CBPF-NF-028/96

Matrix Treatment of Electrical Current: Current Echoes*

by

I.S. Oliveira

Centro Brasileiro de Pesquisas Físicas - CBPF
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brazil

Abstract

A matrix method has been applied to solve the electron equation of motion in electric and magnetic fields. The analogy between this problem and that of an ensemble of nuclear spins in a magnetic field, described by the Bloch equations, leads to transient solutions similar to spin echoes, which we shall call *current echoes*, a phenomenon not yet observed experimentally. In a configuration of static and oscillating fields, the components of the Hall current are calculated in the rotating reference frame. Then we consider transverse pulsed magnetic fields and derive expressions for the transient effects.

Key-words: Bloch equations; Spin Echoes; Free Induction Decay.

*Phys. Rev. Lett. (1996) in press.

Spin Echo pulse NMR, discovered by Hahn in 1950 [1], is today one of the most important tools in experimental Solid State Physics. The technique allows the study of magnetic and transport phenomena in solids through the measurement of NMR spectra, relaxation times, Knight and chemical-shifts, etc. In 1954 E.T. Jaynes developed a very simple and elegant matrix method to the study of the solutions of the Bloch equations [2]. In a subsequent paper A.L. Bloom applied the method to investigate spin echo shapes and amplitudes in the presence of inhomogeneous magnetic fields [3]. Here we show, by means of the same formalism, that a similar phenomenon may exist in conducting materials. The idea is to apply Jaynes method to solve the equation of motion of electrons in electric and magnetic fields. First we analyze the case where a static magnetic field is applied *parallel* to the electric field and another continuous AC magnetic field exists *perpendicularly* to both. The rotating reference frame is introduced and the Hall current is calculated in this system of coordinates. Then we follow to investigate the case where the AC field is applied as a sequence of pulses.

Our starting point is the classical equation of motion for an electron in an electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields, which in standard units is written as [4]:

$$\frac{\partial \mathbf{p}}{\partial t} = -\frac{\mathbf{p}}{\tau} - e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1}$$

where $\mathbf{p} = m\mathbf{v}$ is the electron momentum and τ is the *relaxation time*, which takes into account the interactions between the electron and the lattice.

Defining the electric current $\mathbf{J}$ by

$$\mathbf{J} = -ne\mathbf{v} = -\frac{ne}{m}\mathbf{p}$$

where n is the conduction electron density we can rewrite equation (1) as

$$\frac{\partial \mathbf{J}}{\partial t} + \frac{\mathbf{J} - \sigma\mathbf{E}}{\tau} + \gamma(\mathbf{B} \times \mathbf{J}) = 0 \tag{2}$$

with $\gamma \equiv -e/m$ and $\sigma = ne^2\tau/m$ is the classical electrical conductivity at $T = 0$.

Equation (2) has the same structure as the Bloch equation for the motion of the nuclear magnetization from an spin ensemble in a magnetic field, with the spin-spin and spin-lattice relaxation times being the same [5]. It can be regarded as a *Bloch equation for the electrical current.* Following Jaynes [2], we can write (2) in matrix form by defining a matrix $\tilde{\beta}$ which performs the cross product, as follows:

$$\tilde{\beta} \equiv \gamma\mathbf{B}\times = \gamma \begin{pmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{pmatrix} \tag{3}$$

The current equation becomes:

$$\frac{\partial \mathbf{J}}{\partial t} + \left[\frac{1}{\tau} + \tilde{\beta}\right]\mathbf{J} = \mathbf{A}(t) \tag{4}$$

where $\mathbf{A}(t) = \sigma\mathbf{E}(t)/\tau$.

The general solution of this equation can be written as

$$\mathbf{J}(t) = \mathbf{U}(t,0)\mathbf{J}(0) + \int_0^t \mathbf{U}(t,t')\mathbf{A}(t')dt' \tag{5}$$

where $\mathbf{U}(t,t')$ is a time-developing matrix which satisfies the homogeneous equation (4) of Jaynes [2]. For the case where the matrix $\tilde{\beta}$ is time-independent, $\mathbf{U}(t,t')$ takes the form:

$$\mathbf{U}(t,t') = exp\left[\frac{1}{\tau} + \tilde{\beta}\right](t-t') \tag{6}$$

We can now check the applicability of these results by applying (5) and (6) to some simple situations.

(i) $\mathbf{B} = (0,0,B_o)$ and $\mathbf{A}(t) = (\sigma/\tau)(E_o,0,0)$. That is, a static magnetic field along the z-axis and a static electric field along the x-axis. Taking $\mathbf{J}(0) = 0$ as initial condition, the second term of (5) can be readily calculated yielding:

$$\mathbf{J}(t) = \frac{\sigma}{\tau}\left(\frac{1}{\tau} + \tilde{\beta}\right)^{-1}\mathbf{E} - \frac{\sigma}{\tau}exp\left(-\left[\frac{1}{\tau} + \tilde{\beta}\right]t\right)\left(\frac{1}{\tau} + \tilde{\beta}\right)^{-1}\mathbf{E}$$

We wish to analyze the first term from this expression which represents the stationary solution. It is a straightforward matter to calculate the matrix operator appearing in this term:

$$\left(\frac{1}{\tau} + \tilde{\beta}\right)^{-1} = \frac{1}{\tau^{-2} + \omega_c^2}\begin{pmatrix} \tau^{-1} & -\omega_c & 0 \\ \omega_c & \tau^{-1} & 0 \\ 0 & 0 & (\tau^{-2} + \omega_c^2)\tau \end{pmatrix}$$

where $\omega_c = \gamma B_o$ is the cyclotron frequency of the electron. Applying this matrix to the vector $\mathbf{E}$ defined above we find the solutions:

$$J_x(\infty) = \frac{\sigma E_o}{1 + \omega_c^2\tau^2}$$

$$J_y(\infty) = \frac{\omega_c \tau}{1 + \omega_c^2\tau^2}\sigma E_o \tag{7}$$

$$J_z(\infty) = 0$$

which are the expected results [4].

(ii) Let us now consider a more interesting case where DC electric and magnetic fields exist along the z-axis and one AC magnetic field is applied along x, that is:

$$\mathbf{B} = (2B_1 cos\omega t, 0, B_o)$$

$$\mathbf{E} = (0, 0, E_o)$$

With this form for $\mathbf{B}$ the solution (6) is no longer valid, since $\tilde{\beta}$ will not be time-independent. We can however follow the usual procedure which consists in analyzing the problem in a rotating reference frame where $\mathbf{B}$ is stationary. On such a frame $\mathbf{B}$ takes the form [5]:

$$\mathbf{B} = (B_1, 0, \frac{\omega}{\gamma} - B_o)$$

with the Bloch equation (2) remaining unaltered.

On the rotating frame $\tilde{\beta}$ is time-independent and the solution (6) can be again applied. The stationary solution will be given by:

$$\mathbf{J}(\infty) = \frac{\sigma}{\tau}\left(\frac{1}{\tau} + \tilde{\beta}\right)^{-1}\mathbf{E}$$

where

$$\left(\frac{1}{\tau} + \tilde{\beta}\right) = \begin{pmatrix} \tau^{-1} & -(\omega - \omega_c) & 0 \\ (\omega - \omega_c) & \tau^{-1} & -\omega_1 \\ 0 & \omega_1 & \tau^{-1} \end{pmatrix}$$

with $\omega_c = \gamma B_o$ and $\omega_1 = \gamma B_1$.

As before, by applying the inverse of the matrix above to the vector $\mathbf{E}$ one finds:

$$\mathbf{J}(\infty) = \frac{\sigma\tau E_o}{1 + [\omega_1^2 + (\omega - \omega_c)^2]\tau^2}\begin{pmatrix} (\omega - \omega_c)\omega_1\tau \\ \omega_1 \\ \frac{1+(\omega-\omega_c)^2\tau^2}{\tau} \end{pmatrix}$$

At the resonance, $\omega = \omega_c$ and the components of $\mathbf{J}$ in the rotating frame become:

$$J_x(\infty) = 0$$

$$J_y(\infty) = \frac{\omega_1\tau}{1 + \omega_1^2\tau^2}\sigma E_o$$

$$J_z(\infty) = \frac{\sigma E_o}{1 + \omega_1^2\tau^2}$$

These expressions can be regarded as the components of the *Hall current in the rotating frame* (compare with equation 7). Note that if we switch off the AC field by making $\omega_1 = 0$ we find $J_x = J_y = 0$ and $J_z = \sigma E_o$, which is the expression for the current on the laboratory frame.

The most interesting results emerge from the above formalism when we consider pulsed magnetic fields. As it happens in the usual pulse NMR, the observation of transient effects (FID's, spin echoes, etc.) is only possible if the system does not relax too fast. Roughly speaking, the relaxation times must be long compared to the length of time of one experiment. In the NMR of magnetic metals, for instance, the time scale of experiments is of the order of tens of microseconds, whereas the relaxation times are usually hundreds of microseconds [6]. The applicability of the following results to metallic systems (and eventually other conducting media) depends upon the electron total scattering rate τ^{-1}, which should be about 20 MHz or less (corresponding to a relaxation time of 50 ns or more). In metals, at temperatures well below the Debye temperature, Θ_D, the electron-phonon and impurities scattering rates are the main contributions to τ^{-1}. Whereas the former follows a T^3 law and can be reduced by a factor of about 10^5 by going from 4.2 K to 50 mK, the later depends on the details of the material preparation, its history, and it is difficult to be predicted. This contribution should be minimum in high purity single-crystals. Its not of the author knowledge any recent report where relaxation times have been measured in simple metal single-crystals at low temperatures. However, it is our belief that with the modern ultra high vacuum techniques of deposition and crystal growing, metallic single-crystals can be prepared in a degree of purity such that τ could be $50ns$ or more at, say, 50 mK.

Following the description of Jaynes [2] the components of the current after any sequence of pulses can be written as

$$\begin{pmatrix} J'_+ \\ J'_- \\ J'_z \end{pmatrix} = \begin{pmatrix} \alpha^{*2} & -\beta^{*2} & -2\alpha^*\beta^* \\ -\beta^2 & \alpha^2 & -2\alpha\beta \\ \alpha^*\beta & \alpha\beta^* & |\alpha|^2 - |\beta|^2 \end{pmatrix} \begin{pmatrix} J_+ \\ J_- \\ J_z \end{pmatrix} \tag{8}$$

where α and β are the elements of the 2×2 unimodular matrix $\mathbf{Q}$ given by [7] [1]

$$\mathbf{Q} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \tag{9}$$

Thus, if we start with $J = J_z$, after a pulse sequence the transverse component of the current, $J_-(t)$, will be proportional to the product $-2\alpha\beta$.

Let us first exam the situation where only one pulse of duration τ_p is applied. We want to calculate $J_-(t)$ after the pulse as a function of the applied fields and pulse width. We consider again static magnetic B_o and electric E_o fields along the z-axis. A pulsed magnetic field of amplitude B_1 is applied along the x-axis. It is easy to show that during the application of the pulse, the elements of $\mathbf{Q}$ are given by [3]:

$$\alpha = cos\left(\frac{\gamma B\tau_p}{2}\right) - icos(\theta)sin\left(\frac{\gamma B\tau_p}{2}\right)$$

$$\beta = -isin\theta sin\left(\frac{\gamma B\tau_p}{2}\right) \tag{10}$$

where $\theta = tg^{-1}(\omega_1/\Delta\omega)$, $\gamma B = (\omega_1^2 + \Delta\omega^2)^{1/2}$ and $\Delta\omega = \omega - \omega_c$. After the pulse these matrix elements will become

$$\alpha = exp\left(-i\frac{\Delta\omega t}{2}\right)$$

$$\beta = 0 \tag{11}$$

The resultant $\mathbf{Q}$-matrix will be given by the product of the individual matrices describing the evolution of the current during each interval of time [3]:

$$\mathbf{Q} = \mathbf{Q}_2\mathbf{Q}_1$$

Performing this matrix product we find for the transverse component of the current:

$$J_-(t) = \frac{i}{2}e^{-i\Delta\omega t}\left[sin\theta sin(\gamma B\tau_p) - sin2\theta sin^2\left(\frac{\gamma B\tau_p}{2}\right)\right] \tag{12}$$

[1] Here the matrix element β is not to be confused with the time-evolution matrix $\tilde{\beta}$. As in Jaynes we keep the same Greek letter but the matrix is distinguished by a tilde.

At the resonance, $\Delta\omega = 0$, $\theta = \pi/2$ and we have

$$J_x = 0$$

$$J_y = \frac{\sigma E_o}{2} sin(\gamma B_1 \tau_p)$$

We see that this *free-current-decay* will be maximum when $\gamma B_1 \tau = \pi/2$. We also see that the current is reversed by the rotating field, when $\gamma B_1 \tau = \pi$.

In order to calculate $J_-(t)$ after a sequence of two pulses, we follow the same procedure as above. Now we just have to calculate the product of four matrices and use (10) and (11):

$$\mathbf{Q} = \mathbf{Q}_4 \mathbf{Q}_3 \mathbf{Q}_2 \mathbf{Q}_1$$

where $\mathbf{Q}_4$ describes the evolution of the current after the second pulse, $\mathbf{Q}_2$ between them and $\mathbf{Q}_3$ and $\mathbf{Q}_1$ during their application. The result becomes simpler if we make the two pulses as having the same width; this means $\mathbf{Q}_1 = \mathbf{Q}_3$. With the same initial conditions as before, we find for $J_-(t)$

$$J_-(t) = -2i\sigma E_o \left[cos\left(\frac{\gamma B \tau_p}{2}\right) + i cos\theta sin\left(\frac{\gamma B \tau_p}{2}\right)\right] sin^3\theta sin^3\left(\frac{\gamma B \tau_p}{2}\right) e^{-i\Delta\omega(t-\Delta\tau_p)} \quad (13)$$

where $\Delta\tau_p$ is the time interval between the pulses. In this expression t is measured from the second pulse.

The above expression represents a *current echo*. At the resonance we have

$$J_- = -i\sigma E_o sin(\gamma B_1 \tau_p) sin^2\left(\frac{\gamma B_1 \tau_p}{2}\right) \quad (14)$$

or, in terms of the components

$$J_x = 0$$

$$J_y = \sigma E_o sin(\gamma B_1 \tau_p) sin^2\left(\frac{\gamma B_1 \tau_p}{2}\right)$$

These expressions are identical to that for the spin echo in the magnetic case, except that here $|\gamma| = e/m \approx 1.76 \times 10^3$ GHz kGauss^{-1}, is much larger than its nuclear counterpart.

Figure 1 shows current echo components calculated for $\tau_p = 1$ ns, $B_1 = 10$ Gauss, $\Delta\tau_p = 100$ ns, $\Delta\omega = 0.0$ GHz (1a) and $\Delta\omega = 0.1$ GHz (1b) [2]. The relaxation time was taken as $\tau = 100$ ns. These curves were obtained under the assumption of a Lorentzian distribution in the magnetic field inhomogeneity.

In this paper we have worked out solutions for the equation of motion of electrons in various configurations of electric and magnetic fields, by applying a matrix method developed by Jaynes [2]. We have shown that the similarity between this problem and that of the motion of the nuclear magnetization in a magnetic field leads to transient

[2] Here we have simplified our notation; it is implicit that $\Delta\omega = 2\pi\Delta\nu$

solutions which we have called *current echoes* (and *current FID's*). The observation of current echoes could in principle be made in a similar way to NMR, by detecting the e.m.f. in a "pick-up" coil, induced by the transient current, with subsequent demodulation of the signal, which then would appear as in figure 1 [8]. If proved to exist, current echoes could become a helpful tool in the investigation of transport properties in various materials through the measurement of their electron cyclotron resonance spectra and relaxation times. The formalism developed by Jaynes can easily be generalized to include more than one relaxation time and, much like in the magnetic case, concepts such as *electron temperature* should be introduced. Current-echoes could then open new possibilities for the study of electron-electron and electron-lattice mechanisms of interaction in solids.

Acknowledgments

The author wishes to express his gratitude to Prof. A.P.Guimarães and Prof. J.S.Helman, for useful discussion and suggestions. This work was supported by CNPq, Brazil.

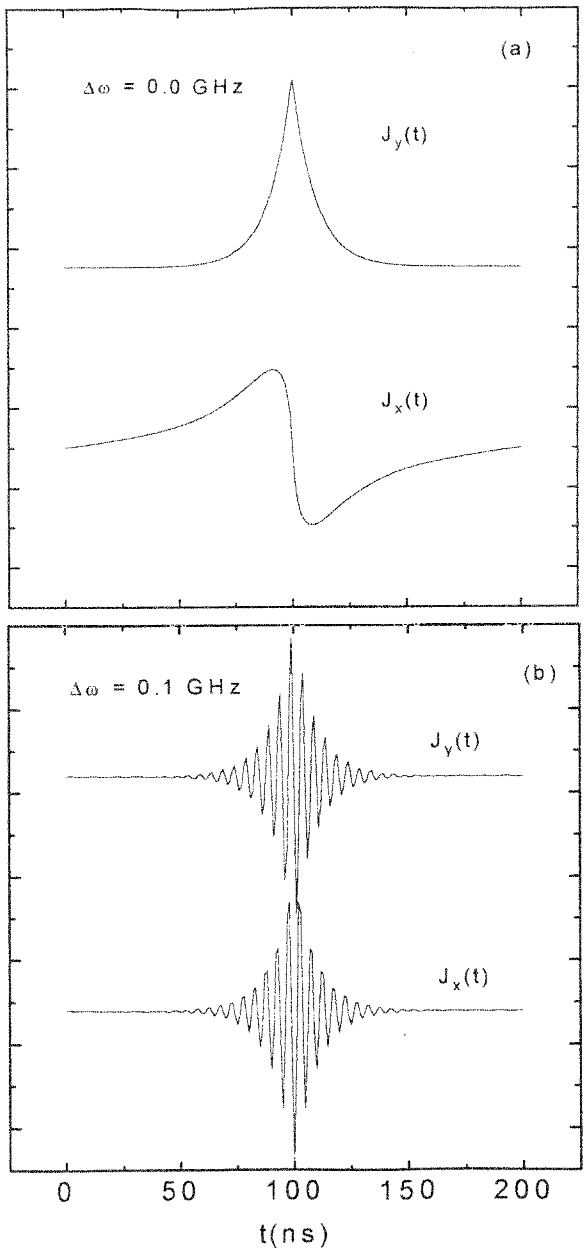

Figure 1 - Calculated current echo components for $\omega = \omega_c$ (a) and $\omega - \omega_c = 0.1$ GHz (b). The relaxation time was taken as $\tau = 100$ ns (see text for details).

References

[1] Hahn, E.L., *Phys. Rev.* **80** (1950) 580

[2] Jaynes, E.T., *Phys. Rev.* **98** (1955) 1099

[3] Bloom, A.L., *Phys. Rev.* **98** (1955) 1105

[4] C. Kittel, *Introduction to Solid State Physics*, 6^{th} edition, John Wiley & Sons, (1986)

[5] C.P. Slichter, *Principles of Magnetic Resonance*, 3^{rd} edition, Springer-Verlag, (1990).

[6] McCausland, M.A.H. and I.S. Mackenzie, *Adv. Phys.* **28** (1979) 305

[7] Goldstein, H., *Classical Mechanics*, 2^{nd} edition, Addison-Wesley (1980).

[8] Fukushima, E. and S.B.W. Roeder, *Experimental Pulse NMR: A Nuts and Bolts Approach*, Addison-Wesley (1981)