (2002), hep-th/0204175.

[15] R. de Lima Rodrigues and A. N. Vaidya, "*SUSY QM for the Two-Component Wave Functions*", XVII Brazilian National Meeting on Particles and Fields, pp. 622 (1996).

[16] R. de Lima Rodrigues, P. Barbosa da Silva Filho, and A. N. Vaidya, *Phys. Rev.* **D58**, 125023 (1998).

[17] R. Rajaraman, *Phys. Rev. Lett.*, **42**, 200 (1979); R. Rajaraman, *Solitons and Instantons*, (North-Holland, Amsterdam, 1982).

[18] J. A. Epichán Carrilo, A. Maia Jr. and V. M. Monstapanenko, *Int J. Mod. Phys.* **A15**, 2645 (2000).

[19] J. A. Epichán Carrilo and A. Maia Jr., *Int. Theor. Phys.* **38**, 2183 (1999), hep-th/9905158; J. A. Epichán Carrilo and A. Maia Jr., *J. Phys.* **A33**, 2081 (2000).

[20] R. de Lima Rodrigues, V. B. Bezerra and A. N. Vaidya, *Phys. Lett* **287A**, 45 (2001).

[21] I. Voronin, *Phys. Rev.* **A43**, 29 (1991).

[22] L. Vestergaard Hau, J. A. Golovchenko and Michael M. Burns, *Phys. Rev. Lett.* **74**, 3138 (1995).

[23] A. Losev, M. Shifman, and A. Vainshtein, *Phys. Rev.*, **B522**, 327, (2001).

[24] M. Shifman, *Phys. Rev.* **D57**, 1258 (1998).
[25] M. Shifman, A. Vainshtein, M. Voloshin, *Phys. Rev.* **D59**, 045016 (1999), hep-th/9810068.
[26] E. B. Bogomol'nyi, *Sov. J. Nucl. Phys.* **24**, 449 (1976).
[27] M. K. Prasad, C. H. Sommerfied, *Phys. Rev. Lett.* **35**, 760 (1975).
[28] A. Ritz, M. Shifman, and A. Vainshtein, *Phys. Rev.*, **D63**, 065018 (2001).
[29] C. A. G. Almeida, D. Bazeia and L. Losano, *J. Phys. A: Math. Gen.* **34**, 3351 (2001), and references therein.
[30] D. Walgraef, *Spatio-Temporal Path Formalism* (New York, Springer-1997).
[31] A. Vilenkin and E. P. S. Shellard, *Cosmic string other topological deffects* (Cambridge, Cambridge University Press-1990).
[32] L. J. Boya and J. Casahorran, *Phys. Rev.* **A39**, 4298 (1989).
[33] J. R. Morris, *Phys. Rev.* **D51**, 697 (1995); J. R. Morris, *Int. J. of Mod. Phys.* **A13**, 1115 (1998).
[34] D. Bazeia and F. A. Brito, *Phys. Rev.* **62D**, 101701 (R)(2000).
[35] S. M. Carroll, S. Hellerman and M. Trodden, *Phys. Rev.* **61D**, 065001 (2000).
[36] D. Bazeia and F. Brito, *Phys. Rev. Lett.* **84**, 1094 (2000).
[37] P. M. Saffin, *Phys. Rev. Lett.* **83** 4249 (1999).
[38] D. Binosi and T. Veldhuis, *Phys. Lett.*, **B476**, 124 (2000).
[39] M. Shifman and T. ter Veldhuis, *Phys. Rev.* **62D**, 065004 (2000).
[40] E. L. Graça and R. de Lima Rodrigues, *Non topological defect associated with two coupled real fields*, Preprint CBPF-NF-013/02 (To appear in the Proceedings of the XXII Brazilian National Meeting on Particles and Fields, into the site www.sbf.if.usp.br.), hep-th/0205088.
[41] N. Grahan and R. L. Jaffe, *Nucl. Phys.* **B544**, 432 (1999), hep-th/9808140.
[42] H. Nastase, M. Stephanov, P. van Nieuwenhuizen and A. Rebhan, *Nucl. Phys.* **B542**, 471-514 (1999).
[43] G. H. Flores and N. F. Svaiter, *Phys. Rev.* **D66**, 025031 (2002), hep-th/0107043; G. Flores-Hidalgo, *Phys. Lett.* **B542**, 282 (2002), hep-th/0206047.
[44] A. Rebhan and P. van Nieuwenhuizen, *Nucl. Phys.* **B508**, 449-467 (1997).
[45] A. Litvintsev and P. van Nieuwenhuizen, "Once more on the BPS bound for the SUSY kink", hep-th/0010051.
[46] A. S. Goldhaber, A. Litvintsev and P. van Nieuwenhuizen, *Phys. Rev.* **D64**, 045013 (2001)
[47] R. Wimmer, "Quantization of supersymmetric solitons", hep-th/0109119.
[48] M. Bordag, A. S. Goldhaber, P. van Nieuwenhuizen and D. Vassilevich, "Heat kernels and zeta-function regularization for the mass of the SUSY kink", hep-th/0203066.
[49] A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, *New Jour. Phys.* **4**, 31 (2002).
[50] A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, "The anomaly in the central charge of the supersymmetric kink from dimensional regularization and reduction," hep-th/0207051.
[51] A. Alonso Izquierdo, W. Gárcia Fuertes, M. A. González León and J. Mateos Guilarte, *Nucl. Phys.* **B635**, 525-557 (2002).
[52] A. P. Balachandran, G. Marmo, B. S. Skagerstam and A. Stern, *Classical Topology and Quantum States*, (World Scientific, Singapore, 1991).

[53] E. B. Bogomol'nyi, *Sov. J. Nucl. Phys.* **24**, 489 (1976).

[54] P. M. Morse and H. Feshbach, **Methods of Mathematical Physics, Vol. II**, McGraw-Hill Book, New York, p. 1650 (1953).

FIGURES

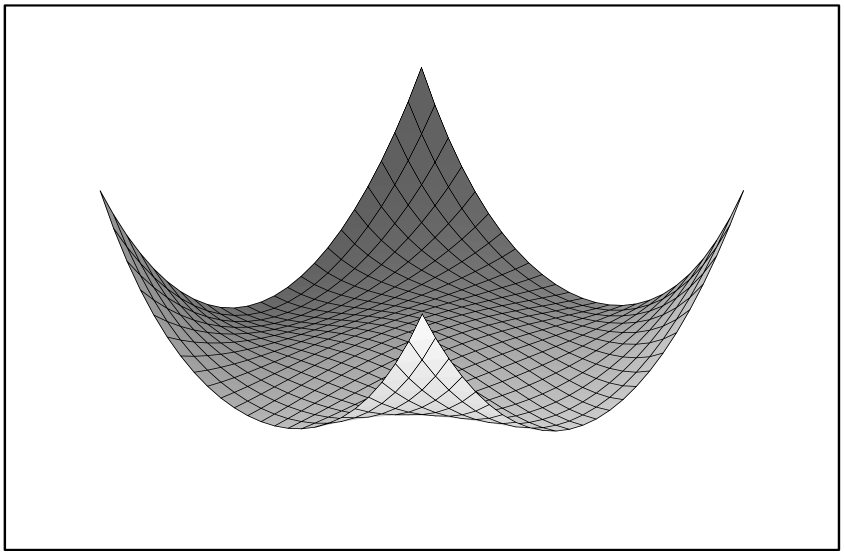

FIG. 1. The two coupled field potential model.

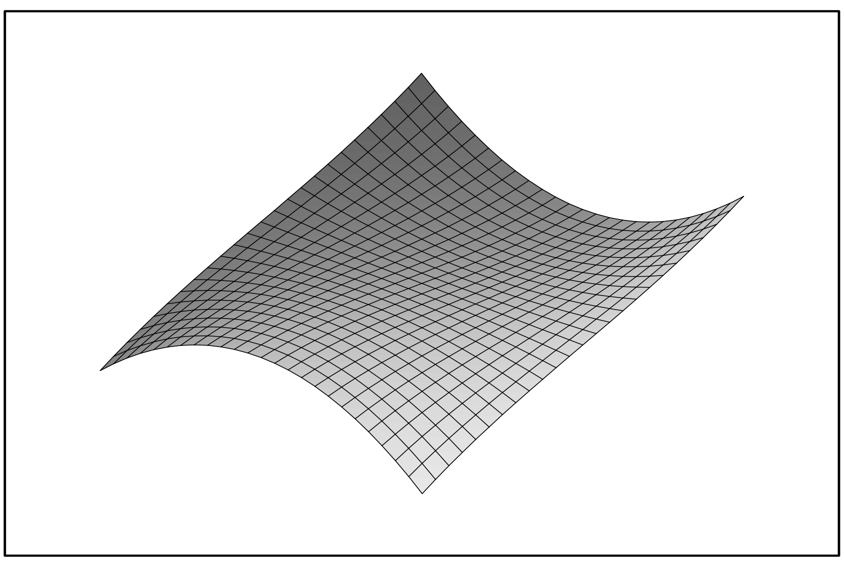

FIG. 2. The polynomial superpotential that generates the BPS domain wall.

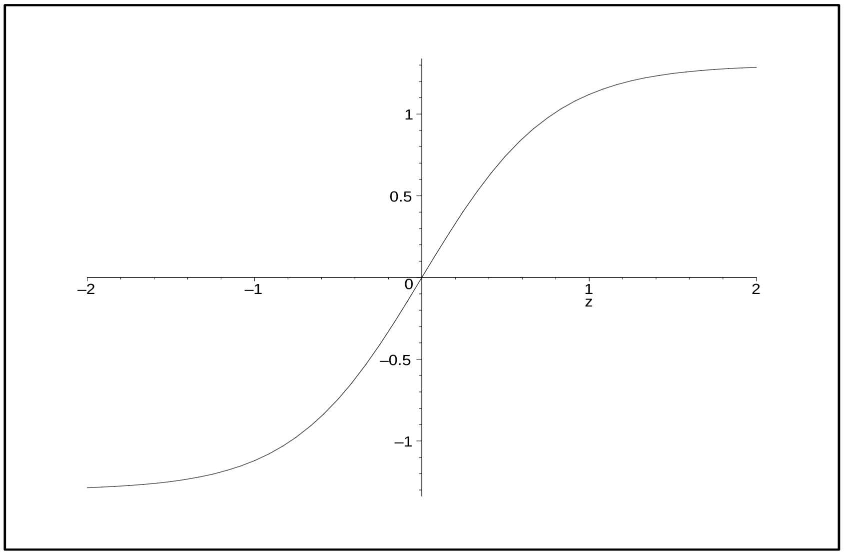

FIG. 3. Static classical configuration which represents the domain wall.

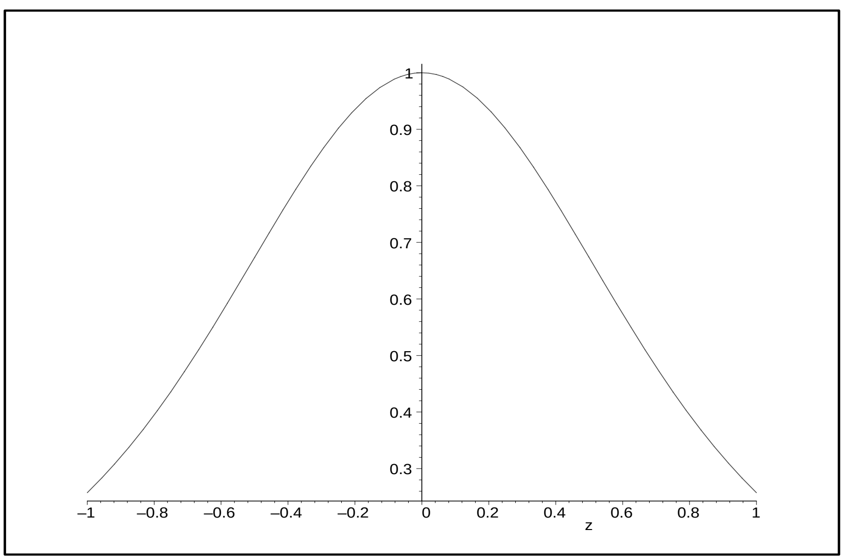

FIG. 4. The eigenfunction of the zero mode.