CBPF-NF-028/08

SUSY SHAPE-INVARIANT HAMILTONIANS FOR THE GENERALIZED DIRAC-COULOMB PROBLEM

R. de Lima Rodrigues

Unidade Acadêmica de Educação, Universidade Federal de Campina Grande

Cuité - PB, 58.175-000 - Brazil

Centro Brasileiro de Pesquisas Físicas (CBPF)

Rua Dr. Xavier Sigaud, 150, CEP 22290-180, Rio de Janeiro, RJ, Brazil

Arvind Narayan Vaidya (In memory)

Instituto de Física - Universidade Federal do Rio de Janeiro

Caixa Postal 68528 - CEP 21945-970, Rio de Janeiro, Brazil

Abstract

A spin $\frac{1}{2}$ relativistic particle described by a general potential in terms of the sum of the Coulomb potential with a Lorentz scalar potential is investigated via supersymmetry in quantum mechanics.

PACS numbers: 03.65.Fd, 03.65.Ge, 11.30.Pb

Key-words: SUSY Quantum Mechanics; Shape-invariant Hamiltonians; Dirac-Coulomb problem.

e-mail: rafaelr@cbpf.br or rafael@df.ufcg.edu.br. This work was presented at XIV EVJAS, section particles and fields, from 21/january to 02/February (2007), in Campos do Jordão-SP, Brazil.

Supersymmetry in Quantum Mechanics (SUSY QM) [1] is of intrinsic mathematical interest in its own as it connects otherwise apparently unrelated second-order differential equations.

The (1+3) and (1+1) dimensional Dirac equations with both scalar-like and vector-like potentials are well known in the literature for a long time [2]. The connection between position-dependent-effective-mass and shape invariant condition under parameter translation has been discussed in non-relativistic quantum mechanics [3, 4]. Recently, some relativistic shape invariant potentials have been investigated [5].

Exact solutions for the bound states in this mixed potential can be obtained by the method of separation of variables [6–8] and also by the use of the dynamical algebra $SO(2,1)$ [9]. In a recent paper the solution of the scattering problem for this potential has been obtained by an analytic method and also by an algebraic method [10], the problem of a relativistic Dirac electron with a $1/r$ scalar potential, as well as a Dirac magnetic monopole and an Aharonov-Bohm potential has also been investigated [11], and the bound eigenfunctions and spectra of a Dirac hydrogen atom have been found via $su(1,1)$ Lie algebra [12].

Recently exact solutions have been found for fermions in the presence of a classical background which is a mixing of the time-dependent of a gauge potential and a scalar potential [13]. Also, exactly solvable Eckart scalar and vector potentials in the Dirac equation have been investigated via SUSY QM [14], the S-wave Dirac equation has been solved exactly for a single particle with spin and pseudospin symmetry moving in a central Woods-Saxon potential [15].

The special case of the non-relativistic [16] and relativistic Coulomb problems have been treated recently via SUSY QM [17–19]. In this work, the relativistic Coulomb potential with a Lorentz scalar potential is investigated via shape invariance conditions of the SUSY QM.

The time independent Dirac equation may be written in the form $H\Psi = E\psi$,where the Hamiltonian is given by

$$H = \rho_1 \otimes \vec{\sigma} \cdot \vec{p} + \left(M - \frac{A_2}{r}\right) \rho_3 \otimes \mathbf{1}_{2X2} - \frac{A_1}{r} \otimes \ \mathbf{1}_{4X4},$$

and we have used a direct product notation in which ρ_i and $\sigma_i, (i = 1, 2, 3)$ are the Pauli spin matrices obeying $[\rho_i, \sigma_j]_- = 0$, with $\hbar = c = 1$.

We consider [20]

$$\Psi = \begin{pmatrix} \frac{iG_{\ell j}}{r} \phi^{\ell}_{jm} \\ \frac{F_{\ell j}}{r} \vec{\sigma} \cdot \vec{n} \phi^{\ell}_{jm} \end{pmatrix}, \tag{1}$$

where $\phi^{\ell}_{jm} = \phi^{(\pm)}_{jm}$, for $j = \ell \pm \frac{1}{2}$. Next, using the relation $[\mathbf{1} + \vec{\sigma} \cdot \vec{L}, \vec{\sigma} \cdot \vec{n}]_+ = 0$ we obtain $K\Psi = -k\Psi$ and the following radial equations

$$\begin{aligned} \frac{dG_{\ell j}}{dr} + \frac{k}{r}G_{\ell j} - \left(E + M - \frac{A_2}{r} + \frac{A_1}{r}\right) F_{\ell j} &= 0, \\ \frac{dF_{\ell j}}{dr} - \frac{k}{r}F_{\ell j} + \left(E - M + \frac{A_2}{r} + \frac{A_1}{r}\right) G_{\ell j} &= 0. \end{aligned} \tag{2}$$

Note that the interaction in these two equations can be diagonalized so that we obtain

$$A^+\hat{G} \propto \hat{F}, \quad A^-\hat{F} \propto \hat{G} \tag{3}$$

where

$$A^{\pm} = \pm\frac{d}{dr} + \frac{\lambda}{r} - \frac{EA_1 + MA_2}{\lambda}. \tag{4}$$

These relations are similar to the relations between the two components of the eigenfunctions of a "supersymmetric" Hamiltonian which satisfies the following Lie graded algebra

$$\mathcal{H} = [\mathbf{Q}, \mathbf{Q}^\dagger]_+ = \mathbf{Q}\mathbf{Q}^\dagger + \mathbf{Q}^\dagger\mathbf{Q}, \quad [\mathcal{H}, \mathbf{Q}^\dagger]_- = 0 = [\mathcal{H}, \mathbf{Q}]_- \tag{5}$$

with the following representation

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} H_- = A^+A^- & 0 \\ 0 & H_+ = A^-A^+ \end{pmatrix}, \quad \Phi_{SUSY} = \begin{pmatrix} F \\ G \end{pmatrix}. \tag{6}$$

Note that the supercharges are nilpotent operators, viz., $(\mathbf{Q}^\dagger)^2 = 0 = \mathbf{Q}^2$.

Thus, using the shape invariant Hamiltonians $H_\pm$ we obtain the energy eigenvalues associated to the component $\hat{F}^n$ given by

$$E_n = \sqrt{\frac{M^2}{1 + \frac{\gamma_n^2}{(\sqrt{k^2 - \gamma_n^2} + n)^2}}} \quad n = 0, 1, 2, \cdots, \quad \gamma_n(E) = A_1 + \frac{MA_2}{E_n}. \tag{7}$$

In conclusion, we obtain the complete set of the energy eigenvalues of the Dirac equation for a potential which is the sum of the Coulomb potential with a Lorentz scalar potential inversely proportional to r via shape invariance property as applied in [17]. One of us (RLR) will make elsewhere a detailed analysis for this problem as applied to the relativistic Coulomb potential via SUSY shape-invariant potentials [17].

Acknowledgments

RLR was supported in part by CNPq (Brazilian Research Agency). He also thanks the staff of the CBPF and CES-UFCG.

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