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A Criticism to the Tsallis Regularizing Entropy Formulae Applied to Lévy Flights in a Harmonic Potential

*Luiz C.L. Botelho**

Centro Brasileiro de Pesquisas Físicas - CBPF
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brasil

*Departamento de Física
Universidade Federal Rural do Rio de Janeiro
23851–180 – Itaguaí, RJ, Brasil

We present an apparent flaw in the proposed Tsallis entropy for a new thermodynamics.

In last years, we have seen the widespread application of the propose of ref. [1] to a new generalized Statistical Physics, the so called non extensive Constantino Tsallis entropy.

We aim at this short note to point out that such C. Tsallis statistics entropy leads, unfortunately, to an incorrect reduction of the ensemble phase-space for a mechanical system in thermal equilibrium with a reservoir for all values of the Constantino parameter q-(with exception of $q = 1$); when applied to the Fokker-Planck equation associated to Lévy flights in a harmonic potential ([2]).

Let us review briefly the C. Tsallis framework. As a first step C. Tsallis has proposed a new "entropy" functional defined over all probability distributions positive functions $P(x, v)$ defined in the mechanical phase-space of system in contact with the thermal reservoir with is always of the form $M \times R^3$, with M a manifold in R^3 (the mechanical configuration space) and R^3 the space of velocities (momenta) which is always unbounded by the very definition of statistical chaos of the system under study. The exact form of this function is obtained from the extremization of the functional below

$$S_q[P(x,v)] = \frac{1 - \int_{M\times R^3} (P(x,v))^q d^3x \times d^3v}{q-1} \tag{1}$$

with the statistics and energy conservation constraints

$$\begin{aligned} &\int_{M\times R^3} P(x,v) d^3v d^3x = 1 \\ &\int_{M\times R^3} (P(x,v))^q E(x,v) d^3v d^3x = c^{TE} \end{aligned} \tag{2}$$

Here q is a arbitrary parameter, called Tsallis parameter.

It is straightforward to see that the Tsallis function eq. (1) has the general expansion

$$S_q[P(x,v)] = -\left\{ \sum_{n=1}^{\infty} \frac{(q-1)^{n-1}}{n} \int_{M\times R^3} d^3x d^3v \; P(x,v)(ln \; P(x,v))^n \right\} \tag{3}$$

Note that for $q \to 1$, one obtain the general Boltzman result for the entropy.

At this point we follow the results presented in ref. [2] to see that the above Tsallis formalis leads to the following expression for the problem of Lévy flight in the presence of a harmonic potential (see eq. (36) – ref. [2])

$$P_q(x,v) = \left(1 - \frac{(1-q)}{2kT}(\lambda \vec{x}^{\,2} + m\vec{v}^{\,2})\right)^{\frac{1}{1-q}} \tag{4}$$

Note that, at this point, *one must consider firstly the correct mathematical domain of the phase-space probability distribution $P(x,v)$ before to proceed in any further calculations.* As a consequence, we have the following constraint for $q \neq 1$ in order to have a *real* function on $M \times R^3$. Namelly,

$$\left(1 - \frac{(1-\varepsilon)}{2kT}(\lambda \vec{x}^{\,2} + \mu \vec{v}^{\,2})\right) > 0 \tag{5}$$

or

$$(\lambda \vec{x}^{\,2} + \mu \vec{v}^{\,2}) < \frac{2kT}{1-q} \tag{6}$$

This (compact) manifold is the mathematical Domain of the function $p(\vec{x}, \vec{v})$ associated to the Lévy flight under the presence of a harmonic potential of strenght λ. Note that only for $q \to 1$, we re-obtain the usual system phase space $M \times R^3$, *and, thus, the correct Boltzman statistics.*

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References

[1] C. Tsallis, J. Stat. Phys. 52, 479 (1988).

[2] S. Jespersen, R. Metzler and Hans C. Fogedby, *"Lévy Flights in External Fields: Langevin and Fractional Fokker-Planck Equations and their Solutions"*, Nordita Pré-Print.

Note added: I have learned from Professor Constantino Tsallis that it should expected on the physical basis the existence of a "natural" cut-off (C. Tsallis et al, Physics A, 261 (1999), 534-554) for the formal mathematical rule eq. (4). Namely, one should have as the true (physical) result for the formal extremization problem (formal in the sense that the functional domain of eq. (1) - eq. (3) is left unspecified!) the following cut-off result: "For values $(\vec{x}, \vec{v})$ which turns the result eq. (4) complex one should take eq. (4) to be zero.

We think that this problem deserves a more *careful mathematical analysis* in order to apply the Tsallis' formulae for physical systems.