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On the Classification of N-extended Supersymmetric Quantum Mechanical Systems

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Abstract

In this paper some properties of the irreducible multiplets of representation for the $N=(p,q)$ – extended supersymmetry in one dimension are discussed. Essentially two results are here presented. At first a peculiar property of the one dimension is exhibited, namely that any multiplet containing $2d$ (d bosonic and d fermionic) particles in M different spin states, is equivalent to a $\{\mathbf{d},\mathbf{d}\}$ multiplet of just 2 spin states (all bosons and all fermions being grouped in the same spin). Later, it is shown that the classification of all multiplets of this kind carrying an irreducible representation of the N – extended supersymmetry is in one-to-one correspondence with the classification of real-valued Clifford Γ-matrices of Weyl type. In particular, $p+q$ is mapped into D, the space-time dimensionality, while $2d$ is determined to be the dimensionality of the corresponding Γ-matrices. The implications of these results to the theory of spinning particles are analyzed.

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1 Introduction

Recently we have assisted at a regain of the interest in the theory of Supersymmetric and Superconformal Quantum Mechanics due to different physical motivations and viewpoints. Supersymmetric and superconformal [1] (see also [2]) quantum mechanical models succeed in describing the low-energy effective dynamics, as well as the moduli space, of a certain class of black holes. Particle models with extended world-line supersymmetries naturally describe the related geometries, see [3]. Another scenario involving Supersymmetric Quantum Mechanics (SQM) concerns the light-cone quantization of supersymmetric theories [4]. Besides that, (SQM) models offer a natural set-up for testing, under a rigorous mathematical framework, some conjectures (like the AdS/CFT correspondence for AdS_2) or properties and consequences of dimensionally reduced supersymmetric field theories [5] and such phenomena as their spontaneous supersymmetry breaking [6], [7], including the partial breaking [8], [9]. Having this in mind, the importance of the investigation of large N-extended (SQM) models cannot be overestimated. Indeed, since the reduced version to a one (temporal) dimension of a supersymmetric $4d$ theory gets 4 times the number of supersymmetries of the original model, $N=2,4$ Super-Yang-Mills are reduced to $N=8$ and respectively $N=16$ SQM models, while the $N=8$ supergravity is associated with the $N=32$ SQM theory.

Not much attention has been paid however to such large–N susy quantum models and only partial results are known [10], [5]. The reason however is clear, $N=4$ is the largest number of extended supersymmetry for which a superfield formalism is known. Investigating the $N>4$ case requires the use of component fields and computations soon become cumbersome.

In this paper we attack the problem of investigating large N SQM models from a different viewpoint. We are able to classify the irreducible multiplets of representations of the N extended supersymmetry. We prove at first that all such multiplets are associated to fundamental short multiplets in which all bosons and all fermions are accommodated into just two spin states. In consequence of that, differently presented dynamical systems turn out to be expression of the same algebraic structure. Later, we give the full classification of the short multiplets. We further mention how the above results find application to the theory of the particles with spin.

The closest references to the results here presented are given by the papers [11, 12] in which the classification of (in our language) the short multiplets for the Euclidean Supersymmetry was derived, i.e. representations of the (1.2) algebra with positive eigenvalues only. See also [13] for explicit constructions of multiplets of extended supersymmetries.

In our work we further prove that all multiplets fit into equivalence classes characterized by the short multiplets. Besides that, we extend the classification of [11, 12] to the pseudo-Euclidean Supersymmetry (arbitrary signatures of the eigenvalues of the ω_{ij} matrix in the formula (1.2)). Indeed, as we will prove in the following, it is in this larger class that symmetries of the particles with spin moving in a Minkowskian or AdS-like background should be looked for. The analysis which follows is based on the results for the classification of real-valued Clifford algebras as presented in [14].

It is well known that the SQM, being the simplest example of a theory which includes simultaneously commuting and anticommuting variables, realizes as its symmetry group the one – dimensional supersymmetry. In general this supersymmetry is generated by N supercharges Q_i, $i = 1, 2, \cdots, N$ and the Hamiltonian

$$H = -i\frac{\partial}{\partial t} \tag{1.1}$$

with the following algebra

$$\{Q_i, Q_j\} = \omega_{ij} H, \tag{1.2}$$

where the constant tensor ω_{ij} has p positive and q negative eigenvalues. Usually all eigenvalues are positive and the above algebra is named the N – extended one – dimensional supersymmetry. Nevertheless, in general, reasons can exist leading to an indefinite tensor ω_{ij} [15]. In the following, without loss of generality, the algebra of supercharges Q_i's will be conveniently diagonalized and normalized in such a way that the tensor ω_{ij} can be expressed as

$$\omega_{ij} = \eta_{ij}, \tag{1.3}$$

where η_{ij} is a pseudo-Euclidean metric with the signature (p, q).

The representation of the algebra (1.2) is formed by commuting (Bosonic) and anticommuting (Fermionic before the quantization and Clifford after it) variables. Some of them are true physical variables, others play an auxiliary role. Usually all these variables are taken to be the components of irreducible superfields.

The simplest way to construct a classical Lagrangian for the SQM in D dimensions is to consider the superfields $(A = 1, 2, \cdots D)$

$$\Phi_A(\tau, \eta^\alpha) = \Phi_A^0(\tau) + \eta^\alpha \Phi_{A\alpha}^1(\tau) + \eta^{\alpha_1}\eta^{\alpha_2}\Phi_{A\alpha_1\alpha_2}^2(\tau) + \cdots + \eta^{\alpha_1}\eta^{\alpha_2}\cdots\eta^{\alpha_N}\Phi_{A\alpha_1\alpha_2\cdots\alpha_N}^N(\tau) \tag{1.4}$$

in the superspace (τ, η^α) with one bosonic coordinate τ and N Grassmann coordinates η^α. Such superfields for general N are highly reducible and only lower values of N were investigated in details. The first components of the superfields are the usual bosonic coordinates $\Phi_A^0(\tau)$, the next ones $\Phi_{A\alpha}^1(\tau)$ are the Grassmann coordinates. All the other components of the superfields are auxiliary. So, the classical Lagrangian of the SQM describes the evolution of bosonic and additional Grassmann degrees of freedom, which after quantization become generators of the Clifford algebra. This fact naturally leads to the matrix realization of the Hamiltonian and supercharges of SQM [6], [16], [17].

The dimensionality of such realization depends on the total number of Grassmann variables. In the case of scalar superfields (1.4) the dimensionality is $2^{[\frac{DN}{2}]}$. So, it rapidly growths for extended supersymmetry. The way out of this difficulty is to use more complicated representations of the extended supersymmetry [18] - [23]. The simplest of them is given by the chiral superfield, which contains one complex bosonic and $\frac{N}{2}$ complex Grassmann fields. The Lagrangian for such superfield naturally describes the two - dimensional SQM. The ratio of numbers (fermi/boson) in this case is $\frac{N}{2}$ instead of N as for scalar superfields. For more complicated representations this ratio grows even more slowly [19] - [23]. This fact has an essential influence on the dimensionality of the matrix realizations of the Hamiltonian and Supercharges - they are smaller for the same number of bosonic coordinates Φ_A^0. The distinguishing feature of such representations is the fact that the lowest components $\Phi_A^0(\tau)$ $(A = 1, 2, \cdots D)$ of the superfields (1.4) *all together* form an irreducible representation of some subgroup of the automorphism group of the algebra (1.2). The corresponding actions are also invariant under the transformations of this subgroup which thus plays the role of space (or space-time) rotations. In particular the inclusion of the time coordinate $t(\tau)$ along with the space ones $x^a(\tau)$ in an irreducible representation $\Phi^0_A(\tau)$ of such subgroup means that such a subgroup, as well as the whole automorphism group of the algebra (1.2), is pseudo-Euclidean [15]. In consequence of that the metric tensor η_{ij} in (1.3) is pseudo-Euclidean too.

So, the numbers of bosonic and fermionic physical components and, correspondingly, the dimensionality of quantum Hamiltonian and supercharges realized as matrices, crucially depend on the choice of the irreducible superfield or, equivalently, irreducible representation of the algebra (1.2). In this sense the classification of all such representations is very useful.

2 The equivalence relations

In this section we analyze the structure of the supermultiplets of the N – extended supersymmetry in one dimensional space. We will show in which sense all irreducible representations are equivalent to the representations with a definite structure of the multiplets. To fix the notation, let N denotes the number of extended supersymmetries and $2d$ (d bosons and d fermions) the dimensionality of the corresponding multiplet, (may be reducible), carrying the N-extended SUSY representation. In general such multiplet can be represented in the form of a chain

$$\Phi^0_{a_0}, \quad \Phi^1_{a_1}, \quad \cdots, \quad \Phi^{M-1}_{a_{M-1}}, \quad \Phi^M_{a_M} \tag{2.1}$$

whose components $\Phi^I_{a_I}$, $(a_I = 1, 2, \cdots, d_I)$ are real. All the components with even I have the same grassmann parity as that of $\Phi^0_{a_0}$, while the components with odd values of I have the opposite one. Such structure of the multiplet is closely related with the superfield representations; the upper index I, numbering the elements of the chain, corresponds to their place in the superfield expansion (1.4) or to their dimensionality which decreases by 1/2 at each step along the chain if one chooses

$$dim\ (\tau) = 1, \quad dim\ (\eta) = \frac{1}{2}. \tag{2.2}$$

The number $M+1$ is the length of the supermultiplet, M being subjected to the constraint $M \leq N$ since, for example in the case of irreducible representations, not all the components of the superfield (1.4) are independent - some of the higher components are expressed in terms of time derivatives of the lower ones.

For the supermultiplet (2.1) we will also use the short notation $\{\mathbf{d_0}, \mathbf{d_1}, \cdots, \mathbf{d_M}\}$. As an example, in the case of $N=2$ one can consider the irreducible representations $\{\mathbf{1}, \mathbf{2}, \mathbf{1}\}$ (real superfield)

$$\Phi = \Phi(\tau, \eta_1, \eta_2) = \Phi^0(\tau) + i\eta^\alpha \Phi^1_\alpha(\tau) + i\eta^1\eta^2\Phi^2(\tau), \tag{2.3}$$

and $\{\mathbf{2}, \mathbf{2}\}$ (chiral superfield)

$$\tilde{\Phi}(\tau, \eta, \overline{\eta}) = \tilde{\Phi}^0(\tau) + \eta\tilde{\Phi}^1(\tau) + \frac{i}{2}\overline{\eta}\eta\dot{\tilde{\Phi}}^0(\tau), \tag{2.4}$$

where $\eta = \eta_1 + i\eta_2, \quad \overline{\eta} = \eta_1 - i\eta_2$ are complex Grassmann coordinates. The last component in the expression (2.4) is proportional to the time derivative of the first one. Both $\tilde{\Phi}^0(\tau)$ and $\tilde{\Phi}^1(\tau)$ in (2.4) are complex. Obviously, in both cases (2.3) and (2.4), $\sum_I d_I = 2$ separately for real bosonic and fermionic components.

Due to a dimensionality arguments the supersymmetry transformation law for the components $\Phi^I_{a_I}$ is of the following form (ε^i are infinitesimal Grassmann parameters)

$$\delta_\varepsilon \Phi^I_{a_I} = \varepsilon^i (C^I_i)_{a_I}{}^{a_{I+1}} \Phi^{I+1}_{a_{I+1}} + \varepsilon^i (\tilde{C}^I_i)_{a_I}{}^{a_{I-1}} \frac{d}{d\tau} \Phi^{I-1}_{a_{I-1}}. \tag{2.5}$$

Evidently, due to the absence in (2.1) of the components with $I = -1, \quad I = M+1$, the transformation laws for the end components of the chain are simpler:

$$\delta_\varepsilon \Phi^0_{a_0} = \varepsilon^i (C^0_i)_{a_0}{}^{a_1} \Phi^1_{a_1}, \quad \delta_\varepsilon \Phi^M_{a_M} = \varepsilon^i (\tilde{C}^M_i)_{a_M}{}^{a_{M-1}} \frac{d}{d\tau} \Phi^{M-1}_{a_{M-1}}. \tag{2.6}$$

The transformation law for the last component of the multiplet reads that it transforms as a total derivative. It is a very essential property, because the integral of this component is invariant under the supersymmetry transformations and can be used to construct invariant actions.

Another very important consequence of the transformation law of the last components is present only in one dimension. Just in this case one can redefine this component

$$\Phi^M_{a_M} = \frac{d}{d\tau} \Psi^{M-2}_{a_M} \tag{2.7}$$

in terms of some functions $\Psi^{M-2}_{a_M}$. This correspondence is exact up to some constants $C^M_{a_M}$ which describe the trivial representations of the supersymmetry algebra. The dimensionality of the new components $\Psi^{M-2}_{a_M}$ coincides with the dimensionality of the components $\Phi^{M-2}_{a_{M-2}}$. Moreover, their transformation law is of the same type – they transform through the components $\Phi^{M-1}_{a_{M-1}}$ and $\Phi^{M-3}_{a_{M-3}}$ (with vanishing coefficients before the time derivative of the last ones)

$$\delta_\varepsilon \Psi^{M-2}_{a_M} = \varepsilon^i (\tilde{C}^M_i)_{a_M}{}^{a_{M-1}} \Phi^{M-1}_{a_{M-1}}. \tag{2.8}$$

So, we have shown that up to trivial representations of the supersymmetry algebra the supermultiplet $\{\mathbf{d_0}, \mathbf{d_1}, \cdots, \mathbf{d_{M-2}}, \mathbf{d_{M-1}}, \mathbf{d_M}\}$ is equivalent to the supermultiplet $\{\mathbf{d_0}, \mathbf{d_1}, \cdots, \mathbf{d_{M-2}} + \mathbf{d_M}, \mathbf{d_{M-1}}, \mathbf{0}\}$. It means that the initial supermultiplet of length $M+1$ is equivalent to a shorter multiplet of length M. Evidently, the total number of bosonic and fermionic components in both supermultiplets is the same. This procedure can obviously be repeated $M-1$ times, so that at the end one reaches the shortest multiplet of length 2 - the multiplet $\{\mathbf{d}, \mathbf{d}\}$.

The simplest example of such shortening of the length is given in the case of $N = 2$. The component $\Phi^2(\tau)$ in the real superfield (2.3) transforms as a total derivative of some new field $\Psi^0(\tau)$ which, together with $\Phi^0(\tau)$, forms the complex $\tilde{\Phi}^0(\tau)$ of (2.4). A more complicated example is furnished by the $N = 4$ representation multiplets $\{\mathbf{1}, \mathbf{4}, \mathbf{3}\}$, $\{\mathbf{2}, \mathbf{4}, \mathbf{2}\}$ and $\{\mathbf{3}, \mathbf{4}, \mathbf{1}\}$ which were used in [19, 20, 8] for one-dimensional, [23] for two-dimensional and [21, 22] for three-dimensional SQM respectively. The corresponding components of these representations are interconnected by the transformation (2.7). To our knowledge, the multiplet $(4,4)$ which should be useful in four-dimensional case was not considered in the literature.

In principle, one can consider the inverse procedure - the lengthening of the multiplet starting from the $\{\mathbf{d},\mathbf{d}\}$'s one. Only the first step is trivial - the transition from the $\{\mathbf{d},\mathbf{d}\}$ multiplet to the $\{\mathbf{d}-\mathbf{d_1},\mathbf{d},\mathbf{d_1}\}$ multiplet can always be done with the help of the transformation inverse to (2.7) applied to an arbitrary number $d_1 \leq d$ of the first components of the initial multiplet. The possibility of further lengthening must be analyzed separately in each particular case and will be shortly discussed at the end of the paper.

It should be noticed that $d_1 = d$ is allowed. The corresponding transformation links two length-2 multiplets, the one in which bosons have higher spin, to the one in which fermions have higher spins. Explicitly we have

$$\delta_\varepsilon \Phi_a = \varepsilon^i (C_i)_a{}^b \Psi_b, \quad \delta_\varepsilon \Psi_a = \varepsilon^i (\tilde{C}_i)_a^b \frac{d}{d\tau}\Phi_b, \tag{2.9}$$

while for $\Xi_a = \frac{d}{d\tau}\Psi_a$ we get

$$\delta_\varepsilon \Xi_a = \varepsilon^i (C_i)_a{}^b \Phi_b, \quad \delta_\varepsilon \Phi_a = \varepsilon^i (\tilde{C}_i)_a^b \frac{d}{d\tau}\Xi_b. \tag{2.10}$$

We finally comment that, due to the previous considerations, the classification of all supermultiplets of length 2 automatically provides the classification of all supermultiplets of length 3. In many physical application of interest, this is quite sufficient.

3 Extended supersymmetries and real-valued Clifford algebras

The main result of the previous Section is that the problem of classifying all N-extended supersymmetric quantum mechanical systems is reduced to the problem of classifying the irreducible representations (2.1) of length 2. Having this in mind we simplify the notations. Let the indices $a,\alpha = 1,\cdots,d$ number the bosonic (and respectively fermionic) elements in the SUSY multiplet. All of them are assumed to depend on the time coordinate τ ($X_a \equiv X_a(\tau)$, $\theta_\alpha \equiv \theta_\alpha(\tau)$).

In order to be definite and without loss of generality let us take the bosonic elements to be the first ones in the chain $\{\mathbf{d},\mathbf{d}\}$, which can be conveniently represented also as a column

$$\Psi = \begin{pmatrix} X_a \\ \theta_\alpha \end{pmatrix}, \tag{3.1}$$

the equations (2.5) are reduced to the following set of equations

$$\begin{aligned} \delta_\varepsilon X_a &= \varepsilon^i (C_i)_a{}^\alpha \theta_\alpha \equiv i(\varepsilon^i Q_i \Psi)_a \\ \delta_\varepsilon \theta_\alpha &= \varepsilon^i (\tilde{C}_i)_\alpha{}^b \frac{d}{d\tau} X_b \equiv i(\varepsilon^i Q_i \Psi)_\alpha \end{aligned} \tag{3.2}$$

where, as a consequence of (1.2),

$$C_i \tilde{C}_j + C_j \tilde{C}_i = i\eta_{ij} \tag{3.3}$$

and

$$\tilde{C}_i C_j + \tilde{C}_j C_i \;\; = \;\; i\eta_{ij} \tag{3.4}$$

Since $\varepsilon_i, X_a, \theta_\alpha$ are real, the matrices C_i's, $\tilde{C}_i$'s have to be respectively imaginary and real. If we set (just for normalization)

$$\begin{aligned} C_i &= \frac{i}{\sqrt{2}}\sigma_i \\ \tilde{C}_i &= \frac{1}{\sqrt{2}}\tilde{\sigma}_i \end{aligned} \tag{3.5}$$

and accommodate $\sigma_i, \tilde{\sigma}_i$ into a single matrix

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \tilde{\sigma}_i & 0 \end{pmatrix}, \tag{3.6}$$

they form a set of real-valued Clifford Γ-matrices of Weyl type (i.e. block antidiagonal), obeying the (pseudo-) Euclidean anticommutation relations

$$\{\Gamma_i, \Gamma_j\} \;\; = \;\; 2\eta_{ij}. \tag{3.7}$$

Conversely, given a set of (pseudo-) Euclidean real-valued Clifford Γ-matrices of Weyl type, one can invert the above procedure and reconstruct the supercharges Q_i

$$Q_i = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_i \\ \tilde{\sigma}_i \cdot H & 0 \end{pmatrix} \tag{3.8}$$

in the basis (3.1).

In addition to the matrices Γ^i (3.6) in the space of vectors (3.1) the further matrix Γ^{N+1}, which anticommutes with the supercharges and corresponds to the fermionic number, exists

$$\Gamma_{N+1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.9}$$

Altogether the matrices (3.6) and (3.9) form the real-valued representation Γ_I of the (pseudo-) Euclidean Clifford algebra with the signature $(p+1, q)$.

Instead of (3.5) one can take

$$\begin{aligned} C_i &= \frac{i}{\sqrt{2}}\sigma_i \\ \tilde{C}_i &= -\frac{1}{\sqrt{2}}\tilde{\sigma}_i \end{aligned} \tag{3.10}$$

and accommodate $\sigma_i, \tilde{\sigma}_i$ into the matrices (3.6) which now obey the (pseudo-) Euclidean anticommutation relations

$$\{\tilde{\Gamma}_i, \tilde{\Gamma}_j\} \;\; = \;\; -2\eta_{ij} \tag{3.11}$$

with opposite to (3.7) sign of the righthand side. Together with fermion number matrix (3.9) new matrix $\tilde{\Gamma}_i$ form the real-valued representation of the (pseudo-) Euclidean Clifford algebra with the signature $(q+1,p)$. This fact means that the representations of $C_{p+1,q}$ and $C_{q+1,p}$ should be connected one with the other. Indeed, this connection is established by the correspondence

$$\tilde{\Gamma}_i = \Gamma_{N+1}\Gamma_i. \tag{3.12}$$

Thus, the representations of the (p,q)- extended supersymmetry algebra (1.2) are in one-to-one correspondence with the real-valued representations of the Clifford algebra $C_{p+1,q} \sim C_{q+1,p}$.

In general the real Clifford algebras were classified in [24] (for the compact case $q=0$) and in [25] (for the noncompact case). The construction along the lines (3.2)-(3.9) for representations of the type $\{\mathbf{d},\mathbf{d}\}$ in the case of positively definite signature $(p,q)=(N,0)$ was performed in [11] (see also [12]) where the dimensionalities as well as realizations of the Γ-matrices (3.6) were described. In the case of pseudo-Euclidean metric with signature (p,q) such construction extensively uses the considerations of the papers [14]. The results will be presented in the next section.

4 Classification of the irreducible representations.

According to the previous Section results, the classification of irreducible multiplets of representation of a (p,q) extended supersymmetry is in one – to – one correspondence with the classification of the real Clifford algebras $C_{p,q}$ with the further property that the Γ matrices can be realized in Weyl (i.e. block antidiagonal) form.

For what concerns real matrix representations of the Clifford algebras we borrow the results of [14]. Three cases have to be distinguished for real representations, specified by the type of most general solution allowed for a real matrix S commuting with all the Clifford Γ_i matrices, i.e.

i) the normal case, realized when S is a multiple of the identity,
ii) the almost complex case, for S being given by a linear combination of the identity and of a real $J^2=-\mathbf{1}$ matrix,
iii) finally the quaternionic case, for S being a linear combination of real matrices satisfying the quaternionic algebra.

Real irreducible representations of normal type exist whenever the condition $p-q=0,1,2 \quad mod \quad 8$ is satisfied (their dimensionality being given by $2^{[\frac{N}{2}]}$, where $N=p+q$), while the almost complex and the quaternionic type representations are realized in the $p-q=3,7 \quad mod \quad 8$ and in the $p-q=4,5,6 \quad mod \quad 8$ cases respectively. The dimensionality of these representations is given in both cases by $2^{[\frac{N}{2}]+1}$.

We further require the extra-condition that the real representations should admit a block antidiagonal realization for the Clifford Γ matrices. This condition is met for $p-q=0 \quad mod \quad 8$ in the normal case (it corresponds to the standard Majorana-Weyl requirement), $p-q=7 \quad mod \quad 8$ in the almost complex case and $p-q=4,6 \quad mod \quad 8$ in the quaternionic case. In all these cases the real irreducible representation is unique.

The above results can be summarized as follows, expressing the dimensionality of the irreducible representations of the algebra (1.2) (independently of the length $M+1$ of the chain (2.1)) as function of the signature (p,q). Let $q = 8k+m,\quad 0 \le m \le 7$ and $p = 8l+m+n,\quad 1 \le n \le 8$ ($l=-1$ when $k=0$ and $p \le q$). Then, the dimensionalities of the bosonic (fermionic) spaces are given by the expression

$$d = 2^{4k+4l+m} \cdot G(n), \tag{4.1}$$

where the so called Radon-Hurwitz function $G(n)$ is defined with the help of the table which can be encountered in[12]

n	1	2	3	4	5	6	7	8
$G(n)$	1	2	4	4	8	8	8	8

(4.2)

By words, $G(n) = 2^r$, where r is the nearest integer which is greater or equal to $\log_2 n$.

A second useful table expresses conversely which kind of signatures (p,q) are possible for a given dimensionality of the bosonic and fermionic spaces. In order to do so it is convenient to introduce the notion of maximally extended supersymmetry. The $C_{p,q}$ ($p-q=6 \quad mod \quad 8$) real representation for the quaternionic case can be recovered from the $7 \quad mod \quad 8$ almost complex $C_{p+1,q}$ representation by deleting one of the Γ matrices; in its turn the latter representation is recovered from the $C_{p+2,q}$ normal Majorana-Weyl representation by deleting another Γ matrix. The dimensionality of the three representations above being the same, the normal Majorana-Weyl representation realizes the maximal possible extension of supersymmetry compatible with the dimensionality of the representation. In search for the maximal extension of supersymmetry we can therefore limit ourselves to consider the normal Majorana-Weyl representations, as well as the quaternionic ones satisfying the $p-q=4 \quad mod \quad 8$ condition.

Let therefore be $p = 8l+m+8+4\epsilon$ and $q = 8k+m$, where the range of values for k,l,m is the same as before, while ϵ assumes two values, distinguishing the Majorana-Weyl ($\epsilon=0$) and the quaternionic case ($\epsilon=1$). A space of $d=2^t$ bosonic and $d=2^t$ fermionic states can carry the following set of maximally extended supersymmetries

$$(p = t-4z+5-3\epsilon, q = t+4z+\epsilon-3) \tag{4.3}$$

where the integer $z = k-l$ must take values in the interval

$$\frac{1}{4}(3-t-\epsilon) \le z \le \frac{1}{4}(t+5-3\epsilon) \tag{4.4}$$

in order to guarantee the $p \ge 0$ and $q \ge 0$ requirements. It is convenient also to represent the answer by the following table

d	(p,q)
2^{4l}	$(8l-4k+1,4k+1),(8l-4k-2,4k+2)$
2^{4l+1}	$(8l-4k+2,4k+2),(8l-4k-1,4k+3)$
2^{4l+2}	$(8l-4k+4,4k),(8l-4k+3,4k+3)$
2^{4l+3}	$(8l-4k+8,4k),(8l-4k+5,4k+1)$

(4.5)

where k is integer satisfying the only conditions $p \geq 0, \quad q \geq 0$.

For the lowest values of dimensionality d the solutions are given by the table:

d	(p,q)
1	$(1,1)$
2	$(2,2)$
4	$(4,0),(3,3),(0,4)$
8	$(8,0),(5,1),(4,4),(1,5),(0,8)$
16	$(9,1),(6,2),(5,5),(2,6),(1,9)$
32	$(10,2),(7,3),(6,6),(3,7),(2,10)$

(4.6)

As already recalled, obviously the representations (p',q') with $p' \leq p, q' \leq q$ also exist for the same dimensionality d. These representations are also irreducible unless either p' or q' become too small. For example, the $d = 16$-dimensional representations are irreducible not only for the signature $(p,q) = (5,5)$, but also for the pairs $(5,4),(5,3),(5,2),(4,5),(3,5),(2,5)$, while the irreducible representations for the signatures $(5,1),(4,4),(1,5)$ are encountered in $d = 8$ dimensions.

5 Examples of representations for supercharges.

For the case $(p,q) = (4,0)$ the following matrices realize four supercharges:

$$Q_1 = \frac{1}{\sqrt{2}}\left|\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ \hline 0&0&0&H&0&0&0&0\\ 0&0&H&0&0&0&0&0\\ 0&H&0&0&0&0&0&0\\ H&0&0&0&0&0&0&0 \end{array}\right| \qquad Q_2 = \frac{1}{\sqrt{2}}\left|\begin{array}{cccc|cccc} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ \hline 0&0&H&0&0&0&0&0\\ 0&0&0&-H&0&0&0&0\\ H&0&0&0&0&0&0&0\\ 0&-H&0&0&0&0&0&0 \end{array}\right| \qquad (5.1)$$

$$Q_3 = \frac{1}{\sqrt{2}}\left|\begin{array}{cccc|cccc} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ \hline H&0&0&0&0&0&0&0\\ 0&H&0&0&0&0&0&0\\ 0&0&-H&0&0&0&0&0\\ 0&0&0&-H&0&0&0&0 \end{array}\right| \qquad Q_4 = \frac{1}{\sqrt{2}}\left|\begin{array}{cccc|cccc} 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ \hline 0&-H&0&0&0&0&0&0\\ H&0&0&0&0&0&0&0\\ 0&0&0&H&0&0&0&0\\ 0&0&-H&0&0&0&0&0 \end{array}\right|$$

These supercharges act in the space with 4 bosonic and 4 fermionic coordinates forming the representation $\{\mathbf{4},\mathbf{4}\}$. The automorphism group $SO(p,q)$ of the algebra (1.2) is now $SO(4)$.

Besides the transformations of the automorphism group $Q'_i = \Lambda_i^j Q_j$ the algebra of supercharges is invariant under the more general transformations of the type

$$Q'_i = U Q_i U^{-1} \tag{5.2}$$

with block-diagonal 8×8 matrices U. When the matrix U is nonsingular and real the transformation (5.2) simply means a change of basis in bosonic and fermionic sectors. On the other hand this transformation drastically changes the representation when U depends on the operator $H = -id/d\tau$. In this case the transformation (5.2) is in general nonlocal. Nevertheless, transformations exist which do not lead to any nonlocality. In particular, in the framework of the example (5.1) one can take

$$U_1 = diag\{1,1,1,H,1,1,1,1\} \tag{5.3}$$

and obtain the new realization for the operators Q_i

$$Q_1 = \frac{1}{\sqrt{2}} \left|\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&H&0&0&0\\ \hline 0&0&0&1&0&0&0&0\\ 0&0&H&0&0&0&0&0\\ 0&H&0&0&0&0&0&0\\ H&0&0&0&0&0&0&0 \end{array}\right| \quad Q_2 = \frac{1}{\sqrt{2}} \left|\begin{array}{cccc|cccc} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-H&0&0\\ \hline 0&0&H&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ H&0&0&0&0&0&0&0\\ 0&-H&0&0&0&0&0&0 \end{array}\right|$$

$$Q_3 = \frac{1}{\sqrt{2}} \left|\begin{array}{cccc|cccc} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-H\\ \hline H&0&0&0&0&0&0&0\\ 0&H&0&0&0&0&0&0\\ 0&0&-H&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0 \end{array}\right| \quad Q_4 = \frac{1}{\sqrt{2}} \left|\begin{array}{cccc|cccc} 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&H&0\\ \hline 0&-H&0&0&0&0&0&0\\ H&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-H&0&0&0&0&0 \end{array}\right| \tag{5.4}$$

in which all the elements of the last column in the left off-diagonal block have lost the multiplier H. Instead, all the elements of the last arrow in the right off-diagonal block acquired H as a multiplier. This representation of the supercharges corresponds to the irreducible supermultiplet $\{\mathbf{3},\mathbf{4},\mathbf{1}\}$ which was used in [21, 22] for constructing the 3-dimensional $N = 4$ extended SQM. The supermultiplets $\{\mathbf{2},\mathbf{4},\mathbf{2}\}$ and $\{\mathbf{1},\mathbf{4},\mathbf{3}\}$ are derived with the help of the following matrices U

$$U_2 = diag\{1,1,H,H,1,1,1,1\}, \; U_3 = diag\{1,H,H,H,1,1,1,1\}. \tag{5.5}$$

The next one in this sequence

$$U_4 = diag\{H,H,H,H,1,1,1,1\} \tag{5.6}$$

gives again the supermultiplet $\{\mathbf{4},\mathbf{4}\}$ but with the opposite grading - the first in the chain is the fermionic subspace. This completes the classification of the irreducible supermultiplets of the $N = 4$ extended SQM. One can show that all the irreducible supermultiplets of the (p,q) extended SQM are of length which does not exceed 3 when the constraint

$$d \leq p + q \tag{5.7}$$

is fulfilled. The determination of the possible values of the lengths of irreducible supermultiplets, as well as their detailed structure, in the case when (5.7) is not fulfilled needs a separate investigation.

A simple example of an irreducible supermultiplet of length 4 is given by the $(p,q) = (3,0)$ case, in which the irreducible representation has also $d = 4$ and supercharges in the $\{\mathbf{4},\mathbf{4}\}$ representation are given by Q_1, Q_2, Q_3 in (5.1). Taking

$$U_5 = diag\{1, H, H, H, 1, H, 1, 1\} \quad (5.8)$$

one derives the expressions for all 4 supercharges

$$Q_1 = \frac{1}{\sqrt{2}} \left|\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&H&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&H&0&0&0\\ \hline 0&0&0&1&0&0&0&0\\ 0&0&H&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ H&0&0&0&0&0&0&0 \end{array}\right| \qquad Q_2 = \frac{1}{\sqrt{2}} \left|\begin{array}{cccc|cccc} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-H\\ 0&0&0&0&H&0&0&0\\ 0&0&0&0&0&-1&0&0\\ \hline 0&0&1&0&0&0&0&0\\ 0&0&0&-H&0&0&0&0\\ H&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0 \end{array}\right|$$

$$Q_3 = \frac{1}{\sqrt{2}} \left|\begin{array}{cccc|cccc} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&-H&0\\ 0&0&0&0&0&0&0&-H\\ \hline H&0&0&0&0&0&0&0\\ 0&H&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0 \end{array}\right| \qquad Q_4 = \frac{1}{\sqrt{2}} \left|\begin{array}{cccc|cccc} 0&0&0&0&0&H^{-1}&0&0\\ 0&0&0&0&-H&0&0&0\\ 0&0&0&0&0&0&0&-H\\ 0&0&0&0&0&0&H&0\\ \hline 0&-1&0&0&0&0&0&0\\ H^2&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0 \end{array}\right| \quad (5.9)$$

from which one can easily see that the fourth supercharge Q_4 becomes singular after the transformation. Indeed, just the first three supercharges in (5.9) are realized in the irreducible representation $\{\mathbf{1},\mathbf{3},\mathbf{3},\mathbf{1}\}$ of length 4.

The Weyl-type $C_{0,4}$ representation has been explicitly presented in [14]. Due to the $mod \;\; 8$ property of Γ matrices, it allows, together with $C_{4,0}$, to construct all quaternionic representations of Weyl type for the allowed values of (p,q). For what concerns the Majorana-Weyl representations, an algorithm to explicitly construct them can be found e.g. in [26]. Moreover, the following symmetry property

$$\Gamma_i^T = \begin{cases} \Gamma_i, & i \leq p \\ \\ -\Gamma_i, & (p+1) \leq i \leq (p+q) \end{cases} \quad (5.10)$$

can always assumed to be valid.

6 An application: supersymmetries of the free kinetic lagrangians of the 'spinning' particle.

We present for completeness the analysis of the extended supersymmetric invariances for the simplest action of the 'spinning' particle model, given by the free kinetic term. We use the quotation marks in the word 'spinning' because actually the considered actions describe particles with spin and are different from the spinning particle models in which both fermions and bosons are space-time vectors and bosons, in addition, are scalars with respect to the supersymmetry transformations.

In general the most significant dynamical systems are σ-models presenting a non-linear kinetic term; for such systems the extended supersymmetries put constraints on the metric of the target. We avoid entering this problem here and just limit ourselves to illustrate how invariances under pseudo-Euclidean supersymmetries can arise. We show in fact that a 'spinning' particle evolving in a non-Euclidean background in general admits invariances under pseudo-Euclidean supersymmetries.

We consider the models involving d bosonic fields X_a and d spinors Θ_α collected in the vector Ψ (3.1) (no auxiliary fields are present).

The free kinetic action is given by

$$S_K = \int dt\mathcal{L} = \frac{1}{2}\int dt\Psi^T\Lambda\Psi = \frac{1}{2}\int dt(X,\Theta)\begin{pmatrix}\lambda_1 H^2 & 0\\ 0 & \lambda_2 H\end{pmatrix}\begin{pmatrix}X\\ \Theta\end{pmatrix} \qquad (6.1)$$

$$= \frac{1}{2}\int dt(\dot{X}_a\lambda_1^{ab}\dot{X}_b - i\Theta_\alpha\lambda_2^{\alpha\beta}\dot{\Theta}_\beta) \qquad (6.2)$$

where the structure of the matrix Λ is dictated by the conservation of the fermion number and by dimensional arguments. Both λ_1, λ_2 should be symmetrical in addition: $\lambda_M^T = \lambda_M$.

The invariance of the action under the supersymmetry transformations (3.2)

$$\delta S_K = \frac{i}{\sqrt{2}}\varepsilon^i X_a(\lambda_1\sigma_i - \tilde{\sigma}_i^T\lambda_2)^{a\alpha}H^2\Theta_\alpha = 0, \qquad (6.3)$$

means that the following property of λ's

$$\lambda_1\sigma_i - \tilde{\sigma}_i^T\lambda_2 = 0 \qquad (6.4)$$

should be valid, in accordance with (5.10)

$$\tilde{\sigma}_i^T = \eta^{ii}\sigma_i. \qquad (6.5)$$

It means that in the case of euclidean supersymmetry ($q = 0$) we get

$$\lambda_1 = \lambda_2 = I, \qquad (6.6)$$

where I is a d dimensional identity matrix (see also [12]). In the general case ($q \geq 1$) the following representation for the Γ_i matrices [14] is useful

$$\Gamma_\mu = \begin{pmatrix}0 & \gamma_\mu\\ \gamma_\mu & 0\end{pmatrix}, \quad \mu = 1, 2\cdots, p+q-1, \quad \Gamma_{p+q} = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}, \qquad (6.7)$$

where γ_μ form a real valued representation of the Clifford algebra $C_{p,q-1}$ with the symmetry property (5.10). So, the conditions (6.4) give, in particular, $\lambda_1 = -\lambda_2 \equiv C$ and

$$C\gamma_\mu + \gamma_\mu^T C = 0, \qquad (6.8)$$

which means that the matrix C is the charge conjugation matrix for the Clifford algebra $C_{p,q-1}$. The additional property of symmetry for this matrix $C^T = C$ limits the possible signatures (p,q) for which the free action (6.1) is invariant under all $p+q$ supersymmetries. These possible signatures (p,q) can be represented by the following table

$p\backslash$ q	0	1	2	3	4	5	6	7	8
0	+	+	+	+	+	+	+	+	+
1	+		+		+		+	+	+
2	+			+	+	+	+	+	+
3	+				+	+	+		+
4	+	+	+	+	+	+	+	+	+
5	+		+	+	+		+		+
6	+	+	+	+	+			+	+
7	+	+	+		+				+
8	+	+	+	+	+	+	+	+	+

(6.9)

which together with the modulo-8 periodicity gives the total answer. For the cases of empty entries of the table it should be checked therefore separately for each specific choice of the matrices λ_1, λ_2 which supersymmetries survive as invariances of the action.

The first non-trivial example concerns a 2-dimensional 'spinning' particle ($d = 2$). Its two bosonic and two fermionic degrees of freedom carry the $\{\mathbf{2}, \mathbf{2}\}$ representation of $(2, 2)$ extended supersymmetry. However, due to the condition (6.4) only half of these supersymmetries can be invariances of the action. We obtain in fact invariance under either the $(2, 0)$ or the $(1, 1)$ extended supersymmetries, whether the target space is respectively Euclidean or Minkowskian. Therefore already for the 2-dimensional Minkowskian 'spinning' particle we observe the arising of the pseudo-Euclidean supersymmetry invariance.

More generally, in all the cases except the euclidean one ($q = 0 \quad or \quad p = 0$), exactly half of the eigenvalues of the charge conjugation matrix C are negative. It means that the action (6.1) describes the free motion in the spacetime with signature $(d/2, d/2)$ with equal numbers of spacelike and timelike coordinates. Both of them transform as irreducible spinors of the isomorphisms group $SO(p, q)$ generated by

$$J_{ik} = \frac{1}{4} [\Gamma_i, \Gamma_k] . \tag{6.10}$$

If one wants to have another spacetime signature, some of the bosonic coordinates can be converted into the auxiliary ones with the help of the procedure described at the end of the previous Section. Formally it means that in the action (6.1) the time derivatives of some bosonic coordinates $\dot{X}_a$ are replaced by new auxiliary variables F_a.

The resulting representation $\{\mathbf{D}, \mathbf{d}, \mathbf{d} - \mathbf{D}\}$ can, for example, contain only 1 timelike and $D - 1$ spacelike bosonic dynamical coordinates. Its corresponding action describes the 'spinning' particle with all its spacetime coordinates belonging to *one* irreducible representation of the extended supersymmetry. All the additional $d - D$ bosonic coordinates are auxiliary. The example of such description of the 4 dimensional spinning particle with $(4, 4)$ extended supersymmetry was given in [15].

7 Conclusions

In this paper we presented some results concerning the representation theory for irreducible multiplets of the one-dimensional $N = (p, q)$ – extended supersymmetry. As pointed out in the text, a peculiar feature of the one-dimensional supersymmetric algebras consists in the fact that the supermultiplets formed by d bosonic and d fermionic degrees of freedom accommodated in a chain with $M+1$ ($M \geq 2$) different spin states such as (2.1) uniquely determines a 2-chain multiplet of the form $\{\mathbf{d}, \mathbf{d}\}$ which carries a representation of the N extended supersymmetry. Furthermore, it is shown that all such 2-chain irreducible multiplets of the (p,q) extended supersymmetry are fully classified; when e.g. the condition $p-q=0$ mod 8 is satisfied, their classification is equivalent to those of Majorana-Weyl spinors in any given space-time, the number $p+q$ of extended supersymmetries being associated to the dimensionality D of the spacetime, while the $2d$ supermultiplet dimensionality is the dimensionality of the corresponding Γ matrices. The more general case for arbitrary values of p and q has also been fully discussed.

These mathematical properties can find a lot of interesting applications in connection with the construction of Supersymmetric and Superconformal Quantum Mechanical Models. These theories are vastly studied due to their relevance in many different physical domains. To name just a few we mention the low-energy effective dynamics of black-hole models, the dimensional reduction of higher-dimensional superfield theories, which are a laboratory for the investigation of the spontaneous breaking of the supersymmetry (for such investigations the extended supersymmetry is an essential ingredient), as well as many others. As recalled in the introduction, it is very crucial to build extended supersymmetric models realized with the lowest-dimensional representations.

Another area in which we have started applying the tools here elaborated is that one of supersymmetric integrable hierarchies in $1+1$ dimensions. They are globally supersymmetric non-linear non-relativistic theories, the one-dimensional susies being realized through charges obtained by integrating the supercurrents along the spatial line.

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