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Vacuum Quantum Effects of Nonconformal Scalar Field in a Nonsingular Cosmological Model

M. Novello, V.B. Bezerra[1] and *V.M. Mostepanenko*[1,*]

Centro Brasileiro de Pesquisas Físicas - CBPF
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brazil

[1]Departamento de Física,
Universidade Federal da Paraíba
Caixa Postal 5008, CEP 58.059-970
João Pessoa, Pb-Brazil

Abstract

The total vacuum stress-energy tensor of nonconformal scalar field is calculated in a nonsingular metric determined by some background matter with the effective negative energy density and pressure. The corrections due to the field nonconformity are shown to dominate the conformal contributions for some cases. The back reaction problem of vacuum stress-energy tensor upon the background metric is also discussed.

Key-words: Quantum vacuum; Scalar field; Nonsingular cosmology.

*On leave from *Alexander Friedmann Laboratory for Theoretical Physics (Russia).*

1. Introduction

The vacuum quantum effects of matter fields in gravitational background were the subject of great number of research in the last few decades (see the monographs[1–4] and references therein). They become relevant between the Planckean time $t_{Pl} = \sqrt{G} \sim 10^{-43}$ s and the Compton time $t_C = m^{-1} \sim 10^{-23}$ s of the matter field (here G is the gravitational constant and the units are used in which $\hbar = c = 1$).

The most important vacuum quantum effects are the particle creation from vacuum by gravitational field, vacuum polarization and the Casimir effect which takes place in the spaces with non-trivial topology. As it is known[1–3] the concept of particle in curved space-time is defined not uniquely. By this reason the total vacuum stress-energy tensor (SET) of quantized field in gravitational background is the quantity giving the most adequate description of vacuum quantum effects instead of the number of created particles. Using this quantity the back reaction of the quantized fields on the gravitational field may be accounted by solving the semiclassical Einstein equations[1–4]

$$G_{ik} = -8\pi G \left(T_{ik}^b + <T_{ik}>\right), \tag{1}$$

where T_{ik}^b is the SET of the classical background matter or of condensate. A number of papers were devoted recently to the investigation of the role of vacuum quantum effects in cosmology, solution of Eq. (1), specifically in the inflationary scenario of the evolution of the Universe (see, e.g., Refs. 5–8).

One of the most complicated problems in the theory of vacuum quantum effects in curved space-time is the calculation of the total vacuum SET. This problem was solved long ago (see, e.g., Ref. 3) for the conformally coupled scalar and spinor fields in a homogeneous and isotropic gravitational background. The solutions of Eq. (1) also were obtained[3]. For the scalar field with arbitrary coupling only several examples were considered analytically. Among other things, in Ref. 9 the total vacuum SET of nonconformal scalar field in de Sitter space-time was calculated and in Ref. 10 — in radiation dominated Friedmann model. At the same time the fields with nonconformal coupling with gravitation have the important applications in cosmology (see, e.g., Ref. 11). The most important of them lies in the fact that such a field drives the inflation process[8].

In two recent papers[12,13] the total vacuum SET of the nonconformal scalar field in homogeneous and isotropic space was derived analytically without specifying the scale factor of the metric. As a ground state the adiabatic vacuum[1] was used. This result was achieved with the help of, as it is called, early time approximation. The important advantage of this approach is that it offers explicit expressions of the total vacuum SET not only for the arbitrary scale factor but for the arbitrary coupling coefficient and mass of the quantized field as well. In Refs. 12,13 the example of the degree-type scale factors was investigated in detailes.

The aim of the present paper is to calculate the total vacuum SET of nonconformal scalar field in one more specific situation. It is a nonsingular cosmological model determined by the background matter (condensate) with the effective negative energy density and pressure. In this case all the results may be obtained analytically following the general expressions of Ref. 13, and their dependence on the coupling coefficient of the scalar field to gravitation may be demonstrated.

The outline of the paper is the following. In Sec. 2 we briefly formulate the necessary results of Ref. 13 which are needed in later sections. Sec. 3 specifies the nonsingular cosmological model under consideration. Sec. 4 contains the calculation of the total vacuum SET of nonconformal scalar field in the early time approximation. The back reaction problem of the vacuum quantum effects upon the background metric is being investigated in Sec. 5. We conclude with the discussion of the obtained results in Sec. 6.

2. Total vacuum SET of nonconformal scalar field in isotropic background

We consider the nonconformal, complex scalar field satisfying the equation with arbitrary coupling

$$\left(\nabla_i\nabla^i + m^2 + \xi R\right)\varphi(x) = 0, \tag{2}$$

where R is the scalar curvature of the space-time and ξ is the coupling coefficient which may take arbitrary values including $\xi = 1/6$ (conformal coupling).

The metric of space-time is given by

$$ds^2 = g_{ik}dx^i dx^k = dt^2 - a^2(t)dl^2, \tag{3}$$

where dl is the line element of a 3-space

$$dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta = d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{4}$$

Here the function $f(\chi) = \sinh\chi, \chi$ or $\sin\chi$ correspondingly to the sign of the 3-space curvature $\kappa = -1, 0, +1$.

In metric $(3), (4)$ the scalar curvature is

$$R = \frac{6}{a^2}\left(\frac{a''}{a} + \kappa\right), \tag{5}$$

where the prime here and below denotes the derivative with respect to the variable $\eta = \int dt/a$.

The separation of variables in Eq. (2) may be performed according to (see, e.g., Ref. 3)

$$\varphi(x) = \frac{1}{\sqrt{2}} \frac{1}{a(\eta)} g_\lambda(\eta) \Phi_J(\boldsymbol{x}), \tag{6}$$

where $\Phi_J(\boldsymbol{x})$ with $J = (\lambda, l, m)$ are the eigenfunctions of the Laplace-Beltrami operator on 3-space with metric (4). The time dependent multiple g_λ is the solution of oscillator equation

$$g_\lambda''(\eta) + \Omega^2(\eta) g_\lambda(\eta) = 0 \tag{7}$$

with the notations

$$\begin{aligned} \Omega^2(\eta) &= \lambda^2 + m^2 a^2(\eta) - q(\eta), \\ q(\eta) &= \Delta\xi a^2 R, \qquad \Delta\xi = \frac{1}{6} - \xi. \end{aligned} \tag{8}$$

Momentum quantum number λ takes arbitrary non-negative values for $\kappa = -1, 0$ and $\lambda = 0, 1, 2, \ldots$ for $\kappa = 1$.

In Ref. 13 the vacuum expectation value of the metrical SET of the field $\varphi(x)$ in metric (3) was calculated without specifying of the scale factor $a(\eta)$. For generality the values of the space-time parameters $a(\eta)$, $q(\eta)$ at some initial moment η_0 were chosen freely:

$$a(\eta_0) = a_0, \qquad q(\eta_0) = q_0. \tag{9}$$

The positive- and negative-frequency solutions to Eq. (7) at initial moment were defined[13] in the sense of frequency Ω

$$g_\lambda^{(+)}(\eta_0) = \Omega^{-\frac{1}{2}}(\eta_0), \quad g_\lambda^{(+)'}(\eta_0) = i\Omega(\eta_0) g_\lambda^{(+)}(\eta_0), \quad g_\lambda^{(-)} = g_\lambda^{(+)^*}. \tag{10}$$

Such initial conditions correspond to the choice of adiabatic vacuum state at the moment η_0 [1].

Renormalization of the vacuum SET in Ref. 13 was performed by the n-wave regularization procedure which is equivalent to adiabatic regularization. The results for the renormalized vacuum SET were expressed in terms of the integrals over λ of the bilinear combinations of function g_λ and its derivative and also of some local counterterms subtracted according to renormalization procedure. It is necessary to stress that each one of these contributions is divergent, giving together the finite expression for the vacuum SET.

As it is shown in Refs. 12,13 by exploiting the early time approximation which reduces to two inequalities

$$mt \ll 1, \qquad \int_{t_0}^{t} dt_1 \sqrt{|\Delta\xi R(t_1)|} \ll 1 \tag{11}$$

one may obtain the asymptotic representations for g_λ in two overlaping momentum regions $(0, \lambda_0)$ and (λ_0, ∞). This, in its turn, gives the possibility to calculate all the momentum integrals in the vacuum SET and to get the explicit cancellation of all the infinities. The first of the inequalities (11) simply means one of the conditions of applicability of semiclassical theory with Eq.(1). The second inequality of (11) restricts the value of the time t to which the obtained results may be applied for a given $\Delta\xi$.

With inequalities (11) the total renormalized vacuum SET was calculated for the arbitrary $a(\eta)$. The result can be displayed as[13]

$$<T_{ik}>=\sum_{a=1}^{5} <T_{ik}^{(a)}>, \tag{12}$$

where every contribution is covariantly conserved:

$$\left(\frac{d}{d\eta}+c\right)<T_{00}^{(a)}>+c\gamma^{\alpha\beta}<T_{\alpha\beta}^{(a)}>=0, \qquad c \equiv \frac{a'}{a}. \tag{13}$$

The explicit form of different contribution to (12) is as follows. The first of them is:

$$\begin{aligned} <T_{ik}^{(1)}>= &-\frac{m^4}{16\pi^2}\left(C+\frac{1}{4}\right)g_{ik}+\frac{m^2}{144\pi^2}\left[1-36\Delta\xi\left(C+\frac{3}{2}\right)\right]G_{ik} \\ &+\frac{\Delta\xi}{144\pi^2}\left[-1+18\Delta\xi\,(C+1)\right]{}^{(1)}H_{ik}+\frac{1}{1440\pi^2}\left(-\frac{1}{6}\,{}^{(1)}H_{ik}+{}^{(3)}H_{ik}\right), \end{aligned} \tag{14}$$

where C is the Euler constant, ${}^{(1)}H_{ik}$, ${}^{(3)}H_{ik}$ are quadratic in the curvature tensors (for their definition see in Refs. 1–3). This addend to (12) consists of generally covariant tensors only.

The second contribution to the total vacuum SET is given by

$$\begin{aligned} <T_{00}^{(2)}>= &\frac{1}{4\pi^2}\left[-\frac{m^4}{4}g_{00}-\Delta\xi m^2 G_{00}+\frac{1}{2}(\Delta\xi)^2\,{}^{(1)}H_{00}\right]\ln(ma), \\ <T_{\alpha\beta}^{(2)}>= &\frac{1}{4\pi^2}\left[-\frac{m^4}{4}g_{\alpha\beta}-\Delta\xi m^2 G_{\alpha\beta}+\frac{1}{2}(\Delta\xi)^2\,{}^{(1)}H_{\alpha\beta}\right]\ln(ma) \\ &-\frac{\gamma_{\alpha\beta}}{12\pi^2}\left[-\frac{m^4}{4}g_{00}-\Delta\xi m^2 G_{00}+\frac{1}{2}(\Delta\xi)^2\,{}^{(1)}H_{00}\right] \end{aligned} \tag{15}$$

and contains logarithmic terms describing the dependence of the vacuum SET on the remormalization point[6].

The third contribution contains geometrical terms connected with non-zero value of κ:

$$
\begin{aligned}
<T_{00}^{(3)}>= & \frac{3m^2\kappa}{144\pi^2} - \frac{3\kappa^2}{720\pi^2 a^2} + \frac{\kappa}{4\pi^2}\Delta\xi\left[-3m^2 + \frac{1}{2a^2}(c^2-\kappa)\right] \\
& - \frac{9}{2\pi^2 a^2}(\Delta\xi)^2 c^2(c'+c^2+\kappa), \\
<T_{\alpha\beta}^{(3)}>= & \gamma_{\alpha\beta}\left\{-\frac{m^2\kappa}{144\pi^2} - \frac{\kappa^2}{720\pi^2 a^2} + \frac{\kappa}{4\pi^2}\Delta\xi\left[m^2 + \frac{1}{6a^2}(-2c'+c^2-\kappa)\right]\right. \\
& \left. + \frac{3}{2\pi^2 a^2}(\Delta\xi)^2\left[c''c + 2c'^2 + c'c^2 + (c^2+\kappa)(2c'-c^2)\right]\right\}. \qquad (16)
\end{aligned}
$$

The fourth contribution to (12) describes dependence on initial conditions:

$$
\begin{aligned}
<T_{00}^{(4)}>= & -\frac{Q_0^4}{64\pi^2 a^2} + \frac{Q_0^2}{16\pi^2 a^2}\left(C + \frac{1}{2} + \ln Q_0\right) \\
& \times\left[2m^2a^2 + 12\Delta\xi(c^2-\kappa) - Q_0^2\right], \\
<T_{\alpha\beta}^{(4)}>= & \gamma_{\alpha\beta}\left\{-\frac{Q_0^4}{192\pi^2 a^2} + \frac{Q_0^2}{48\pi^2 a^2}\left(C + \frac{1}{2} + \ln Q_0\right)\right. \\
& \left. \times\left[-2m^2a^2 + 12\Delta\xi(-2c'+c^2-\kappa) - Q_0^2\right]\right\}, \qquad (17)
\end{aligned}
$$

where $Q = \sqrt{m^2a^2 - q}$, $Q_0 = Q(\eta_0)$.

The last, fifth contribution consists of nonlocal integral terms and may be associated with the SET of particles created from vacuum by the gravitational field:

$$
\begin{aligned}
<T_{00}^{(5)}>= & -\frac{1}{16\pi^2 a^2}\int_{\eta_0}^{\eta} d\eta_1\, Q^{2'}(\eta_1)\int_{\eta_0}^{\eta} d\eta_2\, Q^{2'}(\eta_2)\ln|\eta_1-\eta_2| \\
& + \frac{3}{4\pi^2 a^2}\Delta\xi\left[cQ_0^{2'}\ln|\eta-\eta_0| + c\int_{\eta_0}^{\eta} d\eta_1\, Q^{2''}(\eta_1)\ln|\eta-\eta_1|\right. \\
& \left. - (c'+2c^2)\int_{\eta_0}^{\eta} d\eta_1\, Q^{2'}(\eta_1)\ln|\eta-\eta_1|\right], \\
<T_{\alpha\beta}^{(5)}>= & \frac{\gamma_{\alpha\beta}}{48\pi^2 a^2}\left[4m^2a^2\int_{\eta_0}^{\eta} d\eta_1\, Q^{2'}(\eta_1)\ln|\eta-\eta_1|\right. \qquad (18) \\
& \left. - \int_{\eta_0}^{\eta} d\eta_1\, Q^{2'}(\eta_1)\int_{\eta_0}^{\eta} d\eta_2\, Q^{2'}(\eta_2)\ln|\eta_1-\eta_2|\right] + \frac{\gamma_{\alpha\beta}}{4\pi^2 a^2}\Delta\xi\left[3cQ_0^{2'}\ln|\eta-\eta_0|\right.
\end{aligned}
$$

$$-\frac{Q_0^{2\prime}}{|\eta-\eta_0|} - Q_0^{2\prime\prime}\ln|\eta-\eta_0| - \int_{\eta_0}^{\eta} d\eta_1\, Q^{2\prime\prime\prime}(\eta_1)\ln|\eta-\eta_1|$$
$$\left. + 3c\int_{\eta_0}^{\eta} d\eta_1\, Q^{2\prime\prime}(\eta_1)\ln|\eta-\eta_1| + (c'-2c^2)\int_{\eta_0}^{\eta} d\eta_1\, Q^{2\prime}(\eta_1)\ln|\eta-\eta_1| \right] .$$

In Ref. 13 the results (14)–(18) were applied to the specific case of degree-type scale factors containing singularity. Here we will calculate the total vacuum SET of nonconformal scalar field in a nonsingular cosmological model where it is possible to put $\eta_0 = 0$.

3. Nonsingular cosmological solution

Let us consider the space-time metric (3),(4) with $\kappa = -1$ and a scale factor

$$a(\eta) = a_0 \cosh\eta, \tag{19}$$

where a_0 is an arbitrary constant.

Putting $\eta_0 = t_0 = 0$, we get in terms of a proper synchronous time t

$$a(t) = \sqrt{a_0^2 + t^2}, \qquad t = a_0 \sinh\eta. \tag{20}$$

The scale factor (19),(20) describes nonsingular cosmological solution coinciding asymptotically with the Milne model when $t, \eta \to \infty$. This scale factor may be obtained as the solution of Einstein equations (1) with the SET of background matter

$$T^b_{00} = -\frac{3b_0}{a^2}, \qquad T^b_{\alpha\beta} = -\frac{b_0}{a^2}\gamma_{\alpha\beta}, \qquad b_0 > 0 \tag{21}$$

in the absence of quantum corrections ($<T_{ik}>= 0$). As a result $a_0^2 = 8\pi G b_0$.

It is notable also that the scale factor (19),(20) is a self-consistent solution of Einstein equations with a condensate SET of a conformal self-interacting massless scalar field in the right-hand side. The equation for the operator of the self-interacting field with arbitrary coupling is

$$\left(\nabla_i\nabla^i + \xi R\right)\varphi(x) + \frac{\Lambda}{3}\varphi^*(x)\varphi^2(x) = 0. \tag{22}$$

From the spatial homogeneity of the metric it follows that the vacuum expectation value φ (if nonzero) can depend only on η:

$$<0|\varphi(\eta,\boldsymbol{x})|0>=<0|\varphi(\eta,0)|0>\equiv \sqrt{\frac{3}{\Lambda}}\,\frac{f(\eta)}{a(\eta)}. \tag{23}$$

In the tree approximation it is valid

$$<0|\varphi^*\varphi^2|0>\approx<0|\varphi^*|0><0|\varphi|0>^2=\left(\frac{3}{\Lambda}\right)^{\frac{3}{2}}\frac{f^3}{a^3}. \tag{24}$$

Averaging Eq. (22) in vacuum state for $\kappa=-1$ one obtains the equation for a function f:

$$f''+\left[\frac{a''}{a}(6\xi-1)-6\xi\right]f+f^3=0. \tag{25}$$

For conformal coupling ($\xi=1/6$) Eq. (25) coincides with the Duffing equation

$$f''-f+f^3=0 \tag{26}$$

for the arbitrary scale factor. This equation has non-trivial stable solutions $f=\pm1$ (trivial solution $f=0$ being unstable) corresponding to a formation of a condensate.

In this case the SET of a condensate is[3,14,15]

$$<T_{00}>=-\frac{3}{2\Lambda a^2},\qquad <T_{\alpha\beta}>=-\frac{1}{2\Lambda a^2}\gamma_{\alpha\beta} \tag{27}$$

or of the same form as in (21) with $b_0\equiv1/(2\Lambda)$. Then, substituting (27) into the right-hand side of Einstein equations one gets[3,16,17] the self-consistent solution in the form of (19),(20) with $a_0=\sqrt{4\pi G/\Lambda}$. In the other context the expression (20) for the radius of the Universe was found in Ref. 18.

Note that for the scale factor (19), $a''/a=1$. On the other hand if $a''/a=1$ (so that the scale factor is the arbitrary linear combination of $\exp\eta$ and $\exp(-\eta)$), Eq. (26) follows automatically from (25) for arbitrary ξ not only for $\xi=1/6$. This means the formation of a condensate for a nonconformal field in space-times with such type of scale factors. Unfortunately for $\xi\neq1/6$ the scale factors which lead to formation of a condensate are not self-consistent, i.e. they are not the solutions of Einstein equations with a condensate SET as a source in the right-hand side.

Nevertheless, the scale factor (19),(20) may be used for modeling the nonsingular initial stage of the open Universe expansion driven by the background matter with a SET (21) or by a condensate of a conformal self-interacting scalar field. Also the non-minimally coupled photons lead to effective negative energy density being a source of such expansion[18,19]. In the next section the vacuum quantum effects in such a metric will be investigated.

4. Total vacuum SET of nonconformal field in a nonsingular cosmological model

Let us calculate now all the contributions (14)–(18) to the total vacuum SET of nonconformal scalar field (12) in metric (3),(4) with a scale factor (19). For this scale factor, in the open Universe $R = q = 0$, according to (5),(8). As a consequence the conditions (11) under which the early time approximation is valid reduce to

$$ma_0 \sinh \eta \ll 1. \tag{28}$$

The calculation of the quantities (14)–(17)is straightforward. Substituting (19) into (14) one has:

$$\begin{aligned}
<T_{00}^{(1)}> &= \frac{m^2}{48\pi^2 \cosh^2 \eta}\left[1 - 3(ma_0)^2\left(C + \frac{1}{4}\right)\cosh^4 \eta + \frac{1}{10(ma_0)^2}\cosh^{-4}\eta \right.\\
&\quad \left. -36\Delta\xi\left(C + \frac{3}{2}\right)\right],\\
<T_{\alpha\beta}^{(1)}> &= \frac{m^2\gamma_{\alpha\beta}}{144\pi^2 \cosh^2 \eta}\left[1 + 9(ma_0)^2\left(C + \frac{1}{4}\right)\cosh^4 \eta \right.\\
&\quad \left. + \frac{1}{2(ma_0)^2}\cosh^{-4}\eta - 36\Delta\xi\left(C + \frac{3}{2}\right)\right].
\end{aligned} \tag{29}$$

To get these results we used that for the scale factor under consideration

$${}^{(1)}H_{ik} = 0, \qquad {}^{(3)}H_{00} = \frac{3}{a_0^2 \cosh^6 \eta}, \qquad {}^{(3)}H_{\alpha\beta} = \frac{5\gamma_{\alpha\beta}}{a_0^2 \cosh^6 \eta}. \tag{30}$$

In the same way contribution (15) to the total vacuum SET is:

$$\begin{aligned}
<T_{00}^{(2)}> &= -\frac{m^2}{16\pi^2}\left[(ma_0)^2\cosh^2\eta + 12\Delta\xi\cosh^{-2}\eta\right]\ln(ma_0\cosh\eta),\\
<T_{\alpha\beta}^{(2)}> &= \frac{m^2\gamma_{\alpha\beta}}{48\pi^2}\left\{(ma_0)^2\cosh^2\eta\left[3\ln(ma_0\cosh\eta) - 1\right]\right.\\
&\quad \left. -12\Delta\xi\cosh^{-2}\eta\left[\ln(ma_0\cosh\eta) - 1\right]\right\}.
\end{aligned} \tag{31}$$

In the massless limit, (31) gives zero due to the absence of the term proportional to ${}^{(1)}H_{ik}$ for the scale factor (19).

Substitution of (19) into (16) leads to the expressions

$$\begin{aligned}
<T_{00}^{(3)}> &= -\frac{m^2}{48\pi^2} - \frac{1}{240\pi^2 a_0^2\cosh^2\eta} + \frac{\Delta\xi}{8\pi^2}\left(6m^2 - \frac{\cosh 2\eta}{a_0^2\cosh^4\eta}\right),\\
<T_{\alpha\beta}^{(3)}> &= \gamma_{\alpha\beta}\left[\frac{m^2}{144\pi^2} - \frac{1}{720\pi^2 a_0^2\cosh^2\eta} - \frac{\Delta\xi}{4\pi^2}\left(m^2 + \frac{\cosh 2\eta - 2}{6a_0^2\cosh^4\eta}\right)\right],
\end{aligned} \tag{32}$$

containing contributions connected with the non-zero curvature of 3-space.

The addend (17) connected with the chosen initial conditions is:

$$
\begin{aligned}
<T_{00}^{(4)}>= &\frac{m^4a_0^2}{64\pi^2\cosh^2\eta}\left[-1+2\,(2C+1+2\ln ma_0)\cosh 2\eta\right]\\
&+\frac{3m^2\Delta\xi}{8\pi^2\cosh^4\eta}\,(2C+1+2\ln ma_0)\cosh 2\eta,\\
<T_{\alpha\beta}^{(4)}>= &\frac{m^2\gamma_{\alpha\beta}}{192\pi^2\cosh^2\eta}\left\{-(ma_0)^2\left[1+2\,(2C+1+2\ln ma_0)\,(\cosh 2\eta+2)\right]\right.\\
&\left.+24\Delta\xi\cosh^{-2}\eta\,(2C+1+2\ln ma_0)\,(\cosh 2\eta-2)\right\}. \qquad (33)
\end{aligned}
$$

Here it was used that $Q_0=Q(0)=ma_0$.

The calculation of the last nonlocal contribution $<T_{\alpha\beta}^{(5)}>$ containing integrals is mostly cumbersome. Nevertheless all the internal integrals in (18) may be expressed in terms of the exponential-integral function $\mathrm{Ei}(x)$ using the standard relations from Ref. 20. The second integration in (18) may be performed by the use of the formula

$$
\begin{aligned}
\int_0^x \mathrm{e}^{-\beta x}\mathrm{Ei}(-\alpha x)\,dx = -\frac{1}{\beta}&\left\{\mathrm{e}^{-\beta x}\mathrm{Ei}(-\alpha x)\right.\\
&\left.-\ \mathrm{Ei}[-(\alpha+\beta)x]+\ln\left|1+\frac{\beta}{\alpha}\right|\right\} \qquad (34)
\end{aligned}
$$

(compare with 5.231(2) from Ref. 20 where the brackets under the logarithm instead of modulus should be considered as an error).

The final results for (18) are:

$$
\begin{aligned}
<T_{00}^{(5)}>= &\frac{m^4a_0^2}{256\pi^2\cosh^2\eta}\Big[4(2\ln\eta+1)\cosh 2\eta+2\,(C+\ln 2)\cosh 4\eta\\
&-\left(\mathrm{e}^{-4\eta}+4\eta+3\right)\mathrm{Ei}(2\eta)-\left(\mathrm{e}^{4\eta}-4\eta+3\right)\mathrm{Ei}(-2\eta)+2\,(3C+3\ln 2-2)\Big]\\
&+\frac{3m^2\Delta\xi}{16\pi^2\cosh^4\eta}\left[2\cosh 2\eta\,\ln\eta-\mathrm{Ei}(2\eta)-\mathrm{Ei}(-2\eta)+2\,(C+\ln 2)\right],\\
<T_{\alpha\beta}^{(5)}>= &\frac{m^4a_0^2\gamma_{\alpha\beta}}{768\pi^2\cosh^2\eta}\Big[-16\ln\eta-4\,(2\ln\eta+4C+4\ln 2-1)\cosh 2\eta\\
&-6\,(C+\ln 2)\cosh 4\eta+\left(3\mathrm{e}^{-4\eta}+8\mathrm{e}^{-2\eta}-4\eta+1\right)\mathrm{Ei}(2\eta)\\
&+\left(3\mathrm{e}^{4\eta}+8\mathrm{e}^{2\eta}+4\eta+1\right)\mathrm{Ei}(-2\eta)-2\,(C+\ln 2+2)\Big]\\
&+\frac{m^4\Delta\xi\gamma_{\alpha\beta}}{16\pi^2\cosh^4\eta}\Big[-4\ln\eta+2\cosh 2\eta\,\ln\eta-3\mathrm{Ei}(2\eta)-3\mathrm{Ei}(-2\eta) \qquad (35)\\
&+6\,(C+\ln 2)\Big].
\end{aligned}
$$

It is checked by direct calculation that for each of five addends (29), (31)–(33) and (35) the covariant conservation condition (13) is valid separately.

Let us discuss now the relative role of the obtained contributions to the total vacuum SET. As it follows from (28) the inequality $\eta \ll 1$ should be valid with great supply (for the usual masses of elementary particles and realistic cosmological models $ma_0 \gg 1$). We expand the quantities (29), (31)–(33) and (35) into the series in powers of η preserving the main terms which are important for the validity of covariant conservation condition in the lowest order. The result for $<T_{ik}^{(1)}>$ is:

$$\begin{aligned}
<T_{00}^{(1)}> &\approx \frac{m^2}{48\pi^2}\left\{1 - 3(ma_0)^2\left(C+\frac{1}{4}\right) + \frac{1}{10(ma_0)^2}\right.\\
&- \left[1 + 3(ma_0)^2\left(C+\frac{1}{4}\right) + \frac{3}{10(ma_0)^2}\right]\eta^2\\
&\left. - 36\Delta\xi\left(C+\frac{3}{2}\right)(1-\eta^2)\right\},\\
<T_{\alpha\beta}^{(1)}> &\approx \frac{m^2\gamma_{\alpha\beta}}{144\pi^2}\left[1 + 9(ma_0)^2\left(C+\frac{1}{4}\right) + \frac{1}{2(ma_0)^2}\right.\\
&\left. - 36\Delta\xi\left(C+\frac{3}{2}\right)\right].
\end{aligned} \tag{36}$$

For $<T_{ik}^{(2)}>$ one obtains:

$$\begin{aligned}
<T_{00}^{(2)}> &\approx -\frac{m^2}{16\pi^2}\left\{(ma_0)^2\left[\ln ma_0 + \left(\frac{1}{2}+\ln ma_0\right)\eta^2\right]\right.\\
&\left. -12\Delta\xi\left[\ln ma_0 + \left(\frac{1}{2}-\ln ma_0\right)\eta^2\right]\right\},\\
<T_{\alpha\beta}^{(2)}> &\approx \frac{m^2\gamma_{\alpha\beta}}{48\pi^2}\left[(ma_0)^2(1+3\ln ma_0) - 12\Delta\xi(-1+\ln ma_0)\right].
\end{aligned} \tag{37}$$

In the same way for the third contribution it is valid:

$$\begin{aligned}
<T_{00}^{(3)}> &\approx -\frac{m^2}{48\pi^2}\left\{1 + \frac{1-\eta^2}{5(ma_0)^2} - 6\Delta\xi\left[6 - \frac{1}{(ma_0)^2}\right]\right\},\\
<T_{\alpha\beta}^{(3)}> &\approx \frac{m^2\gamma_{\alpha\beta}}{720\pi^2}\left\{5 - \frac{1}{(ma_0)^2} - 180\Delta\xi\left[1 - \frac{1}{6(ma_0)^2}\right]\right\}.
\end{aligned} \tag{38}$$

For $<T_{ik}^{(4)}>$ one has:

$$\begin{aligned}
<T_{00}^{(4)}> &\approx -\frac{m^2}{64\pi^2}\left\{(ma_0)^2\left[1 + 4C + 4\ln ma_0 + (3 + 4C + 4\ln ma_0)\eta^2\right]\right.\\
&\left. +24\Delta\xi(2C + 1 + 2\ln ma_0)\right\},\\
<T_{\alpha\beta}^{(4)}> &\approx -\frac{m^2\gamma_{\alpha\beta}}{192\pi^2}\left[(ma_0)^2(7 + 12C + \ln ma_0)\right.\\
&\left. +24\Delta\xi(2C + 1 + 2\ln ma_0)\right].
\end{aligned} \tag{39}$$

At last the fifth contribution to the total vacuum SET for small η is:

$$
\begin{aligned}
<T_{00}^{(5)}> &\approx \frac{m^2}{64\pi^2}\left[(ma_0)^2(7-4\ln\eta)\eta^4 + 24\Delta\xi(-1+2\ln\eta)\eta^2\right], \\
<T_{\alpha\beta}^{(5)}> &\approx \frac{m^2\gamma_{\alpha\beta}}{24\pi^2}\left[(ma_0)^2(-3+2\ln\eta)\eta^2 - 12\Delta\xi\ln\eta\right].
\end{aligned} \tag{40}
$$

It is seen from (36),(37),(39) that for a large parameter ma_0 conformal contributions dominate nonconformal ones for not too large values of nonconformity parameter $\Delta\xi$. On the other hand in (38),(40) nonconformal contributions dominate conformal ones, even for $\Delta\xi \sim 1$. As the initial moment $\eta = 0$ is approached the total vacuum SET has the values (here we preserve only the main terms)

$$
\begin{aligned}
<T_{00}>_{\eta\to 0} &= -\frac{1}{8\pi^2 a_0^2}\left(\frac{1}{60}+\Delta\xi\right), \\
<T_{\alpha\beta}>_{\eta\to 0} &= \frac{m^2\gamma_{\alpha\beta}}{1440\pi^2}\left[20 + \frac{3}{(ma_0)^2}\right. \\
&\left. -1220\Delta\xi\left(\ln\eta + C + 1 + \ln ma_0 - \frac{1}{12(ma_0)^2}\right)\right].
\end{aligned} \tag{41}
$$

The presence of a logarithmic divergency here is a consequence of the second equation from (40) which, in its turn, is connected with the first one by the conservation condition.

It is the subject of the next section to estimate the back reaction of the total vacuum SET which is the sum of (36)–(40), on the background space-time.

5. The back reaction problem

We now investigate the approximate solutions of Eq. (1) with the SET of background matter (21) and vacuum quantum corrections $<T_{ik}>$. Due to the conservation condition it will suffice to analyse 00-componens of (1) only (the equation for the space components will be satisfied automatically). Substituting the scale factor (19) we obtain from (21)

$$
T_{00}^b = -\frac{3b_0}{a_0^2\cosh^2\eta} = -\frac{3}{8\pi G\cosh^2\eta}. \tag{42}
$$

The sum of (36)–(40), up to the second order in η, is

$$
<T_{00}> \approx A + B\eta^2 + D\eta^2\ln\eta, \tag{43}
$$

where

$$
\begin{aligned}
A &\equiv -\frac{1}{64\pi^3 G b_0}\left(\frac{1}{60}+\Delta\xi\right), \qquad D \equiv \frac{3m^2\Delta\xi}{4\pi^2}, \\
B &\equiv -\frac{m^2}{48\pi^2}\left[1 + \frac{1}{10(ma_0)^2} - 36\Delta\xi\left(\frac{1}{2} + C + \ln ma_0\right)\right].
\end{aligned} \tag{44}
$$

For $b_0 \gg 1$ vacuum energy density $< T_0{}^0 >$ is much smaller than the energy density of the background matter and may be considered as perturbation (the inequality $Gm^2 \ll 1$ is also taken into account which is valid for all known elementary particles).

Substituting the sum of (42) and (43) into the 00-component of Einstein equations (1) one gets

$$\frac{\tilde{a}'^2}{\tilde{a}^2} = h_0(a) + h_1, \tag{45}$$

where $\tilde{a}$ is the perturbed scale factor with

$$h_0(a) = 1 - \frac{1}{\cosh^2 \eta}, \qquad h_1 = \frac{8\pi G}{3}\left(A + B\eta^2 + D\eta^2 \ln \eta\right). \tag{46}$$

In (46) we preserve the notation "a" for the non-perturbed scale factor (19).

Let us find the solution of Eq. (45) in the first order in perturbation h_1 and in the lowest order in η. Evidently

$$\tilde{a}(\eta) = a_0 \exp\left[\int_0^\eta \sqrt{h_0 + h_1}\, d\eta\right]. \tag{47}$$

Taking into account that h_0 goes to zero when η decreases we will use the identity

$$\begin{aligned}\sqrt{h_0 + h_1} &= \left(\sqrt{h_0} + \sqrt{h_1}\right)\left[1 - \frac{2\sqrt{h_0 h_1}}{\left(\sqrt{h_0} + \sqrt{h_1}\right)^2}\right]^{\frac{1}{2}} \\ &\approx \sqrt{h_0} + \sqrt{h_1} - \frac{\sqrt{h_0 h_1}}{\sqrt{h_0} + \sqrt{h_1}}.\end{aligned} \tag{48}$$

Integrating (47) with account of (48) one gets the corrected scale factor in the lowest order in η

$$\tilde{a}(\eta) \approx a_0 \left(1 + \sqrt{\frac{8\pi G A}{3}}\eta\right)\cosh \eta, \tag{49}$$

where the coefficient A is defined in (44).

Notice that due to (44) the corrected scale factor (49) exists only under condition

$$\frac{1}{60} + \Delta\xi \leq 0, \qquad \xi \geq \frac{11}{60}, \tag{50}$$

otherwise the Einstein equation (45) is contradictory. The case $\xi = 11/60$ is isolated: here $A = 0$ and there is no first order corrections in η to the background metric due to the vacuum quantum effects. These features of the obtained solution are connected with the choice of the adiabatic vacuum as the initial quantum state of nonconformal scalar field.

6. Conclusion and discussion

In the foregoing we calculated the total vacuum stress-energy tensor of nonconformal scalar field in nonsingular homogeneous isotropic space with a scale factor represented by Eq. (19). This scale factor was discussed previously in the literature and may be determined by the background matter (or condensate) with the effective negative energy density and pressure.

For the calculation of the total SET in adiabatic initial vacuum state the results of Refs. 12,13 were used. According to these results in the early time approximation (inequalities (11)) the vacuum SET may be represented as an explicit functional of the metric. Such representation reduces investigation of the vacuum SET for the specific scale factors to the calculation of some geometrical quantities and repeated integrals. It was shown that for the nonsingular scale factor (19) all these integrals may be calculated analytically. Some of the nonconformal contributions to the vacuum SET turn out to be dominant campared with conformal ones.

The representation for the vacuum SET of nonconformal scalar field obtained in Refs. 12,13 is very well adapted for the analytical investigation of the back reaction of vacuum quantum effects on the space-time metric (several related numerical results were obtained in Ref. 21). In the present paper only a little step in this direction was made: we calculated the first order correction to the scale factor caused by the back reaction of vacuum SET on the background metric. It was shown that for some values of coupling coefficient Einstein equations are contradictory but for the other ones there exists a consistent solution taking into account quantum effects of nonconformal scalar field. A complete investigation of Einstein equations with account of vacuum quantum corrections shows promise of solving some problems of the standard cosmological model[22] in description of the very early Universe.

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